Applications of Facial Reduction to Compressed Sensing, Sensor Network Localization, and Molecular Conformation

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Motivation: Loss of Slater CQ/Facial reduction

- Slater condition – existence of a strictly feasible solution – is at the heart of convex optimization.
- Without Slater: first-order optimality conditions may fail; dual problem may yield little information; small perturbations may result in infeasibility; many software packages can behave poorly.
- a pronounced phenomenon: though Slater holds generically, surprisingly many models arising from hard nonconvex problems show loss of strict feasibility, e.g., Matrix completions/compressive sensing, sensor network localization, SNL, EDM, POP, Molecular Conformation, QAP, GP, strengthened MC
- We concentrate on Semidefinite Programming, SDP.
  We look at various reasons and how to take advantage using
  two views of FACIAL REDUCTION, FR

  Refs: Borwein, W. '79-81'; Cheung, Schurr, W.'11 ; Krislock, W.'10 ; Drusvyatskiy, Pataki, W.'15 ; Cheung, Drusvyatskiy, Krislock, W.'14

Abstract convex program

$$\text{(ACP)} \quad \inf_x f(x) \text{ s.t. } g(x) \preceq_K 0, x \in \Omega$$

where:

- $f : \mathbb{R}^n \to \mathbb{R}$ convex; $g : \mathbb{R}^n \to \mathbb{R}^m$ is K-convex
    - $K \subset \mathbb{R}^m$ closed convex cone; $\Omega \subseteq \mathbb{R}^n$ convex set
    - $a \preceq_K b \iff b - a \in K$, $a \prec_K b \iff b - a \in \operatorname{int} K$
    - $g(\alpha x + (1 - \alpha y)) \preceq_K \alpha g(x) + (1 - \alpha) g(y)$,
      $\forall x, y \in \mathbb{R}^n, \forall \alpha \in [0, 1]$

Slater's CQ: $\exists \hat{x} \in \Omega$ s.t. $g(\hat{x}) \in -\operatorname{int} K$ $(g(x) \prec_K 0)$

- guarantees strong duality
- (near) loss of strict feasibility, nearness to infeasibility, correlates with number of iterations & loss of accuracy

Faces of Convex Sets - Useful for Charact. of Opt.

Face of C, $\boxed{F \trianglelefteq C}$

- $F \subseteq C$ is a face of C if F contains any line segment in C whose relative interior intersects F.
- A convex cone $F \subseteq K$ is a face of a convex cone K, $F \trianglelefteq K$, if

$$\boxed{x, y \in K \text{ and } x + y \in F \implies x, y \in F}$$

Polar (Dual) Cone/Conjugate Face

- polar cone $\quad K^* := \{\phi : \langle \phi, k \rangle \geq 0, \ \forall k \in K\}$
- If $F \trianglelefteq K$, the conjugate face of F is

$$\boxed{F^c := F^\perp \cap K^* \trianglelefteq K^*}$$

FACES of CONES

General case

- A face of a face is a face
- intersection of a face with a face is a face.
- Let $C \subseteq K$, then face(C) denotes the minimal face
  (intersection of faces) containing C.

SDP case/Replicating cone

- Let $X \in K = \mathcal{S}^n_+$, PSD cone; with spectral decomposition,
  $$X = [P\ Q] \begin{bmatrix} D_+ & 0 \\ 0 & 0 \end{bmatrix} [P\ Q]^T, \quad D_+ \in \mathbb{S}^r_{++} \quad (\mathrm{rank}\, X = r)$$
- Then $\mathrm{Range}(X) = \mathrm{Range}(P), \quad \mathrm{Null}(X) = \mathrm{Range}(Q)$
  $$\mathrm{face}(X) = P\mathbb{S}^r_+ P^T = (QQ^T)^\perp \cap \mathcal{S}^n_+ .$$
  ($Z = QQ^T$ an *exposing vector/matrix* for the face.)
- $\mathrm{face}(X)^c = Q\mathbb{S}^{n-r}_+ Q^T$

- polar cone: $K^* = \{\phi : \langle \phi, y \rangle \geq 0, \forall y \in K\}$.
- $K^f :=$ face(F) minimal face containing feasible set F.

