Coordinate shadows of semi-definite and Euclidean distance matrices

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Outline

Projections (shadows) of PSD and EDM cones wrt matrix graph

- Basic question: closure of projections/feasible sets
- Motivation: e.g., sparse PSD algorithms that evaluate *subset of elements* of e.g., $Y = RR^T \in \mathcal{S}^n$, $R \in \mathbb{R}^{nr}$

PSD and EDM completions of partial matrices

- finding minimal face containing feasible sets
- Motivation: e.g., stability/robustness, reduction in size

Eg: Find Positive Semidefinite Completion, PSD

Example (graph edges correspond to matrix nonzeros)

graph $G = (V, E)$, nodes $V = \{1, 2, 3, 4\}$, edges $E = \{12, 23, 34, 14\} \cup \{11, 22, 33, 44\}$ (include all self-loops)

$$C(\epsilon), \epsilon \geq 0 : \quad \begin{bmatrix} 1+\epsilon & 1 & ? & -1 \\ 1 & 1+\epsilon & 1 & ? \\ ? & 1 & 1+\epsilon & 1 \\ -1 & ? & 1 & 1+\epsilon \end{bmatrix}.$$

- For $\epsilon > 0$ sufficiently large, $C(\epsilon)$ with $? = 0$ is positive definite (by diagonal dominance).
- By Grone-Johnson-Sa-W. (GRSW) 1984 Lemma 6, [8], $\exists!$ PSD matrix A satisfying $A_{ij} = 1, \forall |i - j| \leq 1$, namely matrix of all 1's. Hence $C(0)$ is infeasible, i.e., NOT PSD completable.
- What can we say about the boundary of the feasible set? Is the feasible set closed? (Important for stability/convergence questions/constraint qualifications.)

Backgr./Notat.: (i) ($\mathcal{S}^n_+$ *PSD*); (ii) ($\mathcal{E}^n$ *EDM*) cones

- symmetric matrix $X \in \mathcal{S}^n$ psd if $v^T X v \geq 0, \forall v$.
- $D \in \mathcal{E}^n \subset \mathcal{S}^n$ if there exist n points $p_i \in \mathbb{R}^k$ (for $i = 1, \ldots, n$) satisfying $\boxed{D_{ij} = \|p_i - p_j\|^2, \forall i, j}$.

Consider undirected graph: $G = (V, E), |V| = n$, L self-loops

- classical semi-definite (PSD, $\succeq$) completion problem: given data $a \in \mathbb{R}^E$: $\exists ? n \times n \ X \succeq 0$ completing a meaning: $\boxed{0 \preceq X = X^T, X_{ij} = a_{ij}, \forall ij \in E}$
- Euclidean distance (EDM, $\mathcal{E}$) completion problem: given such data vector a, does there exist a Euclidean distance matrix, EDM, $\mathcal{E}^n$, completing it.
- surveys, many applications, and parallel results: Laurent/96/98, Alfakih-Khandani-W./97, Alfakih-W./00, Floudas/01, Netzer/12.

Here: projections of PSD cone $\mathcal{S}_+^n$ and EDM cone $\mathcal{E}^n$

Projections onto matrix entries indexed by edge set E

"coordinate shadows", denoted by

$$\mathcal{P}(\mathcal{S}_+^n), \mathcal{P}(\mathcal{E}^n) \subseteq \mathbb{R}^E$$

These are precisely the sets of data vectors $a \in \mathbb{R}^E$ that render the corresponding completion problems feasible.

("spectrahedral shadows" e.g., Gouveia-Parrilo-Thomas/13, Helton-Nie/09/10, Auslander/96, .)

Two goals

1. Highlight Geometry of $\mathcal{P}(\mathcal{S}_+^n)$ and $\mathcal{P}(\mathcal{E}^n)$
2. geometry leads to simplified and transparent analysis and important conclusions for the Krislock-W. EDM completion algorithm

We start with a basic question:

Under what conditions are coordinate shadows $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$ closed?

