Facial Reduction in Cone Optimization with Applications to Sensor Network Localization and Low Rank Matrix Completion

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** Motivation: Loss of Slater CQ/Facial reduction

- Slater condition – existence of a strictly feasible solution – is at the heart of convex optimization.
- Without Slater: first-order optimality conditions may fail; dual problem may yield little information; small perturbations may result in infeasibility; many software packages can behave poorly.
- a pronounced phenomenon: though Slater holds generically, surprisingly many models arising from relaxations of hard nonconvex problems show loss of strict feasibility, e.g., Matrix completions/compressive sensing, sensor network localization, SNL, EDM, POP, Molecular Conformation, QAP, GP, strengthened Max-Cut, and for constraints and objective in Quantum Computing (QKD). ($\approx$ 70% of NETLIB LP problems fail strict feasibility)
- We concentrate on Semidefinite Programming, SDP. We look at various reasons and how to take advantage using two views of FACIAL REDUCTION, FR

Main Ref. for Facial Reduction (FR)
"The many faces of degeneracy in conic optimization", Drusvyatskiy, Wolkowicz 2016 [5]

Outline

1. Facial reduction/preproc. for LP (intro to FR) (P4)
2. FR in General and abstract convex programming (P8)
3. Semidefinite programming case (P16)
4. Application to EDM, SNL (Krislock et W. et al ’10,’15, [3, 9] P29)
5. Application to Low-Rank Matrix Completion, LRMC, (Huang-W.’16 [8], P54)

** Facial Reduction/Preprocessing for LP

Primal-Dual Pair: A onto, $m \times n$, $\mathcal{P} = \{1, \ldots, n\}$

(LP-P) $\max \; b^\top y$ s.t. $A^\top y \le c$

(LP-D) $\min \; c^\top x$ s.t. $Ax = b$, $x \ge 0$.

- dual x shadow prices of *resources* c,
- internal, better indicator than market prices
- x_i >market price$_i$ implies it is worth paying more for resource$_i$
- strict feasiblity fails $\Longrightarrow$ shadow prices lose proper meaning

Slater's CQ for (LP-D) / Theorem of alternative

Exactly One is True:

(I) $\exists \hat{x}$ s.t. $A\hat{x} = b, \hat{x} > 0$ $\quad (\hat{x} \in \operatorname{ri} F)$
Slater point

(II) $0 \ne z = A^\top y \ge 0,\; b^\top y = 0$ $\quad (\langle z, F\rangle = 0)$
exposing vector

Linear Programming Example, $x \in \mathbb{R}^5$

$$\min \quad \begin{pmatrix} 2 & 6 & -1 & -2 & 7 \end{pmatrix} x$$
$$\text{s.t.} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \end{bmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$x \geq 0$$

Sum the two constraints (multiply by: $y^T = (1\ 1)$):
get: $2x_1 + x_4 + x_5 = 0 \implies x_1 = x_4 = x_5 = 0$
i.e., equiv. simplified problem/smaller face/ fewer constr.

$$\boxed{\min 6x_2 - x_3 \text{ s.t. } x_2 + x_3 = 1, x_2, x_3 \geq 0,}$$
$$(x_1 = x_4 = x_5 = 0)$$

Theorem

Strict feasibility fails imples EVERY BFS is degenerate.
And, there is an implicit singularity.

Linear Programming, LP

Slater's CQ for (LP-P) / Theorem of alternative

$\exists \hat{y}$ s.t. $c - A^\top \hat{y} > 0, \qquad \left(\left(c - A^\top \hat{y} \right)_i > 0, \forall i \in \mathcal{P} =: \mathcal{P}^l \right)$

iff

$Ad = 0, \ c^\top d = 0, \ d \geq 0 \implies d = 0 \qquad (*)$

implicit equality constraints: $i \in \mathcal{P}^e$

Find $0 \neq d^*$ to $(*)$ with max number of non-zeros
(exposes minimal face containing feasible slacks)

$$d_i^* > 0 \implies (c - A^\top y)_i = 0, \forall y \in \mathcal{F}^y \quad i \in \mathcal{P}^e)$$

(where $\mathcal{F}^y$ is primal feasible set)

Make implicit-equalities explicit/ Regularizes LP

Facial Reduction: $A^\top y \leq_f c$; minimal face $f \trianglelefteq \mathbb{R}^n_+$
proper primal-dual pair; dual of dual is primal

$$(\mathrm{LP}_{reg}\text{-P}) \qquad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & (A^l)^\top y \leq c^l \\ & (A^e)^\top y = c^e \end{array}$$

$$(\mathrm{LP}_{reg}\text{-D}) \qquad \begin{array}{ll} \min & (c^l)^\top x^l + (c^e)^\top x^e \\ \text{s.t.} & \begin{bmatrix} A^l & A^e \end{bmatrix} \begin{pmatrix} x^l \\ x^e \end{pmatrix} = b \\ & x^l \geq 0,\ x^e \text{ free} \end{array}$$

Generalized Slater CQ holds - And!

after deleting redundant equality constraints! (at least one)
Mangasarian-Fromovitz CQ (MFCQ) holds

$$\left(\exists \hat{y} : \quad (A^l)^\top \hat{y} < c^l, \quad (A^e)^\top \hat{y} = c^e \right) \qquad (A^e)^\top \text{ is onto}$$

MFCQ holds iff dual optimal set is compact

Numerical difficulties if MFCQ fails; in particular for interior point methods! Modelling issue!

** General convex programming

Ordinary convex programming, (OCP)

$$\text{(CP)} \qquad \sup_y \; b^\top y \text{ subject to } \; g(y) \le 0$$

$b \in \mathbb{R}^m$; $g(y) = \big(g_i(y)\big) \in \mathbb{R}^n$, $g_i : \mathbb{R}^m \to \mathbb{R}$ convex, $\forall i \in \mathbb{P}$

Slater's CQ; strict feasibility

$\exists\, \hat{y}$ s.t. $g_i(\hat{y}) < 0, \forall i$ (implies MFCQ)

Slater's CQ fails $\boxed{\iff}$ implicit equality constraints exist

$\mathcal{P}^e := \{i \in \mathcal{P} : g(y) \le 0 \implies g_i(y) = 0\} \neq \emptyset$

Let $\mathcal{P}^l := \mathcal{P} \backslash \mathcal{P}^e$ and

$$g^l := (g_i)_{i \in \mathcal{P}^l}, \qquad g^e := (g_i)_{i \in \mathcal{P}^e}$$

implicit equalities to *equalities*/ Regularize OCP

Minimal face f

$f = \{z \in \mathbb{R}^m_+ \ : \ z_i = 0, \forall i \in \mathcal{P}^e\} \trianglelefteq \mathbb{R}^m_+$

(OCP) is equivalent to $g(y) \leq_f 0$

$$(\mathrm{OCP_{reg}}) \qquad \begin{array}{ll} \sup & b^\top y \\ \text{s.t.} & g^l(y) \leq 0 \\ & y \in \mathcal{F}^e \end{array}$$

where $\mathcal{F}^e := \{y : g^e(y) = 0\}$.

