A RECIPE FOR BEST SEMIDEFINITE RELAXATION FOR (0,1)-QUADRATIC PROGRAMMING

Henry Wolkowicz
Department of Combinatorics and Optimization
University of Waterloo
(anonymous ftp at orion.uwaterloo.ca)

work with:
Svata Poljak (University Passau)
Franz Rendl (University of Graz)

OUTLINE

- Hidden semidefinite constraints in quadratic programming

- Best relaxations for (0,1)-quadratic programming (with linear constraints)

- Applications:

  - Quadratic Assignment Problem

  - Graph Partitioning Problem

  - Max-clique Problem

(-1,+1)-quadratic programming problem

$$(P) \quad \mu^* := \max\ q(x), \quad x \in F \cap S,$$

where:
$F = \{-1, 1\}^n$, $S \subset \Re^n$, and $F \cap S \neq \emptyset$

quadratic objective function

$$q(x) := x^t Q x - 2c^t x,$$

Q $n \times n$ symmetric matrix, $c \in \Re^n$

EXAMPLES

Quadratic Assignment Problem
(in the trace formulation)

$$(QAP) \quad \max_{X \in \Pi} \; q(X) := \operatorname{trace}(AXB - 2C)X^t$$

Max-Cut Problem

$$(MC) \quad \max \; \frac{1}{2} \sum_{i<j} w_{ij}(1 - x_i x_j), \quad x \in F$$

HIDDEN SEMIDEFINITE CONSTRAINTS

Trust Region Subproblem (TRS)

$$\begin{aligned}
\mu^* &= \min_x q(x) \text{ s.t. } x^t x = s^2 \ (\leq s^2) \\
&= \min_x \max_\lambda L(x,\lambda) \\
&\geq \max_\lambda \min_x L(x,\lambda) \\
&= \max_{Q-\lambda \succeq 0} \min_x L(x,\lambda) \\
&= \max_{Q-\lambda \succeq 0} h(\lambda) \\
&= \mu^*
\end{aligned}$$

where

$$x_\lambda = (Q - \lambda I)^\dagger c$$

$$h(\lambda) = L(x,\lambda) = -c^t (Q - \lambda I)^\dagger c + \lambda s^2$$

:nonconvex objective but min-max = max-min
:hidden convexity provides the hidden convex dual program

RELAXATIONS OF (-1,1)-QUADRATIC PROGRAMMING

use perturbations

$$q_u(x) := q(x) + x^t \mathrm{Diag}(u) x - u^t e$$

Relaxation 0:

$$f_0(u) := \max_x q_u(x)$$

$$\begin{aligned} \mu^* \le B_0 &:= \min_{u^t e = 0} f_0(u) = \min_u f_0(u) \\ &= \min_{Q + \mathrm{Diag}(u) \preceq 0} f_0(u) \end{aligned}$$

This bound equals the Lagrangian dual of the original (0,1)-quadratic program in the following form - u_i are Lagrange multipliers

$$\min q(x) \text{ s.t. } x_i^2 = 1, \ \forall i$$

Relaxation 1 - sphere of radius $\sqrt{n}$

$$f_1(u) := \max_{||x||^2=n} \; q_u(x)$$

$$\mu^* \leq B_1 := \min_{u^t e=0} f_1(u) = \min_u f_1(u)$$

$$\begin{aligned} B_1 &= \min_u \max_{x^t x=n} q_u(x) \\ &= \min_{u,\lambda} \max_x q_u(x) + \lambda(x^t x - n) \\ &= \min_{v^t e=0} \max_x q_v(x), \text{ with } v = u + \lambda e \\ &= B_0 \end{aligned}$$