Lemma (Facial Reduction (FR); find EXPOSING vector ϕ)

Suppose $\bar{x}$ is feasible. Then the LHS system

$$\left\{ \begin{array}{ll} (\Omega - \bar{x})^+ \cap \partial \langle \phi, g(\bar{x}) \rangle \neq \emptyset & \\ \phi \in K^+, & \langle \phi, g(\bar{x}) \rangle = 0 \end{array} \right\} \quad \textit{implies} \quad K^f \subseteq \phi^\perp \cap K.$$

Proof

line 1 of system implies $\bar{x}$ global min for convex function $\langle \phi, g(\cdot) \rangle$ on Ω; i.e., $0 = \langle \phi, g(\bar{x}) \rangle \leq \langle \phi, g(x) \rangle \leq 0, \forall x \in F$; implies $-g(F) \subseteq \phi^\perp \cap K$. □

Semidefinite Programming, SDP, $\mathcal{S}^n_+$

$K = \mathcal{S}^n_+ = K^*$: nonpolyhedral, self-polar, facially exposed

$$\text{(SDP-P)} \quad v_P = \sup_{y \in \mathbb{R}^m} b^\top y \text{ s.t. } g(y) := \mathcal{A}^* y - c \preceq_{\mathcal{S}^n_+} 0$$

$$\text{(SDP-D)} \quad v_D = \inf_{x \in \mathcal{S}^n} \langle c, x \rangle \text{ s.t. } \mathcal{A}x = b, \; x \succeq_{\mathcal{S}^n_+} 0$$

where:

- PSD cone $\mathcal{S}^n_+ \subset \mathcal{S}^n$ symm. matrices
- $c \in \mathcal{S}^n$, $b \in \mathbb{R}^m$
- $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ is an onto linear map, with adjoint $\mathcal{A}^*$
- $\mathcal{A}x = (\operatorname{trace} A_i x) = (\langle A_i, x \rangle) \in \mathbb{R}^m, \quad A_i \in \mathcal{S}^n$
  $\mathcal{A}^* y = \sum_{i=1}^m A_i y_i \in \mathcal{S}^n$

Slater's CQ/Theorem of Alternative

Assume feasibility: $\exists\, \tilde{y}$ s.t. $c - \mathcal{A}^*\tilde{y} \succeq 0$.

Exactly one of the following alternatives holds/is consistent:

$$(I) \quad \exists\, \hat{y} \text{ s.t. } s = c - \mathcal{A}^*\hat{y} \succ 0 \qquad (\textit{Slater})$$

<u>or</u>

$$(II) \quad \mathcal{A}d = 0,\ \langle c, d\rangle = 0,\ 0 \neq d \succeq 0 \quad (*)$$

(II) finds exposing vector: $0 \neq d \succeq 0$

d exposes a proper face containing all the feasible slacks

$$z = c - \mathcal{A}^*y \succeq 0 \implies zd = 0. \quad (\text{equiv. trace } zd = 0)$$

Regularization Using Minimal Face

Borwein-W.'81 , $f_P = \text{face}\, \mathcal{F}_P^s$; min. face of feasible slacks

(SDP-P) is equivalent to the regularized

$$(\text{SDP}_{reg}\text{-P}) \qquad v_{RP} := \sup_y \{\langle b, y\rangle \ : \ \mathcal{A}^* y \preceq_{f_P} c\}$$

f_p is mininimal face of primal feasible slacks

$\{s \succeq 0 : s = c - \mathcal{A}^* y\} \subseteq f_p \trianglelefteq \mathcal{S}_+^n$

Lagrangian dual of regularized problem satisfies strong duality:

$$(\text{SDP}_{reg}\text{-D}) \quad v_{DRP} \quad := \inf_x \{\langle c, x\rangle \ : \ \mathcal{A} x = b, \ x \succeq_{f_P^*} 0\}$$

$v_P = v_{RP} = v_{DRP}$ and v_{DRP} is attained.

regularized primal-dual pair

If we take the dual of $(\text{SDP}_{reg}\text{-D})$ we recover the primal regularized problem $(\text{SDP}_{reg}\text{-P})$.

Slater's CQ/Theorem of Alternative for Dual

Assume feasibility: $\exists\, \tilde{x}$ s.t. $\mathcal{A}\,\tilde{x} = b, \tilde{x} \succeq 0$.