Then $\boxed{\mathcal{F}^e = \{y : g^e(y) \leq 0\}, \quad \text{so is a convex set!!}}$

Slater's CQ holds for (OCP_{reg}) $\boxed{\exists \hat{y} \in \mathcal{F}^e : g^l(\hat{y}) < 0}$

modelling issue again! (BBZ Conditions '80)

FYI Aside: Faithfully convex case

Faithfully convex function f (Rockafellar'70)

f affine on a line segment only if affine on complete line containing the segment
(e.g. analytic convex functions)

$\mathcal{F}^e = \{y : g^e(y) = 0\}$ is an affine set

Then:
$\mathcal{F}^e = \{y : Vy = V\hat{y}\}$ for some $\hat{y}$ and full-row-rank matrix V.

Then MFCQ holds for regularized

$$
(\text{OCP}_{\text{reg}}) \qquad \begin{array}{llll} \sup & b^\top y & & \\ \text{s.t.} & g^I(y) & \leq & 0 \\ & Vy & = & V\hat{y} \end{array}
$$

* (FYI - full generality) Abstract convex program

(ACP) $\inf_x f(x)$ s.t. $g(x) \preceq_K 0, x \in \Omega$

where:

- $f : \mathbb{R}^n \to \mathbb{R}$ convex; $g : \mathbb{R}^n \to \mathbb{R}^m$ is K-convex
  - $K \subset \mathbb{R}^m$ closed convex cone; $\Omega \subseteq \mathbb{R}^n$ convex set
  - $a \preceq_K b \iff b - a \in K$, $a \prec_K b \iff b - a \in \operatorname{int} K$
  - $g(\alpha x + (1 - \alpha y)) \preceq_K \alpha g(x) + (1 - \alpha) g(y)$,
    $\forall x, y \in \mathbb{R}^n, \forall \alpha \in [0, 1]$

Slater's CQ: $\exists \hat{x} \in \Omega$ s.t. $g(\hat{x}) \in -\operatorname{int} K$ $(g(x) \prec_K 0)$

- guarantees strong duality
  (zero duality gap AND dual attainmment)
- (near) loss of strict feasibility, nearness to infeasibility, correlates with number of iterations & loss of accuracy
- Recall that Slater (M-F) is equivalent to a nonempty bounded dual optimal set.

Faces of Convex Sets - Useful for Charact. of Opt.

Face of C, $\boxed{F \trianglelefteq C}$

- $F \subseteq C$ is a face of C if F contains any line segment in C whose relative interior intersects F.
- A convex cone $F \subseteq K$ is a face of a convex cone K, $F \trianglelefteq K$, if (simplified)
$$\boxed{x, y \in K \text{ and } x + y \in F \implies x, y \in F}$$

Polar (Dual) Cone/Conjugate Face

- polar cone $\quad K^* := \{\phi : \langle \phi, k \rangle \geq 0, \ \forall k \in K\}$
- If $F \trianglelefteq K$, the conjugate face of F is
$$\boxed{F^c := F^\perp \cap K^* \trianglelefteq K^*}$$

Properties of Faces

General case

- A face of a face is a face
- intersection of a face with a face is a face.
- Let $C \subseteq K$, then $\mathrm{face}(C)$ denotes the minimal face (intersection of faces) containing C.

$F \trianglelefteq K$ is an **exposed face** if there exists $\phi \in K^*$ with

$$F = K \cap \phi^{\perp}$$

F^c is always exposed by $x \in \mathrm{ri}\, F$.

The SDP cone is facially exposed, all its faces are exposed. (In fact like $\mathbb{R}^n_+$: $\mathcal{S}^n_+$ is a proper closed convex cone, self-dual and facially exposed.)

Regularize abstract convex program (full generality)

$$\text{(ACP)} \quad \inf_x f(x) \text{ s.t. } g(x) \preceq_K 0, x \in \Omega$$

(Borwein-W.'81)

$$(\text{ACP}_R) \quad \inf_x f(x) \text{ s.t. } g(x) \preceq_{K^f} 0, x \in \Omega$$

where: K^f is the minimal face

Like LP, it is simple if we use the minimal face K^f.
We get a proper primal-dual pair?

Recall: (ACP) $\inf_x f(x)$ s.t. $g(x) \preceq_K 0, x \in \Omega$

- polar cone: $K^* = \{\phi : \langle \phi, y \rangle \geq 0, \forall y \in K\}$.
- $K^f := \mathrm{face}(F)$ minimal face containing feasible set F.

Lemma (Facial Reduction (FR); find EXPOSING vector ϕ)

Suppose $\bar{x}$ is feasible. Then the LHS system

$$\left\{ \begin{array}{ll} (\Omega - \bar{x})^* \cap \partial \langle \phi, g(\bar{x}) \rangle \neq \emptyset \\ \phi \in K^*, & \langle \phi, g(\bar{x}) \rangle = 0 \end{array} \right\} \quad \textit{implies} \quad K^f \subseteq \phi^\perp \cap K,$$

where: ∂ is subgradient; $\langle \cdot \rangle$ is inner-product.

Proof

line 1 of system implies $\bar{x}$ global min for convex function $\langle \phi, g(\cdot) \rangle$ on Ω; i.e., $0 = \langle \phi, g(\bar{x}) \rangle \leq \langle \phi, g(x) \rangle \leq 0, \forall x \in F$; implies $-g(F) \subseteq \phi^\perp \cap K$. □

* SDP Case/Replicating Cone/Faces

SDP case/Replicating cone

- Let $X \in \mathcal{S}^n_+$ with spectral decomposition,

$$X = [P\ Q] \begin{bmatrix} D_+ & 0 \\ 0 & 0 \end{bmatrix} [P\ Q]^T, \quad D_+ \in \mathbb{S}^r_{++} \quad (\operatorname{rank} X = r)$$

- Then

$$\text{Range}(X) = \text{Range}(P), \quad \text{Null}(X) = \text{Range}(Q)$$

$$\text{face}(X) = P\mathbb{S}^r_+ P^T = (QQ^T)^\perp \cap \mathcal{S}^n_+ .$$

($Z = QQ^T$ *exposing vector/matrix* for face.)