Relaxation 2 - Unit Box

$$f_1(u) := \max_{|x_i| \leq = 1} \quad q_u(x)$$

$$\mu^* \leq B_2 := \min_{Q + \mathrm{Diag}(u) \preceq 0} f_2(u) = \min_u f_2(u)$$

$$\begin{aligned} B_2 &= \min_u \max_{x_i^2 \leq 1} q_u(x) \\ &= \min_u \min_{\lambda \geq 0} \max_x q_u(x) + \sum_i \lambda_i (1 - x_i^2) \\ &= B_0 \quad \text{after} \quad v = u - \lambda \end{aligned}$$

again

Relaxation 1^c - Homogenization,

- -sphere radius $= \sqrt{n+1}$

$$Q^c := \begin{bmatrix} 0 & -c^t \\ -c & Q \end{bmatrix}$$

$$q_u^c(y) := y^t(Q^c + \text{diag}(u))y - u^t e$$

$$\begin{aligned} f_1^c(u) &:= \max_{\|y\|^2 = n+1} q_u^c(y) \\ &= (n+1)\lambda_{\max}(Q^c + \text{diag}(u)) - u^t e \end{aligned}$$

$$\mu^* \leq B_1^c := \min_{u^t e = 0} f_1^c(u) = \min_u f_1^c(u)$$

$$\begin{aligned} B_1^c &= \min_v \max_{y^t y = n+1} q_v^c(y) = \min_v \max_y q_v^c(y) \\ &= \min_{u,u_0} \max_{x,x_0} u_0(x_0^2 - 1) + x^t(Q + \text{Diag}(u))x \\ &\qquad - 2x_0 c^t x - u^t e \\ &= \min_u \max_{x, x_0^2 = 1} x^t(Q + \text{Diag}(u))x - 2x_0 c^t x - u^t e \\ &= B_0 \end{aligned}$$

again

Similarly for B_2^c

Relaxation 3 - semidefinite SDP ($c = 0$)

$$q(x) = x^t Q x = \text{trace}\, Q x x^t$$

with $Y = xx^t$

$$\begin{array}{rcc} B_3 := & \max & \text{trace}\, QY \\ & \text{subject to} & \text{diag}(Y) = e \\ & & Y \succeq 0 \end{array}$$

The dual is

$$\begin{array}{rcc} B_3 := & \text{minimize} & y^t e \\ & \text{subject to} & Q - \text{Diag}(y) \preceq 0 \end{array}$$

———————–

$$Q - \text{Diag}(y - \frac{e^t y}{n} e) \preceq \frac{e^t y}{n} I$$

with $w = y - \frac{e^t y}{n} e$ and $z = \frac{e^t y}{n}$

$$\begin{array}{rcc} B_3 := & \text{minimize} & nz \\ & \text{subject to} & Q - \text{Diag}(w) \preceq zI \\ & & w^t e = 0 \\ = & B_1^c & \end{array}$$

All the bounds are equal to the Lagrangian relaxation of the equivalent quadratically constrained program.

The SDP relaxation is the dual of the Lagrangian dual.

What happens if we allow more general perturbations?

More General Perturbations

$$q_{V,d}(x) := x^t(Q+V)x + (c+d)^t x$$

Theorem 1 *Suppose that*

$$q_{V,d}(x) \geq q(x), \quad \forall x \in F.$$

Then

$$V = P + U, \text{ with } P \succeq 0, \; U \text{ is diagonal, and } \operatorname{trace} U = 0.$$

Moreover, there exists a diagonal matrix W, *with* $\operatorname{trace} W = 0$, *such that*

$$\max_x q_{V,d}(x) \geq \max_x q_{W,0}(x).$$

□

Therefore, we need only consider diagonal perturbations, i.e. we have the best quadratic approximation - by duality we have the best SDP relaxation.

General Case - allow linear constraints

Equivalent quadratic program

$$\begin{array}{rcc} \mu^* = & \max & q(x) = x^t Q x - 2c^t x \\ & \text{subject to} & \|Ax - b\|^2 = 0 \\ & & x_i^2 = 1, \quad \forall i. \end{array}$$

RECIPE for SDP relaxation

1. Replace (P) by the equivalent quadratic program
2. Take Lagrangian dual - get min-max of Lagrangian
3. Homogenize Lagrangian
4. Use hidden semidefinite constraint to get SDP
5. Take Lagrangian dual of SDP to get desired SDP

The Lagrangian relaxation of the equivalent quadratic program yields the best possible quadratic bound.