Exactly one of the following alternatives holds/is consistent:

$$(I) \quad \exists\, \hat{x} \text{ s.t. } \mathcal{A}\,\hat{x} = b, \hat{x} \succ 0 \qquad (\textit{Slater})$$

or

$$(II) \quad 0 \neq z = \mathcal{A}^* y \succeq 0,\ \langle b, y \rangle = 0, \qquad (**)$$

(II) finds exposing vector: $0 \neq z \succeq 0$

z exposes a proper face containing all the dual feasible points

$$\mathcal{A}\,x = b, x \succeq 0 \implies zx = 0. \quad (\text{equiv. trace } zx = 0)$$

Regularization of Dual Using Minimal Face

Borwein-W.'81 , $f_D = \text{face}\,\mathcal{F}_D^x$; min. face of dual feasible set

(SDP-D) is equivalent to the regularized

(SDP$_{reg}$-D) $\quad v_{RD} := \inf_x \{\langle c, x\rangle \; : \; \mathcal{A}x = b, x \succeq_{f_D} 0\}$

f_D is mininimal face of dual feasible set

$\{x \succeq 0 : \mathcal{A}x = b, x \succeq 0\} \subseteq f_D \trianglelefteq \mathcal{S}_+^n$

Lagrang. dual of regulariz. dual problem satisfies strong duality:

(SDP$_{reg}$-DD) $\quad v_{DRD} \quad := \sup_y \{\langle b, y\rangle \; : \; \mathcal{A}^*y \preceq_{f_D^*} c\}$

$v_D = v_{RD} = v_{DRD}$ and v_{DRD} is attained.

regularized primal-dual pair

If we take the dual of (SDP$_{reg}$-DD) we recover the dual regularized problem (SDP$_{reg}$-P).

(SDP_D) $\min\{\operatorname{trace} CX \text{ s.t. } \mathcal{A}X = b, X \in \mathcal{S}^n_+\}$

Step 1: Let $0 \neq Z \succeq 0$ be an exposing vector. Then can add constraint $\operatorname{trace} ZX = 0$. Equivalently, from spectral decomposition of Z, with $\operatorname{Range} P = \operatorname{Null} Z$:

substitute: $X = P\mathbb{S}^{t_1}_+ P^T$

We get the equivalent smaller problem

$$(SDP_{D1}) \qquad \begin{array}{ll} \min & \operatorname{trace}(P^T C P)R \\ \text{s.t.} & \operatorname{trace}(P^T A_i P)R = b_i, i = 1, \dots, m \\ & R \in \mathbb{S}^{t_1}_+ \end{array}$$

Remove/delete redundant linear constraints;
repeat from Step 1.
minimum number of steps is called the singularity degree

View Two for FR in SDP

Lemma: Using exposing vectors

Let

$$Z_i \succeq 0, F_i = \mathcal{S}_+^n \cap Z_i^\perp, i = 1, \ldots, m.$$

Then

$$\cap_{i=1}^m F_i = \mathcal{S}_+^n \cap \left(\sum_{i=1}^m Z_i \right)^\perp$$

i.e., intersection of faces is exposed by sum of exposing vectors. □

Equivalence of exposing vectors with image set

Thm: DPW '15 : $F := F_P = \{x \in \mathcal{K} : \mathcal{A}x = b\} \neq \emptyset$

Vector v exposes a proper face of $\mathcal{A}(\mathcal{K})$ containing b <u>iff</u> v satisfies the auxiliary system

$$0 \neq \mathcal{A}^* v \in \mathcal{K}^* \quad \text{and} \quad \langle v, b \rangle = 0.$$

And the following are true.

(I) We always have:

$$\boxed{\mathcal{K} \cap \mathcal{A}^{-1}(\text{face}(b, \mathcal{A}(\mathcal{K}))) = \text{face}(F, \mathcal{K})}$$

(II) For any vector $w \in \mathbb{Y}$ the following equivalence holds:

$$\boxed{w \text{ exposes } \text{face}(b, A(\mathcal{K})) \iff \mathcal{M}^* w \text{ exposes } \text{face}(F, C)}$$

(III) Consequently Slater condition failing implies: singularity degree $d = 1$ for the system <u>iff</u> the minimal face $\text{face}(b, \mathcal{M}(C))$ is exposed. □

Applications?