- $$\text{face}(X)^c = Q\mathbb{S}^{n-r}_+ Q^T$$

Range/Nullspace representations

$$\textit{face}(X) = \left\{ Y \in \mathcal{S}^n_+ : \text{Range}(Y) \subseteq \text{Range}(X) \right\}$$
$$\textit{face}(X) = \left\{ Y \in \mathcal{S}^n_+ : \text{Null}(Y) \supseteq \text{Null}(X) \right\}$$
$$\text{ri } \textit{face}(X) = \left\{ Y \in \mathcal{S}^n_+ : \text{Range}(Y) = \text{Range}(X) \right\}$$

Semidefinite Programming, SDP, $\mathcal{S}^n_+$

$K = \mathcal{S}^n_+ = K^*$: nonpolyhedral, self-polar, facially exposed

$$\text{(SDP-P)} \quad v_P = \sup_{y \in \mathbb{R}^m} b^\top y \text{ s.t. } g(y) := \mathcal{A}^* y - c \preceq_{\mathcal{S}^n_+} 0$$

$$\text{(SDP-D)} \quad v_D = \inf_{x \in \mathcal{S}^n} \langle c, x \rangle \text{ s.t. } \mathcal{A} x = b, \ x \succeq_{\mathcal{S}^n_+} 0$$

where:

- PSD cone $\mathcal{S}^n_+ \subset \mathcal{S}^n$ symm. matrices
- $c \in \mathcal{S}^n$, $b \in \mathbb{R}^m$
- $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ is an onto linear map, with adjoint $\mathcal{A}^*$
- $\mathcal{A} x = (\operatorname{trace} A_i x) = (\langle A_i, x \rangle) \in \mathbb{R}^m, \quad A_i \in \mathcal{S}^n$
  $\mathcal{A}^* y = \sum_{i=1}^m A_i y_i \in \mathcal{S}^n$

Slater's CQ/Theorem of Alternative simplifies for SDP

Assume feasibility: $\exists\, \tilde{y}$ s.t. $c - \mathcal{A}^*\tilde{y} \succeq 0$.

Exactly one of the following alternatives holds/is consistent:

$$(I) \quad \exists\, \hat{y} \text{ s.t. } s = c - \mathcal{A}^*\hat{y} \succ 0 \qquad (\textit{Slater})$$

or

$$(II) \quad \mathcal{A}d = 0,\ \langle c, d\rangle = 0,\ 0 \neq d \succeq 0 \quad (*)$$

In case (II): - finds exposing vector: $0 \neq d \succeq 0$

d exposes a proper face containing all the feasible slacks

$$z = c - \mathcal{A}^*y \succeq 0 \implies zd = 0. \quad (\text{equiv. trace}\, zd = 0)$$

Regularization Using Minimal Face

Borwein-W.'81 , $f_P = \text{face}\,\mathcal{F}_P^s$; min. face of feasible slacks

(SDP-P) is equivalent to the regularized

$$(\text{SDP}_{reg}\text{-P}) \qquad v_{RP} := \sup_y \{\langle b, y\rangle \;:\; \mathcal{A}^* y \preceq_{f_P} c\}$$

f_p is minimal face of primal feasible slacks

$\{s \succeq 0 : s = c - \mathcal{A}^* y\} \subseteq f_p \trianglelefteq \mathcal{S}_+^n$

Lagrangian dual of regularized problem satisfies strong duality:

$$(\text{SDP}_{reg}\text{-D}) \quad v_{DRP} \quad := \inf_x \{\langle c, x\rangle \;:\; \mathcal{A} x = b,\; x \succeq_{f_P^*} 0\}$$

$v_P = v_{RP} = v_{DRP}$ and v_{DRP} is attained.

regularized primal-dual pair (dual of dual is primal)

If we take the dual of $(\text{SDP}_{reg}\text{-D})$ we recover the primal regularized problem $(\text{SDP}_{reg}\text{-P})$.

Slater's CQ/Theorem of Alternative for Dual

Assume feasibility: $\exists\, \tilde{x}$ s.t. $\mathcal{A}\,\tilde{x} = b, \tilde{x} \succeq 0$.

Exactly one of the following alternatives holds/is consistent:

$$(I) \quad \exists\, \hat{x} \;\text{ s.t. } \mathcal{A}\,\hat{x} = b, \hat{x} \succ 0 \qquad (\textit{Slater})$$

or

$$(II) \quad 0 \neq z = \mathcal{A}^* y \succeq 0, \; \langle b, y \rangle = 0, \qquad (**)$$

(II) finds exposing vector: $0 \neq z \succeq 0$

z exposes a proper face containing all the dual feasible points

$$\mathcal{A}\,x = b, x \succeq 0 \implies zx = 0. \quad (\text{equiv. } \operatorname{trace} zx = 0)$$

Regularization of Dual Using Minimal Face

Borwein-W.'81 , $f_D = \text{face}\,\mathcal{F}^x_D$; min. face of dual feasible set

(SDP-D) is equivalent to the regularized

$$(\text{SDP}_{reg}\text{-D}) \qquad v_{RD} := \inf_x \{\langle c, x\rangle \;:\; \mathcal{A}x = b, x \succeq_{f_D} 0\}$$

f_D is mininimal face of dual feasible set

$\{x \succeq 0 : \mathcal{A}x = b, x \succeq 0\} \subseteq f_D \trianglelefteq \mathcal{S}^n_+$

Lagrang. dual of regulariz. dual problem satisfies strong duality:

$$(\text{SDP}_{reg}\text{-DD}) \quad v_{DRD} \quad := \sup_y \{\langle b, y\rangle \;:\; \mathcal{A}^*y \preceq_{f_D^*} c\}$$

$v_D = v_{RD} = v_{DRD}$ and v_{DRD} is attained.

regularized primal-dual pair

If we take the dual of $(\text{SDP}_{reg}\text{-DD})$ we recover the dual regularized problem $(\text{SDP}_{reg}\text{-P})$.

View One for FR in SDP

(SDP_D) $\min\{\operatorname{trace} CX \text{ s.t. } \mathcal{A}X = b, X \in \mathcal{S}_+^n\}$

Step 1: Let $0 \neq Z \succeq 0$ be an exposing vector.

add constraint $\operatorname{trace} ZX = 0$. (Equivalently $ZX = 0$)

from spectral decomposition of Z, with $\operatorname{Range} P = \operatorname{Null} Z$:

substitute: $X = P\mathbb{S}_+^{t_1}P^T, \quad t_1 = \operatorname{nullity}(Z)$

We get the equivalent smaller problem

$$(SDP_{D1}) \quad \begin{array}{ll} \min & \operatorname{trace}(P^T CP)R \\ \text{s.t.} & \operatorname{trace}(P^T A_i P)R = b_i, i = 1, \ldots, m \\ & R \in \mathbb{S}_+^{t_1} \end{array}$$

Remove/delete redundant linear constraints;

repeat from Step 1.