Part of: deciding if linear image of a general closed convex set is itself closed
(Pataki fundamental closure result is used in our proofs; fundamental connection to constraint qualifications, strong duality in convex opt., e.g., Rockafellar/70, Duffin-Jeroslow-Karlovitz/81, Duffin/56, Pataki/11)

Conditions for closure

Simple example, $n = 2$

$\mathcal{S}_+^2 = \left\{ Z \in \mathcal{S}_+^2 : Z = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \right\}$ and by abuse of notation:

$\mathcal{P}_z(\mathcal{S}_+^2) = \mathbb{R}_+, \quad \mathcal{P}_y(\mathcal{S}_+^2) = \mathbb{R}, \quad \mathcal{P}_{x,z}(\mathcal{S}_+^2) = \mathbb{R}_+^2$ all closed

But

$\mathcal{P}_{x,y}(\mathcal{S}_+^2) = \quad \mathcal{P}_{z,y}(\mathcal{S}_+^2) = \{(0,0)\} \cup (\mathbb{R}_{++} \times \mathbb{R})$ not closed

this example extends to characterization of general case

Surprisingly, combinatorial answer to topological question:

- $\mathcal{P}(\mathcal{S}_+^n)$ is closed iff
  the set vertices attached to self-loops $L = \{i \in V : ii \in E\}$
  is disconnected from its complement L^c
- more surprisingly: $\mathcal{P}(\mathcal{E}^n)$ is always closed

Algorithmic significance of coordinate shadows

When is PSD completion problem feasible region nonempty?

Given data vector $a \in \mathbb{R}^E$, the set of all PSD completions:

$$F_G := \{X \in \mathcal{S}_+^n : X_{ij} = a_{ij}, \forall ij \in E\}$$ PSD feasible region

Necessary conditions for $F_G \neq \emptyset$

data vector $a \in \mathbb{R}^E$ must be a partial PSD matrix (all its principal submatrices are positive semi-definite)

BUT, to guarantee suff. of partial PSD matrix $a \in \mathbb{R}^E$

we need restriction of G to L is chordal (each of its cycles of four or more vertices has a chord) and the self-loop nodes L is disconnected from L^c

Algorithmic significance of coordinate shadows cont...

Failure of Slater/pos. def. completion

Krislock-W.:

- even if $F_G \neq \emptyset$, Slater condition often fails
- i.e., small perturbations to any specified principal submatrix of a having deficient rank can yield the semi-definite completion problem infeasible.
- i.e., the partial matrix a lies on the boundary of $\mathcal{P}(\mathcal{S}^n_+)$;
- we can exploit this!

Analogous results for EDM completion

$$\{X \in \mathcal{E}^n : X_{ij} = a_{ij} \text{ for } ij \in E\} \qquad \text{feasible set}$$

rank of each principal submatrix of $a \in \mathbb{R}^E$ is replaced by its embedding dimension.

Preprocesss in Krislock-W./Combinatorial description

- utilizes cliques in graph G to systematically decrease size of EDM completion problem;
  found to be very efficient;
- In current work: use geometric argument with boundary of $\mathcal{P}(\mathcal{E}^n)$ playing a key role.
  In fact: when G is chordal and all cliques are considered, the preprocessing technique discovers the minimal face of $\mathcal{E}^n$ (respectively $\mathcal{S}_+^n$) containing the feasible region, i.e., a purely combinatorial description.

$\mathcal{P}(\mathcal{S}^n_+)$ - set of all *partial* PSD completable matrices

partial matrix $a \in \mathbb{R}^E$ is a partial PSD matrix if:

all principal submatrices, defined by a, are PSD matrices

G itself is a PSD completable graph

if every partial PSD matrix $a \in \mathbb{R}^E$ is completable to a PSD matrix.

PD completions, partial PD matrices, and PD completable graphs are defined similarly.

Chordality

We call a graph chordal if any cycle of four or more nodes has a chord, i.e., an edge exists joining any two nodes that are not adjacent in the cycle.

Correction of Theorem in GJSW

Theorem (PSD completable matrices & chordal graphs)

The following are true.

1. *The graph G is PD completable if and only if the graph induced by G on L is chordal.*
2. *Supposing equality $L = V$ holds, the graph G is PSD completable if and only if G is chordal.*

Without $L = V$: $\begin{bmatrix} 0 & 1 \\ 1 & ? \end{bmatrix}$ chordal/not psd completable

EDM completable, $\mathcal{P}(\mathcal{E}^n)$

- $L = V$ (the diagonal of an EDM is always fixed at zero)
- a completion $A \in \mathcal{S}^n$ of a partial matrix $a \in \mathbb{R}^E$ is an EDM completion if A is an EDM.
- a partial matrix $a \in \mathbb{R}^E$ is a partial EDM if any existing principal submatrix, defined by a, is an EDM.
- G is an EDM completable graph if any partial EDM is completable to an EDM.

Theorem (Bakonyi-Johnson , EDM complet. & chord. gr.)

The graph G is EDM completable if and only if G is chordal.