- preprocessing is important in e.g., LP.
- Can we do facial reduction in general?
- Is it efficient/worthwhile?
- important applications? .... relation to feasibility questions and iterative methods? convergence rates? (DR, MAP)

Singularity Degree - Minimal Number of FR Steps

Sturm's error bounds Theorem for SDP, 2000

Given an affine subspace $\mathcal{V}$ of $\mathcal{S}^n$, the pair $(\mathcal{V}, \mathcal{S}^n_+)$ is $\frac{1}{2^d}$-Holder regular, $\gamma = \frac{1}{2^d}$, with displacement, where d is the singularity degree of $(\mathcal{V}, \mathcal{S}^n_+)$ with displacement.
(e.g., for intersecting sets, for all compact sets U there exists a constant $c > 0$ such that
$\text{dist}(x, \mathcal{V} \cap \mathcal{S}^n_+) \leq c \left(\text{dist}^\gamma(x, \mathcal{V}) + \text{dist}^\gamma(x, \mathcal{S}^n_+)\right), \quad \forall x \in U$)

Cgnce rate alternating directions (MAP) for SDP

Theorem (Drusvyatskiy, Li, W. 2015) If the sequence X_k, Y_k converges, $d > 0$, then the rate is $\mathcal{O}\left(k^{-\frac{1}{2^{d+1}-2}}\right)$
(If Slater holds then cgnce is R-linear.)

VIEW 2: FR - Motivation; SNL and (Protein Folding)

Highly (implicit) degenerate/low-rank problem

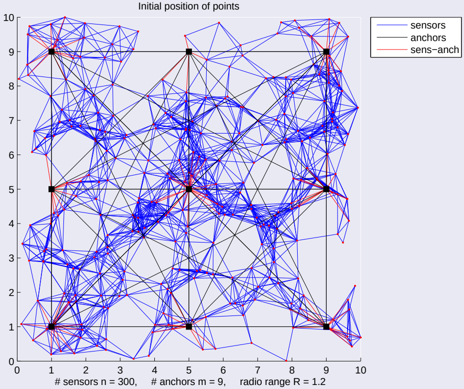
- high (implicit) degeneracy translates to low rank solutions
- take advantage of degeneracy; fast, high accuracy solutions

SNL - a Fundamental Problem of Distance Geometry; easy to describe - dates back to Grasssmann 1886

- r : embedding dimension
- n ad hoc wireless sensors $p_1, \ldots, p_n \in \mathbb{R}^r$ to locate in $\mathbb{R}^r$;
- m of the sensors $p_{n-m+1}, \ldots, p_n$ are anchors (positions known, using e.g. GPS)
- pairwise distances $D_{ij} = \|p_i - p_j\|^2, ij \in E$, are known within radio range $R > 0$
- $$P^\top = \begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix} = \begin{bmatrix} X^\top & A^\top \end{bmatrix} \in \mathbb{R}^{r \times n}$$

Sensor Localization Problem/Partial EDM

Sensors ○ and Anchors ■



Underlying Graph Realization/Partial EDM NP-Hard

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

- node set $\mathcal{V} = \{1, \ldots, n\}$
- edge set $(i, j) \in \mathcal{E}$; $\omega_{ij} = \|p_i - p_j\|^2$ known approximately
- The anchors form a CLIQUE (complete subgraph)
- Realization of $\mathcal{G}$ in $\mathbb{R}^r$: a mapping of nodes $v_i \mapsto p_i \in \mathbb{R}^r$ with squared distances given by ω.

Corresponding Partial Euclidean Distance Matrix, EDM

$$D_{ij} = \begin{cases} d_{ij}^2 & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise (unknown distance)}, \end{cases}$$

$d_{ij}^2 = \omega_{ij}$ are known squared Euclidean distances between sensors p_i, p_j; anchors correspond to a clique.