minimum number of steps is called the singularity degree

View Two for FR in SDP

Lemma: Using exposing vectors

Let

$$Z_i \succeq 0, F_i = \mathcal{S}_+^n \cap Z_i^\perp, i = 1, \ldots, m.$$

Then

$$\cap_{i=1}^m F_i = \mathcal{S}_+^n \cap \left(\sum_{i=1}^m Z_i\right)^\perp$$

intersection of faces is exposed by sum of exposing vectors □

Equivalence of exposing vectors with image set

Thm: DPW '15 : $F := F_P = \{x \in \mathcal{K} : \mathcal{A}\,x = b\} \neq \emptyset$

Vector v exposes a proper face of $\mathcal{A}(\mathcal{K})$ containing b
iff v satisfies the auxiliary system

$$0 \neq \mathcal{A}^* v \in \mathcal{K}^* \quad \text{and} \quad \langle v, b\rangle = 0.$$

And the following are true.
(I) We always have:

$$\boxed{\mathcal{K} \cap \mathcal{A}^{-1}(\operatorname{face}(b, \mathcal{A}(\mathcal{K}))) = \operatorname{face}(F, \mathcal{K})}$$

(II) For any vector $w \in \mathbb{Y}$ the following equivalence holds:

$$\boxed{w \text{ exposes } \operatorname{face}(b, A(\mathcal{K})) \iff \mathcal{A}^* w \text{ exposes } \operatorname{face}(F, C)}$$

(III) Consequently Slater condition failing implies:
singularity degree $d = 1$ for the system
iff the minimal face $\operatorname{face}(b, \mathcal{A}(C))$ is exposed. □

- at most $n-1$ iterations to satisfy Slater's CQ.
- to check Theorem of Alternative

$$\mathcal{A}d = 0,\ \langle c, d\rangle = 0,\ 0 \neq d \succeq_{\mathcal{S}^n_+} 0, \qquad (*)$$

use stable auxiliary problem

$$(AP) \qquad \min_{\delta, d}\ \delta \ \text{ s.t. } \left\| \begin{bmatrix} \mathcal{A}d \\ \langle c, d\rangle \end{bmatrix} \right\|_2 \leq \delta, \quad \operatorname{trace}(d) = \sqrt{n}, \quad d \succeq 0.$$

- Both (AP) with e.g. $d = I, \delta >> 0$, and its dual satisfy Slater's CQ.

Auxiliary Problem

$$(AP) \qquad \min_{\delta,d} \ \delta \ \text{ s.t. } \left\| \begin{bmatrix} \mathcal{A}d \\ \langle c, d \rangle \end{bmatrix} \right\|_2 \leq \delta, \quad \text{trace}(d) = \sqrt{n}, \ d \succeq 0.$$

Both (AP) and its dual satisfy Slater's CQ ... but ...

Cheung-Schurr-W'11, a $k = 1$ step CQ

Strict complementarity holds for (AP)

iff

$k = 1$ steps are needed to regularize (SDP-P).

$k = 1$ always holds in LP case.

($k = 1$ is a special/regular case.)

Singularity Degree d - Minimal Number of FR Steps

Sturm's error bounds Theorem for SDP, 2000

Given an affine subspace $\mathcal{V}$ of $\mathcal{S}^n$, the pair $(\mathcal{V}, \mathcal{S}^n_+)$ is $\frac{1}{2^d}$-Holder regular, $\gamma = \frac{1}{2^d}$, with displacement, where d is the singularity degree of $(\mathcal{V}, \mathcal{S}^n_+)$ with displacement.
(e.g., for intersecting sets, for all compact sets U there exists a constant $c > 0$ such that
$\operatorname{dist}(x, \mathcal{V} \cap \mathcal{S}^n_+) \leq c \left(\operatorname{dist}^\gamma(x, \mathcal{V}) + \operatorname{dist}^\gamma(x, \mathcal{S}^n_+)\right), \quad \forall x \in U$)

Cgnce rate alternating directions (MAP) for SDP

Theorem (Drusvyatskiy, Li, W. 2015) If the sequence X_k, Y_k converges, $d > 0$, then the rate is $\mathcal{O}\left(k^{-\frac{1}{2^{d+1}-2}}\right)$
(If Slater holds then cgnce is R-linear.)

(Paper includes Empirical Confirmation)

Applications?

- preprocessing is essential in commercial LP software.
- Can we do facial reduction in general?
- Is it efficient/worthwhile?
- important applications?
  - relation to feasibility questions, e.g., for matrix completion
  - iterative methods? convergence rates? (DR, MAP)

** FR - Motivation/Application; EDM, SNL

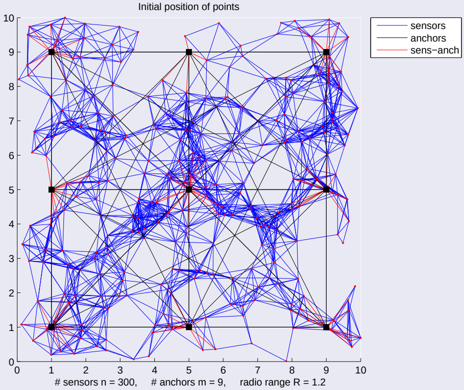
Highly (implicit) degenerate/low-rank problem

- high (implicit) degeneracy translates to low rank solutions
- take advantage of degeneracy; fast, high accuracy solutions

SNL - a Fundamental Problem of Distance Geometry; easy to describe - dates back to Grasssmann 1886

- r : embedding dimension
- n ad hoc wireless sensors $p_1, \ldots, p_n \in \mathbb{R}^r$ to locate in $\mathbb{R}^r$;
- m of the sensors $p_{n-m+1}, \ldots, p_n$ are anchors (positions known, using e.g. GPS)
- pairwise distances $D_{ij} = \|p_i - p_j\|^2, ij \in E$, are known within radio range $R > 0$
- $$P^\top = \begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix} = \begin{bmatrix} X^\top & A^\top \end{bmatrix} \in \mathbb{R}^{r \times n}$$

Sensors ○ and Anchors ■



Underlying Graph Realization/Partial EDM NP-Hard

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

- node set $\mathcal{V} = \{1, \ldots, n\}$
- edge set $(i, j) \in \mathcal{E}$; $\omega_{ij} = \|p_i - p_j\|^2$ known approximately
- The anchors form a CLIQUE (complete subgraph)
- Realization of $\mathcal{G}$ in $\mathbb{R}^r$: a mapping of nodes $v_i \mapsto p_i \in \mathbb{R}^r$ with squared distances given by ω.

Corresponding Partial Euclidean Distance Matrix, EDM

$$D_{ij} = \begin{cases} d_{ij}^2 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise (unknown distance)}, \end{cases}$$

$d_{ij}^2 = \omega_{ij}$ are known squared Euclidean distances between sensors p_i, p_j; anchors correspond to a clique.