Theorem 2 *Suppose that the set S is described by linear equalities as above. Suppose that the general perturbed quadratic function $q_{V,d}$ is defined as above and*

$$q_{V,d,\lambda} := q_{V,d} - \lambda ||Ax - b||^2.$$

If

$$q_{V,d}(x) \geq q(x), \quad \forall x \in F \cap S,$$

then there exists λ, W such that

$$\begin{array}{c} q_{V,d} \equiv q_{W,b,\lambda}, \\ W = P + U, \text{ with } P \succeq 0, \\ U \text{ is diagonal,, } \operatorname{trace} U = 0, \\ \text{and } b = d - 2\lambda A^t b. \end{array}$$

Moreover, there exists a diagonal Z with $\operatorname{trace} Z = 0$ such that

$$\max_x q_{V,d}(x) \geq \max_x q_{Z,0,\lambda}(x).$$

□

QUADRATIC ASSIGNMENT PROBLEM

$$\begin{array}{cc} \max & q(X) = \operatorname{trace}(AXB - 2C)X^t \\ \text{subject to} & XX^t = I \\ & X_{ij}^2 - X_{ij} = 0, \quad \forall i, j. \end{array}$$

Apply the recipe:

$$L_Q := \begin{bmatrix} 0 & -\operatorname{vec}(C)^t \\ -\operatorname{vec}(C) & B \otimes A \end{bmatrix},$$

$$\mathrm{B}^0\mathrm{Diag}(\Lambda) := \begin{bmatrix} 0 & 0 \\ 0 & I \otimes \Lambda \end{bmatrix}.$$

and $\mathrm{b}^0\mathrm{diag} = \mathrm{B}^0\mathrm{Diag}^*$ adjoint operator

---

Then SPD relaxation is:

$$\begin{array}{cc} \max & \operatorname{trace} L_Q Y \\ \text{subject to} & \operatorname{diag}(Y) = (1, Y_{0,1:n^2})^t \\ & \mathrm{b}^0\mathrm{diag}(Y) = I \\ & Y \succeq 0. \end{array}$$

GRAPH PARTITIONING

$$\begin{array}{cc} w(E_{uncut}) = \max & \frac{1}{2}\text{trace}\, X^t A X \\ \text{subject to} & X e_k = e_n \\ & X^t e_n = m \\ & X_{ij} \in \{0,1\},\ \forall ij, \end{array}$$

$$\begin{array}{cc} \max & \frac{1}{2}\text{trace}\, X^t A X \\ \text{subject to} & \|X e_k - e_n\|^2 + \|X^t e_n - m\|^2 = 0 \\ & X_{ij}^2 - X_{ij} = 0,\ \forall ij. \end{array}$$

$$L_A := \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} I \otimes A \end{bmatrix},$$

$$v = \text{vec}\, e_n m^t,$$

$$L_\alpha := \begin{bmatrix} 0 & -(e+v)^t \\ -(e+v) & (e_k e_k^t I \otimes I + I \otimes e_n e_n^t) \end{bmatrix}.$$

The SDP is:

$$\begin{array}{cc} \max & \text{trace}\, L_A Y \\ \text{subject to} & \text{diag}(Y) = (1, Y_{0,1:n})^t \\ & \text{trace}\, Y L_\alpha = 0 \\ & Y \succeq 0. \end{array}$$

MAX-CLIQUE and STABLE SET

$\omega(G)$ size of largest clique in graph G

x is indicator vector for largest clique

$$\begin{array}{rcl} \omega(G) = & \max & x^t x \\ & \text{subject to} & x_i x_j = 0, \text{ if } ij \notin E, \; i \neq j \\ & & x_i^2 - x_i = 0, \; \forall i. \end{array}$$

$$L_A := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

———————-

SDP relaxation is

$$\begin{array}{cc} \max & \operatorname{trace} L_A Y \\ \text{subject to} & \operatorname{diag}(Y) = (1, Y_{0,1:n})^t \\ & Y_{ij} = 0, \; \forall ij \notin E \\ & Y \succeq 0. \end{array}$$