Connections to Semidefinite Programming (SDP)

$D = \mathcal{K}(B) \in \mathcal{E}^n$, $B = \mathcal{K}^\dagger(D) \in \mathcal{S}^n \cap \mathcal{S}_C$ (centered $Be = 0$)

$P^\top = \begin{bmatrix} p_1 & p_2 & \dots & p_n \end{bmatrix} \in \mathcal{M}^{r \times n}$;
$B := PP^\top \in \mathcal{S}^n_+$ (Gram matrix of inner products);
$\operatorname{rank} B = r$; let $D \in \mathcal{E}^n$ corresponding EDM; $e = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^\top$

$$
\begin{array}{lrcl}
(\text{to } D \in \mathcal{E}^n) & D & = & \left(\|p_i - p_j\|_2^2\right)_{i,j=1}^n \\
 & & = & \left(p_i^T p_i + p_j^T p_j - 2p_i^T p_j\right)_{i,j=1}^n \\
 & & = & \boxed{\operatorname{diag}(B)\, e^\top + e \operatorname{diag}(B)^\top - 2B} \\
 & & =: & \mathcal{K}(B) \quad (\text{from } B \in \mathcal{S}^n_+).
\end{array}
$$

Euclidean Distance Matrices; Semidefinite Matrices

Moore-Penrose Generalized Inverse $\mathcal{K}^\dagger$

$$B \succeq 0 \quad \Longrightarrow \quad D = \mathcal{K}(B) = \operatorname{diag}(B)\, e^\top + e \operatorname{diag}(B)^\top - 2B \in \mathcal{E}$$

$$D \in \mathcal{E} \quad \Longrightarrow \quad B = \mathcal{K}^\dagger(D) = -\tfrac{1}{2} J \operatorname{offDiag}(D) J \succeq 0,\ Be = 0$$

$$\left(J = I - \tfrac{1}{n} ee^T\right)$$

Theorem (Schoenberg, 1935)

A (hollow) matrix D (with $\operatorname{diag}(D) = 0, D \in S_H$) is a
Euclidean distance matrix
if and only if

$$B = \mathcal{K}^\dagger(D) \succeq 0. \quad \text{(and centered } Be = 0\text{)}$$

And !!!!

$$\boxed{\operatorname{embdim}(D) = \operatorname{rank}\left(\mathcal{K}^\dagger(D)\right), \quad \forall D \in \mathcal{E}^n} \tag{1}$$

Popular Techniques; SDP Relax.; Highly Degen.

Nearest, Weighted, SDP Approx. (relax/discard $\operatorname{rank} B$)

- $\min_{B \succeq 0} \|H \circ (\mathcal{K}(B) - D)\|$; $\operatorname{rank} B = r$; typical weights: $H_{ij} = 1/\sqrt{D_{ij}}$, if $ij \in E$, $H_{ij} = 0$ otherwise.
- with rank constraint: a non-convex, NP-hard program
- SDP relaxation is convex, BUT: expensive/low accuracy/implicitly highly degenerate (cliques restrict ranks of feasible Bs)

Instead: (Shall) Take Advantage of Degeneracy!

clique α, $|\alpha| = k$ (corresp. $D[\alpha]$) with embed. dim. $= t \leq r < k$
$\implies \operatorname{rank} \mathcal{K}^\dagger(D[\alpha]) = t \leq r \implies \operatorname{rank} B[\alpha] \leq \operatorname{rank} \mathcal{K}^\dagger(D[\alpha]) + 1$
$\implies \operatorname{rank} B = \operatorname{rank} \mathcal{K}^\dagger(D) \leq n - \boxed{(k - t - 1)} \implies$
Slater's CQ (strict feasibility) fails

Basic Single Clique/Facial Reduction

Matrix with Fixed Principal Submatrix

For $Y \in \mathcal{S}^n$, $\alpha \subseteq \{1, \ldots, n\}$: $Y[\alpha]$ denotes principal submatrix formed from rows & cols with indices α.

$\bar{D} \in \mathcal{E}^k$, $\alpha \subseteq 1\!:\!n$, $|\alpha| = k$

Define $\mathcal{E}^n(\alpha, \bar{D}) := \left\{ D \in \mathcal{E}^n : D[\alpha] = \bar{D} \right\}$. (completions)

Given $\bar{D}$; find a corresponding $B \succeq 0$; find the corresponding face; find the corresponding subspace.

if $\alpha = 1 : k$; embedding dim $\mathrm{embdim}(\bar{D}) = t \leq r$

$$D = \begin{bmatrix} \bar{D} & \cdot \\ \cdot & \cdot \end{bmatrix}.$$

BASIC THEOREM for Single Clique FR

Primal View

Let:

- $\bar{D} := D[1:k] \in \mathcal{E}^k$, $k < n$, $\text{embdim}(\bar{D}) = t \le r$ be given;
- $B := \mathcal{K}^\dagger(\bar{D}) = \bar{U}_B S \bar{U}_B^\top$, $\bar{U}_B \in \mathcal{M}^{k\times t}$, $\bar{U}_B^\top \bar{U}_B = I_t$, $S \in \mathbb{S}^t_{++}$ be full rank orthogonal decomposition of Gram matrix;
- $U_B := \begin{bmatrix} \bar{U}_B & \frac{1}{\sqrt{k}} e \end{bmatrix} \in \mathcal{M}^{k\times(t+1)}$, $U := \begin{bmatrix} U_B & 0 \\ 0 & I_{n-k} \end{bmatrix}$, and $\begin{bmatrix} V & \frac{U^\top e}{\|U^\top e\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal.