EDM Connections to SDP

$D = \mathcal{K}(B) \in \mathcal{E}^n$, $B = \mathcal{K}^\dagger(D) \in \mathcal{S}^n \cap \mathcal{S}_C$ (centered $Be = 0$)

$P^\top = \begin{bmatrix} p_1 & p_2 & \dots & p_n \end{bmatrix} \in \mathcal{M}^{r \times n}$;
$B := PP^\top \in \mathcal{S}^n_+$ (Gram matrix of inner products);
$\operatorname{rank} B = r$; let $D \in \mathcal{E}^n$ corresponding EDM; $e = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^\top$

$$
\begin{array}{rcl}
(\text{to } D \in \mathcal{E}^n) \quad D & = & \left(\|p_i - p_j\|_2^2\right)_{i,j=1}^n \\
& = & \left(p_i^T p_i + p_j^T p_j - 2p_i^T p_j\right)_{i,j=1}^n \\
& = & \boxed{\operatorname{diag}(B)\, e^\top + e \operatorname{diag}(B)^\top - 2B} \\
& =: & \mathcal{K}(B) \quad (\text{from } B \in \mathcal{S}^n_+).
\end{array}
$$

Moore-Penrose Generalized Inverse $\mathcal{K}^\dagger$, $J = I - \frac{1}{n}ee^T$

$$B \succeq 0 \quad \Longrightarrow \quad D = \mathcal{K}(B) = \operatorname{diag}(B)\, e^\top + e \operatorname{diag}(B)^\top - 2B \in \mathcal{E}$$

$$D \in \mathcal{E} \quad \Longrightarrow \quad B = \mathcal{K}^\dagger(D) = -\frac{1}{2} J \operatorname{offDiag}(D)\, J \succeq 0,\ Be = 0$$

Theorem (Schoenberg, 1935)

A (hollow) matrix D (with $\operatorname{diag}(D) = 0$, $D \in \mathbb{S}_H$) is a EDM if and only if

$$B = \mathcal{K}^\dagger(D) \succeq 0. \quad (\text{and centered } Be = 0, B \in \mathbb{S}_C)$$

And !!!!

$$\boxed{\operatorname{embdim}(D) = \operatorname{rank}\left(\mathcal{K}^\dagger(D)\right), \quad \forall D \in \mathcal{E}^n} \tag{1}$$

Popular Techniques; SDP Relax.; Highly Degen.

Nearest, Weighted, SDP Approx. (relax/discard $\operatorname{rank} B$)

- $\min_{B \succeq 0} \| H \circ (\mathcal{K}(B) - D) \|$

  $\operatorname{rank} B = r; \qquad H_{ij} = \begin{cases} 1/\sqrt{D_{ij}} & \text{if } ij \in E, \\ H_{ij} = 0 & \text{otherwise} \end{cases}$
- with rank constraint: a non-convex, NP-hard program
- SDP relaxation is convex

  BUT: expensive/low accuracy/implicitly highly degenerate (cliques restrict ranks of feasible B)

Take Advantage of Degeneracy! Krislock W.'10

clique α, $|\alpha| = k$ (corresp. submatrix EDM $D[\alpha]$)

$$\left(\operatorname{rank} \mathcal{K}^{\dagger}(D[\alpha]) = t \leq r\right) \implies \left(\operatorname{rank} B[\alpha] \leq \operatorname{rank} \mathcal{K}^{\dagger}(D[\alpha]) + 1\right)$$

$$\implies \operatorname{rank} B = \operatorname{rank} \mathcal{K}^{\dagger}(D) \leq n - \boxed{(k - t - 1)}$$

implies

Slater's CQ (strict feasibility) fails

Basic Single Clique/Facial Reduction

Matrix with Fixed Principal Submatrix

For $Y \in \mathcal{S}^n$, $\alpha \subseteq \{1,\ldots,n\}$: $Y[\alpha]$ denotes principal submatrix formed from rows & cols with indices α.

$\bar{D} \in \mathcal{E}^k$, $\alpha \subseteq 1:n$, $|\alpha| = k$

$\mathcal{E}^n(\alpha,\bar{D}) := \left\{D \in \mathcal{E}^n : D[\alpha] = \bar{D}\right\}$ (all EDM completions)

Given $\bar{D}$; find corresponding $\bar{B} \succeq 0$;
find corresponding face; find corresponding subspace.

if $\alpha = 1:k$; embedding dim $\operatorname{embdim}(\bar{D}) = t \le r$

$$D = \begin{bmatrix} \bar{D} & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

BASIC THEOREM for Single Clique FR

Primal View

Let:

- $\bar{D} := D[1\!:\!k] \in \mathcal{E}^k$, $k < n$, $\operatorname{embdim}(\bar{D}) = t \leq r$ be given;
- $B := \mathcal{K}^\dagger(\bar{D}) = \bar{U}_B S \bar{U}_B^\top$, $\bar{U}_B \in \mathcal{M}^{k\times t}$, $\bar{U}_B^\top \bar{U}_B = I_t$, $S \in \mathbb{S}^t_{++}$ be full rank orthogonal decomposition of Gram matrix;
- $U_B := \begin{bmatrix} \bar{U}_B & \frac{1}{\sqrt{k}} e \end{bmatrix} \in \mathcal{M}^{k\times(t+1)}$, $U := \begin{bmatrix} U_B & 0 \\ 0 & I_{n-k} \end{bmatrix}$, and $\begin{bmatrix} V & \frac{U^\top e}{\|U^\top e\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal.

Then the minimal face:

- $$\begin{aligned} \operatorname{face} \mathcal{K}^\dagger\left(\mathcal{E}^n(1\!:\!k, \bar{D})\right) &= \left(U \mathbb{S}^{n-k+t+1}_+ U^\top\right) \cap \mathcal{S}_C \\ &= (UV)\mathbb{S}^{n-k+t}_+ (UV)^\top \end{aligned}$$

The minimal face

Aside:

- $$\begin{aligned} \operatorname{face} \mathcal{K}^{\dagger}\left(\mathcal{E}^n(1:k, \bar{D})\right) &= \left(U \mathbb{S}_+^{n-k+t+1} U^\top\right) \cap \mathcal{S}_C \\ &= (UV) \mathbb{S}_+^{n-k+t} (UV)^\top \end{aligned}$$

Note that the minimal face is defined by the subspace $\mathcal{L} = \operatorname{Range}(UV)$. We add $\frac{1}{\sqrt{k}} e$ to represent $\operatorname{Null}(\mathcal{K})$; then we use V to eliminate e to recover a centered face.

Facial Reduction for Disjoint Cliques

Corollary from Basic Theorem

let $\alpha_1,\ldots,\alpha_\ell \subseteq 1{:}n$ pairwise disjoint sets, wlog:
$\alpha_i = (k_{i-1}+1){:}k_i$, $k_0 = 0$, $\alpha := \bigcup_{i=1}^{\ell} \alpha_i = 1{:}|\alpha|$ let
$\bar{U}_i \in \mathbb{R}^{|\alpha_i|\times(t_i+1)}$ with full column rank satisfy $e \in \mathrm{Range}(\bar{U}_i)$ and

$$U_i := \begin{array}{c} \\ k_{i-1} \\ |\alpha_i| \\ n-k_i \end{array} \begin{array}{c} \begin{array}{ccc} k_{i-1} & t_i+1 & n-k_i \end{array} \\ \begin{bmatrix} I & 0 & 0 \\ 0 & \bar{U}_i & 0 \\ 0 & 0 & I \end{bmatrix} \end{array} \in \mathbb{R}^{n\times(n-|\alpha_i|+t_i+1)}$$

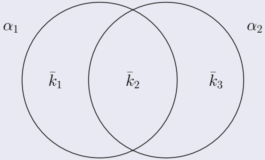
The minimal face is defined by $\mathcal{L} = \mathrm{Range}(U)$:

$$U := \begin{array}{c} \\ |\alpha_1| \\ \vdots \\ |\alpha_\ell| \\ n-|\alpha| \end{array} \begin{array}{c} \begin{array}{cccc} t_1+1 & \ldots & t_\ell+1 & n-|\alpha| \end{array} \\ \begin{bmatrix} \bar{U}_1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \bar{U}_\ell & 0 \\ 0 & \ldots & 0 & I \end{bmatrix} \end{array} \in \mathbb{R}^{n\times(n-|\alpha|+t+1)},$$

where $t := \sum_{i=1}^{\ell} t_i + \ell - 1$. And $e \in \mathrm{Range}(U)$.

Sets for Intersecting Cliques/Faces (subspaces)

$\alpha_1 := 1\!:\!(\bar{k}_1 + \bar{k}_2); \quad \alpha_2 := (\bar{k}_1 + 1)\!:\!(\bar{k}_1 + \bar{k}_2 + \bar{k}_3)$



Two (Intersecting) Clique Reduction/Subsp. Repres.

Let:

- $\alpha_1, \alpha_2 \subseteq 1:n; \quad k := |\alpha_1 \cup \alpha_2|$
- for $i = 1, 2$: $\bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$, embedding dimension t_i;
- $B_i := \mathcal{K}^\dagger(\bar{D}_i) = \bar{U}_i S_i \bar{U}_i^\top$, $\bar{U}_i \in \mathcal{M}^{k_i \times t_i}$, $\bar{U}_i^\top \bar{U}_i = I_{t_i}$, $S_i \in \mathbb{S}_{++}^{t_i}$;
- $U_i := \begin{bmatrix} \bar{U}_i & \frac{1}{\sqrt{k_i}} e \end{bmatrix} \in \mathcal{M}^{k_i \times (t_i+1)}$; and $\bar{U} \in \mathcal{M}^{k \times (t+1)}$

  satisfies $\boxed{\text{Range}(\bar{U}) = \text{Range}\left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{\bar{k}_3} \end{bmatrix}\right) \cap \text{Range}\left(\begin{bmatrix} I_{\bar{k}_1} & 0 \\ 0 & U_2 \end{bmatrix}\right), \text{ with } \bar{U}^\top \bar{U} = I_{t+1}}$
- $U := \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathcal{M}^{n \times (n-k+t+1)}$ and $\begin{bmatrix} V & \frac{U^\top e}{\|U^\top e\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal.

Then
$$\boxed{\begin{aligned} \underline{\underline{\textstyle\bigcap_{i=1}^2 \text{face}\, \mathcal{K}^\dagger\left(\mathcal{E}^n(\alpha_i, \bar{D}_i)\right)}} &= \left(U \mathbb{S}_+^{n-k+t+1} U^\top\right) \cap \mathcal{S}_C \\ &= (UV)\mathbb{S}_+^{n-k+t}(UV)^\top \end{aligned}}$$

Expense/Work of (Two) Clique/Facial Reductions

Subspace Intersection for Two Intersecting Cliques/Faces

Suppose:

$$U_1 = \begin{bmatrix} U_1' & 0 \\ U_1'' & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} I & 0 \\ 0 & U_2'' \\ 0 & U_2' \end{bmatrix}$$

Then:

$$U := \begin{bmatrix} U_1' \\ U_1'' \\ U_2'(U_2'')^\dagger U_1'' \end{bmatrix} \quad \text{or} \quad U := \begin{bmatrix} U_1'(U_1'')^\dagger U_2'' \\ U_2'' \\ U_2' \end{bmatrix}$$

($Q_1 =: (U_1'')^\dagger U_2'', Q_2 = (U_2'')^\dagger U_1''$ orthogonal/rotation)
(Efficiently) satisfies

$$\text{Range}(U) = \text{Range}(U_1) \cap \text{Range}(U_2)$$

Two (Intersecting) Clique Explicit Delayed Completion

Let:

- Hypotheses of intersecting Theorem (Thm 2) holds
- $\bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$, for $i = 1, 2$, $\beta \subseteq \alpha_1 \cap \alpha_2, \gamma := \alpha_1 \cup \alpha_2$
- $\bar{D} := D[\beta]$ with embedding dimension r
- $B := \mathcal{K}^\dagger(\bar{D}), \quad \bar{U}_\beta := \bar{U}(\beta, :)$, where $\bar{U} \in \mathcal{M}^{k \times (t+1)}$ satisfies intersection equation of Thm 2
- $\begin{bmatrix} \bar{V} & \frac{\bar{U}^\top e}{\|\bar{U}^\top e\|} \end{bmatrix} \in \mathcal{M}^{t+1}$ be orthogonal.
- $\boxed{Z := (J\bar{U}_\beta \bar{V})^\dagger B((J\bar{U}_\beta \bar{V})^\dagger)^\top}$.