Then the minimal face:

- $$\begin{aligned} \text{face}\,\mathcal{K}^\dagger\left(\mathcal{E}^n(1:k,\bar{D})\right) &= \left(U \mathbb{S}_+^{n-k+t+1} U^\top\right) \cap \mathcal{S}_C \\ &= (UV)\mathbb{S}_+^{n-k+t}(UV)^\top \end{aligned}$$

The minimal face

Aside:

- $$\begin{aligned} \operatorname{face} \mathcal{K}^{\dagger}\left(\mathcal{E}^n(1:k, \bar{D})\right) &= \left(U \mathbb{S}_+^{n-k+t+1} U^\top\right) \cap \mathcal{S}_C \\ &= (UV)\mathbb{S}_+^{n-k+t}(UV)^\top \end{aligned}$$

Note that the minimal face is defined by the subspace $\mathcal{L} = \mathcal{R}(UV)$. We add $\frac{1}{\sqrt{k}} e$ to represent $\mathcal{N}(\mathcal{K})$; then we use V to eliminate e to recover a centered face.

Facial Reduction for Disjoint Cliques

Corollary from Basic Theorem

let $\alpha_1, \ldots, \alpha_\ell \subseteq 1{:}n$ pairwise disjoint sets, wlog:
$\alpha_i = (k_{i-1}+1){:}k_i$, $k_0 = 0$, $\alpha := \bigcup_{i=1}^{\ell} \alpha_i = 1{:}|\alpha|$ let
$\bar{U}_i \in \mathbb{R}^{|\alpha_i| \times (t_i+1)}$ with full column rank satisfy $e \in \mathcal{R}(\bar{U}_i)$ and

$$U_i := \begin{array}{c} k_{i-1} \\ |\alpha_i| \\ n-k_i \end{array} \overset{\begin{array}{ccc} k_{i-1} & t_i+1 & n-k_i \end{array}}{\begin{bmatrix} I & 0 & 0 \\ 0 & \bar{U}_i & 0 \\ 0 & 0 & I \end{bmatrix}} \in \mathbb{R}^{n \times (n-|\alpha_i|+t_i+1)}$$

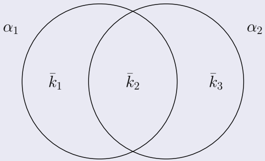
The minimal face is defined by $\mathcal{L} = \mathcal{R}(U)$:

$$U := \begin{array}{c} |\alpha_1| \\ \vdots \\ |\alpha_\ell| \\ n-|\alpha| \end{array} \overset{\begin{array}{cccc} t_1+1 & \ldots & t_\ell+1 & n-|\alpha| \end{array}}{\begin{bmatrix} \bar{U}_1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \bar{U}_\ell & 0 \\ 0 & \ldots & 0 & I \end{bmatrix}} \in \mathbb{R}^{n \times (n-|\alpha|+t+1)},$$

where $t := \sum_{i=1}^{\ell} t_i + \ell - 1$. And $e \in \mathcal{R}(U)$.

Sets for Intersecting Cliques/Faces

$\alpha_1 := 1\!:\!(\bar{k}_1 + \bar{k}_2); \quad \alpha_2 := (\bar{k}_1 + 1)\!:\!(\bar{k}_1 + \bar{k}_2 + \bar{k}_3)$



Let:

- $\alpha_1, \alpha_2 \subseteq 1:n; \quad k := |\alpha_1 \cup \alpha_2|$
- for $i = 1, 2$: $\bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$, embedding dimension t_i;
- $B_i := \mathcal{K}^\dagger(\bar{D}_i) = \bar{U}_i S_i \bar{U}_i^\top$, $\bar{U}_i \in \mathcal{M}^{k_i \times t_i}$, $\bar{U}_i^\top \bar{U}_i = I_{t_i}$, $S_i \in \mathbb{S}^{t_i}_{++}$;
- $U_i := \begin{bmatrix} \bar{U}_i & \frac{1}{\sqrt{k_i}} e \end{bmatrix} \in \mathcal{M}^{k_i \times (t_i+1)}$; and $\bar{U} \in \mathcal{M}^{k \times (t+1)}$

  satisfies $\boxed{\mathcal{R}(\bar{U}) = \mathcal{R}\left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{\bar{k}_3} \end{bmatrix}\right) \cap \mathcal{R}\left(\begin{bmatrix} I_{\bar{k}_1} & 0 \\ 0 & U_2 \end{bmatrix}\right), \text{ with } \bar{U}^\top \bar{U} = I_{t+1}}$
- $U := \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathcal{M}^{n \times (n-k+t+1)}$ and $\begin{bmatrix} V & \frac{U^\top e}{\|U^\top e\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$

  be orthogonal.

Then

$$\boxed{\begin{array}{rcl} \underline{\underline{\bigcap_{i=1}^2 \text{face}\, \mathcal{K}^\dagger\left(\mathcal{E}^n(\alpha_i, \bar{D}_i)\right)}} & = & \left(U \mathbb{S}_+^{n-k+t+1} U^\top\right) \cap \mathcal{S}_C \\ & = & (UV)\mathbb{S}_+^{n-k+t}(UV)^\top \end{array}}$$

Expense/Work of (Two) Clique/Facial Reductions

Subspace Intersection for Two Intersecting Cliques/Faces

Suppose:

$$U_1 = \begin{bmatrix} U_1' & 0 \\ U_1'' & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} I & 0 \\ 0 & U_2'' \\ 0 & U_2' \end{bmatrix}$$

Then:

$$U := \begin{bmatrix} U_1' \\ U_1'' \\ U_2'(U_2'')^\dagger U_1'' \end{bmatrix} \quad \text{or} \quad U := \begin{bmatrix} U_1'(U_1'')^\dagger U_2'' \\ U_2'' \\ U_2' \end{bmatrix}$$

($Q_1 =: (U_1'')^\dagger U_2'', Q_2 = (U_2'')^\dagger U_1''$ orthogonal/rotation)
(Efficiently) satisfies

$$\mathcal{R}(U) = \mathcal{R}(U_1) \cap \mathcal{R}(U_2)$$

Two (Intersecting) Clique Explicit Delayed Completion

Let:

- Hypotheses of intersecting Theorem (Thm 2) holds
- $\bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$, for $i = 1, 2$, $\beta \subseteq \alpha_1 \cap \alpha_2, \gamma := \alpha_1 \cup \alpha_2$
- $\bar{D} := D[\beta]$ with embedding dimension r
- $B := \mathcal{K}^\dagger(\bar{D})$, $\quad \bar{U}_\beta := \bar{U}(\beta, :)$, where $\bar{U} \in \mathcal{M}^{k \times (t+1)}$ satisfies intersection equation of Thm 2
- $\begin{bmatrix} \bar{V} & \frac{\bar{U}^\top e}{\|\bar{U}^\top e\|} \end{bmatrix} \in \mathcal{M}^{t+1}$ be orthogonal.
- $\boxed{Z := (J\bar{U}_\beta \bar{V})^\dagger B((J\bar{U}_\beta \bar{V})^\dagger)^\top}$.

THEN $t = r$ in Thm 2, and $Z \in \mathbb{S}^r_+$ is the unique solution of the equation $(J\bar{U}_\beta \bar{V})Z(J\bar{U}_\beta \bar{V})^\top = B$, and the exact completion is

$$\boxed{D[\gamma] = \mathcal{K}\left(PP^\top\right)} \quad \text{where} \quad \boxed{P := UVZ^{\frac{1}{2}} \in \mathbb{R}^{|\gamma| \times r}}$$

Completing SNL (Delayed use of Anchor Locations)

Rotate to Align the Anchor Positions

- Given $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{R}^{n \times r}$ such that $D = \mathcal{K}(PP^\top)$
- Solve the orthogonal Procrustes problem:

$$\boxed{\begin{array}{ll} \min & \|A - P_2 Q\| \\ \text{s.t.} & Q^\top Q = I \end{array}}$$