<u>THEN</u> $t = r$ in Thm 2, and $Z \in \mathbb{S}^r_+$ is the unique solution of the equation $(J\bar{U}_\beta \bar{V})Z(J\bar{U}_\beta \bar{V})^\top = B$, and the exact completion is

$$\boxed{D[\gamma] = \mathcal{K}\left(PP^\top\right)} \quad \text{where} \quad \boxed{P := UVZ^{\frac{1}{2}} \in \mathbb{R}^{|\gamma| \times r}}$$

Completing SNL (Delayed use of Anchor Locations)

Rotate to Align the Anchor Positions

- Given $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{R}^{n \times r}$ such that $D = \mathcal{K}(PP^\top)$
- Solve the orthogonal Procrustes problem:

$$\boxed{\begin{array}{ll} \min & \|A - P_2 Q\| \\ \text{s.t.} & Q^\top Q = I \end{array}}$$

  $P_2^\top A = U \Sigma V^\top$ SVD decomposition; set $Q = UV^\top$;
  (Golub/Van Loan'79-'12, Algorithm 12.4.1)
- Set $X := P_1 Q$

Random Noisless Problems, Krislock W. '**2010**

- 2.16 GHz Intel Core 2 Duo, 2 GB of RAM
- Dimension $r = 2$
- Square region: $[0, 1] \times [0, 1]$
- $m = 9$ anchors
- Using only Rigid Clique Union and Rigid Node Absorption
- Error measure: Root Mean Square Deviation

$$\text{RMSD} = \left(\frac{1}{n} \sum_{i=1}^{n} \|p_i - p_i^{\text{true}}\|^2 \right)^{1/2}$$

Results - Large n (SDP size $O(n^2)$)

n # of Sensors Located

n # sensors \ R	0.07	0.06	0.05	0.04
2000	2000	2000	1956	1374
6000	6000	6000	6000	6000
10000	10000	10000	10000	10000

CPU Seconds

# sensors \ R	0.07	0.06	0.05	0.04
2000	1	1	1	3
6000	5	5	4	4
10000	10	10	9	8

RMSD (over located sensors)

n # sensors \ R	0.07	0.06	0.05	0.04
2000	$4e{-}16$	$5e{-}16$	$6e{-}16$	$3e{-}16$
6000	$4e{-}16$	$4e{-}16$	$3e{-}16$	$3e{-}16$
10000	$3e{-}16$	$5e{-}16$	$4e{-}16$	$4e{-}16$

Results - N Huge SDPs Solved

Large-Scale Problems

# sensors	# anchors	radio range	RMSD	Time
20000	9	.025	$5e{-16}$	25s
40000	9	.02	$8e{-16}$	1m 23s
60000	9	.015	$5e{-16}$	3m 13s
100000	9	.01	$6e{-16}$	9m 8s

Size of SDPs Solved: $N = \binom{n}{2}$ (# vrbls)

$\mathcal{E}_n(\text{density of } \mathcal{G}) = \pi R^2$; $M = \mathcal{E}_n(|E|) = \pi R^2 N$ (# constraints)

Size of SDP Problems:

$M = [3,078,915 \quad 12,315,351 \quad 27,709,309 \quad 76,969,790]$

$N = 10^9 \,[0.2000 \quad 0.8000 \quad 1.8000 \quad 5.0000]$

View 2: Recall Details with Exposing Vector/Numerics

Thm D.P.W. '15: $\mathcal{M} : \mathbb{E} \to \mathbb{Y}$, K proper convex cone

$\emptyset \neq F = \{X \in K : \mathcal{M}(X) = b\}$. Then a vector v exposes a proper face of $\mathcal{M}(K)$ containing b if, and only if, v satisfies the auxiliary system

$$0 \neq \mathcal{M}^* v \in \mathcal{K}^*, \quad \langle v, b \rangle = 0.$$

Let $N = \mathrm{face}(b, \mathcal{M}(K))$ (smallest face containing b). Then:

- $K \cap \mathcal{M}^{-1}(N) = \mathrm{face}(F, K)$
- v exposes N IFF $\mathcal{M}^*(v)$ exposes $\mathrm{face}(F, K)$.

Corollary

If Slater's condition fails, then $d = 1$ IFF the minimal $\mathrm{face}(b, \mathcal{M}(K))$ is exposed.

Using Exposed Vectors

- Find a set of medium sized cliques $\mathcal{C}$ (e.g. a clique for each node). $r+1 \leq |C| \leq M, \forall C \in \mathcal{C}$.
- Find an exposing vector $Y_C \in \mathbb{S}_+^{|C|}$ and *weight/value* for each $C \in \mathcal{C}$. Fill out $Y_C \in \mathcal{S}_+^n$ with zeros for remaining nodes.
- Find final exposing vector $\sum_{C \in \mathcal{C}} w_C Y_C$ and nullspace V.
- solve the smaller EDM/SNL with $X = VRV^T$.

(Related to Amit Singer '08)

PSD/ EDM Matrix Completions (from GJSW , DPW)

Graph G, vertex set V, edge set E, self-loops L

G is chordal if any cycle of four or more nodes has a chord

Assume *partial* graphs.

Theorem (PSD completable matrices & chordal graphs)

1. *The graph G is PD completable if and only if the graph induced by G on L is chordal.*
2. *Supposing equality $L = V$ holds, the graph G is PSD completable if and only if G is chordal.*

Theorem (Euclidean distance completability & chordal graphs)

The graph G is EDM completable if and only if G is chordal.

Minimal Faces and Chordal Graphs PSD

Theorem (Finding the minimal face on chordal graphs)

Suppose that the graph induced by G on L is chordal. Consider a partial PSD matrix $a \in \mathbb{R}^E$ and the region

$$F = \{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij}, \forall ij \in E\}.$$

Then the equality

$$\operatorname{face}(F, \mathcal{S}^n_+) = \bigcap_{\chi \in \Theta} \operatorname{face}(F_\chi, \mathcal{S}^n_+) \qquad \textit{holds},$$

where Θ denotes the set of all cliques in the restriction of G to L, and for each $\chi \in \Theta$ we define the relaxation

$$F_\chi := \{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij} \textit{ for all } ij \in E(\chi)\}.$$

Facial Reduction for EDM

Theorem (Clique facial reduction for EDM is sufficient)

Let G be chordal, $a \in \mathbb{R}^E$ a partial EDM and let

$$F := \{X \in \mathbb{S}_c \cap \mathcal{S}_+^n \ : [\mathcal{K}(X)]_{ij} = a_{ij} \textit{ for all } ij \in E\}.$$

Let Θ denote the set of all cliques, and for each $\chi \in \Theta$ define

$$F_\chi := \{X \in \mathbb{S}_c \cap \mathcal{S}_+^n \ : [\mathcal{K}(X)]_{ij} = a_{ij} \textit{ for all } ij \in E(\chi)\}.$$

Then the equality

$$\operatorname{face}(F, \mathbb{S}_c \cap \mathcal{S}_+^n) = \bigcap_{\chi \in \Theta} \operatorname{face}(F_\chi, \mathbb{S}_c \cap \mathcal{S}_+^n) \qquad \textit{holds.}$$

Completions/Chordality/Singularity Degree d

Corollary (Singularity degree of chordal completions PSD)

If the restriction of G to L is chordal, then the PSD completion problem has singularity degree at most one.

Corollary (Singularity degree of chordal completions EDM)

If the graph G is chordal, then the EDM completion problem has singularity degree at most one when feasible.

Above explains the success of clique approaches

* FR for Low-Rank Matrix Completion, LRMC, (Huang-W.'16)

Intractable (nonconvex) minimum rank completion

Given partial $m \times n$ real matrix $Z \in \mathbb{R}^{m \times n}$.

$$(LRMC) \quad \begin{array}{ll} \min & \operatorname{rank}(M) \\ \text{s.t.} & \|M_{\hat{E}} - Z_{\hat{E}}\| \leq \delta, \end{array}$$

$\hat{E}$ sampled indices; $Z_{\hat{E}} \in \mathbb{R}^{\hat{E}}$; $\delta > 0$ tuning parameter

convex nuclear norm relaxation

$$\begin{array}{ll} \min & \|M\|_* \\ \text{s.t.} & \|M_{\hat{E}} - Z_{\hat{E}}\| \leq \delta, \end{array}$$

where $\|M\|_* = \sum_i \sigma_i(M)$.

SDP Equivalent to Nuclear Norm Minimization

Trace minimization

$$\begin{array}{ll} \min & \|Y\|_* = \operatorname{trace}(Y) \\ \text{s.t.} & \|Y_{\bar{E}} - Q_{\bar{E}}\| \leq \delta \\ & Y \in \mathbb{S}^{m+n}_+, \end{array}$$

$Q = \begin{bmatrix} 0 & Z \\ Z^T & 0 \end{bmatrix} \in \mathbb{S}^{m+n}_+$ and $\bar{E}$ indices in Y corresponding to $\hat{E}$

Noiseless case: strict feasibility trivially holds

$$Y_{\bar{E}} = Q_{\bar{E}}$$

choose diagonal of Y sufficiently large, positive.
(strict feas. holds for dual as well)

Why consider this here?

It has been shown recently by Huang-W. that one can exploit the structure at the optimum and efficiently apply FR.

Associated Undirected Weighted Graph $G = (V, E, W)$

node set $V = \{1, \ldots, m, m+1, \ldots, m+n\}$ Let:

$$\begin{aligned} E_{1,m} &:= \{ij \in V \times V : i < j \leq m\} \\ E_{m+1,m+n} &:= \{ij \in V \times V : m+1 \leq i < j \leq m+n\} \end{aligned}$$

edge set

$$E := \bar{E} \cup E_{1,m} \cup E_{m+1,m+n}.$$

weights for all $ij \in E$

$$w_{ij} := \begin{cases} Z_{i(j-m)}, & \forall ij \in \bar{E} \\ 0, & \text{otherwise.} \end{cases}$$

Corresponding adjacency matrix A; cliques C

nontrivial cliques of interest (after row/col perms) corresp. to full (specified) submatrix X in Z; $C = \{i_1, \ldots, i_k\}$ with cardinalities

$$|C \cap \{1, \ldots, m\}| = p \neq 0, \quad |C \cap \{m+1, \ldots, m+n\}| = q \neq 0.$$

Exposing Vector for Low-Rank Completions

Clique - X

$$X \equiv \{Z_{i(j-m)} : ij \in C\}, \quad \text{specified } p \times q \text{ submatrix.}$$

let $\operatorname{rank} X = r_X$. Wlog

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ X & Z_3 \end{bmatrix},$$

full rank factorization $X = \bar{P}\bar{Q}^T$ using SVD

$$X = \bar{P}\bar{Q}^T = U_X \Sigma_X V_X^T, \; \Sigma_Z \in \mathbb{S}^{r_X}_{++}, \quad \bar{P} = U_X \Sigma_X^{1/2}, \; \bar{Q} = V_X \Sigma_X^{1/2}.$$

$$C_X = \{i, \ldots, m, m+1, \ldots, m+k\}, \qquad r < \max\{p, q\},$$

target rank r.
In HWY rewrite optimality conditions SDP as

$$0 \preceq Y = \begin{bmatrix} U \\ P \\ Q \\ V \end{bmatrix} D \begin{bmatrix} U \\ P \\ Q \\ V \end{bmatrix}^T = \left[\begin{array}{c|cc|c} UDU^T & UDP^T & UDQ^T & UDV^T \\ \hline PDU^T & PDP^T & PDQ^T & PDV^T \\ QDU^T & QDP^T & QDQ^T & QDV^T \\ \hline VDU^T & VDP^T & VDQ^T & VDV^T \end{array}\right].$$

Using exposing vectors

Lemma (Basic FR)

Let $r < \min\{p, q\}$ and $X = PDQ^T = \bar{P}\bar{Q}^T$ as above. We find a pair of exposing vectors using

$$\mathrm{FR}(\bar{P}, \bar{Q}) : \ \bar{P}\bar{P}^T + \bar{U}\bar{U}^T \succ 0, \ \bar{P}^T\bar{U} = 0,$$

$$\bar{Q}\bar{Q}^T + \bar{V}\bar{V}^T \succ 0, \ \bar{Q}^T\bar{V} = 0.$$

Numerics for Low rank matrix completion

Lemma: Using exposing vectors/average over 5 instances

Table: noisy: $r = 2$; $m \times n$ size ↑; density p ↓; noise ↑.

Specifications				Time (s)		Rank		Residual (%Z)	
m	n	% noise	p	initial	refine	initial	refine	initial	refine
700	1000	0.00	0.40	2.85	2.85	2.00	2.00	0.00000	0.00000
700	1000	0.01	0.40	2.33	2.33	2.00	2.00	0.00011	0.00011
700	1000	0.15	0.40	2.24	2.24	2.00	2.00	0.00168	0.00168
700	1000	0.30	0.40	2.30	2.30	2.00	2.00	0.00336	0.00336
700	1000	0.45	0.40	2.28	2.28	2.00	2.00	0.00504	0.00504
1700	2000	1.00	0.40	8.92	8.92	2.00	2.00	0.00771	0.00771
1700	2000	1.00	0.35	8.41	8.41	2.00	2.00	0.01052	0.01052
1700	2000	1.00	0.30	7.78	12.12	2.20	2.20	0.01326	0.01326
1700	2000	1.00	0.25	7.53	7.53	1.80	1.80	0.17287	0.17287
1700	2000	1.00	0.20	7.87	7.87	1.80	1.80	0.15956	0.15956

** Conclusion

Preprocessing

- Though strict feasibility holds generically, failure appears in many applications. Loss of strict feasibility is directly related to ill-posedness and difficulty in numerical methods.
- Preprocessing based on structure can both *regularize* and simplify the problem. In many cases one gets an optimal solution without the need of any SDP solver.

Expoloit structure at optimum

For low-rank matrix completion the structure at the optimum can be exploited to apply FR.

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Thanks for your attention!

Facial Reduction in Cone Optimization with Applications to Sensor Network Localization and Low Rank Matrix Completion

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