  $P_2^\top A = U \Sigma V^\top$ SVD decomposition; set $Q = UV^\top$;
  (Golub/Van Loan'79-'12, Algorithm 12.4.1)
- Set $X := P_1 Q$

Random Noisless Problems, Krislock W. '**2010**

- 2.16 GHz Intel Core 2 Duo, 2 GB of RAM
- Dimension $r = 2$
- Square region: $[0,1] \times [0,1]$
- $m = 9$ anchors
- Using only Rigid Clique Union and Rigid Node Absorption
- Error measure: Root Mean Square Deviation

$$\text{RMSD} = \left(\frac{1}{n} \sum_{i=1}^{n} \| p_i - p_i^{\text{true}} \|^2 \right)^{1/2}$$

Results - Large n (SDP size $O(n^2)$)

n # of Sensors Located

n # sensors \ R	0.07	0.06	0.05	0.04
2000	2000	2000	1956	1374
6000	6000	6000	6000	6000
10000	10000	10000	10000	10000

CPU Seconds

# sensors \ R	0.07	0.06	0.05	0.04
2000	1	1	1	3
6000	5	5	4	4
10000	10	10	9	8

RMSD (over located sensors)

n # sensors \ R	0.07	0.06	0.05	0.04
2000	$4e{-}16$	$5e{-}16$	$6e{-}16$	$3e{-}16$
6000	$4e{-}16$	$4e{-}16$	$3e{-}16$	$3e{-}16$
10000	$3e{-}16$	$5e{-}16$	$4e{-}16$	$4e{-}16$

Results - N Huge SDPs Solved

Large-Scale Problems

# sensors	# anchors	radio range	RMSD	Time
20000	9	.025	$5e{-}16$	25s
40000	9	.02	$8e{-}16$	1m 23s
60000	9	.015	$5e{-}16$	3m 13s
100000	9	.01	$6e{-}16$	9m 8s

Size of SDPs Solved: $N = \binom{n}{2}$ (# vrbls)

$\mathcal{E}_n(\text{density of } \mathcal{G}) = \pi R^2$; $M = \mathcal{E}_n(|E|) = \pi R^2 N$ (# constraints)

Size of SDP Problems:

$M = \begin{bmatrix} 3,078,915 & 12,315,351 & 27,709,309 & 76,969,790 \end{bmatrix}$

$N = 10^9 \begin{bmatrix} 0.2000 & 0.8000 & 1.8000 & 5.0000 \end{bmatrix}$

View 2: Recall Details with Exposing Vector/Numerics

Thm D.P.W. '15: $\mathcal{M}: \mathbb{E} \to \mathbb{Y}$, K proper convex cone

$\emptyset \neq F = \{X \in K : \mathcal{M}(X) = b\}$. Then a vector v exposes a proper face of $\mathcal{M}(K)$ containing b if, and only if, v satisfies the auxiliary system

$$0 \neq \mathcal{M}^* v \in \mathcal{K}^*, \quad \langle v, b \rangle = 0.$$

Let $N = \text{face}(b, \mathcal{M}(K))$ (smallest face containing b). Then:

- $K \cap \mathcal{M}^{-1}(N) = \text{face}(F, K)$
- v exposes N IFF $\mathcal{M}^*(v)$ exposes $\text{face}(F, K)$.

Corollary

If Slater's condition fails, then $d = 1$ IFF the minimal $\text{face}(b, \mathcal{M}(K))$ is exposed.

Using Exposed Vectors

- Find a set of medium sized cliques $\mathcal{C}$ (e.g. a clique for each node). $r+1 \leq |C| \leq M, \forall C \in \mathcal{C}$.
- Find an exposing vector $Y_C \in \mathbb{S}^{|}C|_+$ and *weight/value* for each $C \in \mathcal{C}$. Fill out $Y_C \in \mathcal{S}^n_+$ with zeros for remaining nodes.
- Find final exposing vector $\sum_{C \in \mathcal{C}} w_C Y_C$ and nullspace V.
- solve the smaller EDM/SNL with $X = VRV^T$.

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Applications of Facial Reduction to Compressed Sensing, Sensor Network Localization, and Molecular Conformation

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Sat. June 25, 2016, noon

at: Informal Workshop on Nonlinear Optimization
University of Western Ontario