Theorem (Main result 1: Closedness of projected PSD cone)

projected set $\mathcal{P}(\mathcal{S}_+^n)$ is closed iff

vertices in L are disconnected from those in complement L^c

Moreover, if latter condition fails, then:
for any edge $i^*j^* \in E$ joining a vertex in L with a vertex in L^c, any partial matrix $a \in \mathbb{R}^E$ satisfying

$$a_{i^*j^*} \neq 0 \text{ and } a_{ij} = 0 \text{ for all } ij \in E \cap (L \times L),$$

lies in $\left(\operatorname{cl} \mathcal{P}(\mathcal{S}_+^n)\right) \setminus \mathcal{P}(\mathcal{S}_+^n)$.

Corollary (PSD completability, chordal graphs, and connectivity)

The graph G is PSD completable if and only if the graph induced by G on L is chordal and L is disconnected from L^c.

Theorem (Main result 2: Closedness of projected EDM cone)

The projected image $\mathcal{P}(\mathcal{E}^n)$ is always closed.

Boundaries/projected sets/facial reduction

Conic system

$F := \{X \in C : \mathcal{M}(X) = b\}$,
C closed convex cone; $\mathcal{M} : \mathbb{E} \to \mathbb{Y}$ surjective linear transformation; $\mathbb{E}, \mathbb{Y}$ Euclidean spaces;

Slater condition

if there exists $X \in \operatorname{int} C$ satisfying system $\mathcal{M}(X) = b$.
Equivalently, (since $\mathcal{M}$ is surjective/open mapping)

$b \in \operatorname{int} \mathcal{M}(C)$.

Theorem (Facial reduction)

For any vector v *exposing* $\operatorname{face}(b, \mathcal{M}(C))$, *the vector* $\mathcal{M}^* v$ *exposes* $\operatorname{face}(F, C)$ *(the minimal face).*

Restrict conic system to linear span of $\operatorname{face}(F, C)$,
where F is minimal face;
then (strict feasibility) Slater's holds

Exploit structure/efficient facial reduction

Consider subproblems using indices $I \subseteq E$

For example I describes a clique in G.
Krislock-W. algorithm:

- Use cliques to facially reduce the problem;
- if two cliques intersect 'rigidly' then take the intersection of faces to find the union of the cliques, i.e., this completes all distances in the union of the cliques

Theorem (Clique facial reduction for PSD completions)

Let $\chi \subseteq L$ be any k-clique in the graph G. Let $a \in \mathbb{R}^E$ be a partial PSD matrix and define

$$F_\chi := \{X \in \mathcal{S}^n_+ \ : X_{ij} = a_{ij}, \forall ij \in E(\chi)\}$$

where $E(\chi)$ denotes edge set in subgraph induced by G on χ. Then for any matrix v_χ exposing $\text{face}(a_\chi, \mathcal{S}^\chi_+)$, the matrix

$$\mathcal{P}^*_\chi v_\chi \quad \textit{exposes} \quad \text{face}(F_\chi, \mathcal{S}^n_+).$$

Find minimal face using only cliques?

Example (Slater condition & nonchordal graphs)

$G = (V, E)$ cycle, $V = \{1, 2, 3, 4\}$, all loops,
$E = \{12, 23, 34, 14\} \cup \{11, 22, 33, 44\}$.

$$C(\epsilon), \epsilon \geq 0: \qquad \begin{bmatrix} 1+\epsilon & 1 & ? & -1 \\ 1 & 1+\epsilon & 1 & ? \\ ? & 1 & 1+\epsilon & 1 \\ -1 & ? & 1 & 1+\epsilon \end{bmatrix}.$$

For $\epsilon > 0$, note all specified principal submatrices are positive definite; all faces arising from cliques are trivial, i.e., facial reduction using <u>only</u> cliques does <u>nothing</u>.
But, Lemma 6 in GJSW '84, implies there exists a unique positive semidefinite matrix A satisfying $A_{ij} = 1, \forall |i - j| \leq 1$, namely the matrix of all 1's. Hence $C(0)$ is infeasible, i.e., $a(0)$ lies outside of $\mathcal{P}(\mathcal{S}^4_+)$.

Example (Slater condition & nonchordal graphs cont...)

i.e., $a(0)$ lies outside of $\mathcal{P}(\mathcal{S}^4_+)$.
But, for large ϵ, partial matrices $a(\epsilon)$ lie in $\mathcal{P}(\mathcal{S}^4_+)$ due to diagonal dominance.
$\mathcal{P}(\mathcal{S}^4_+)$ is closed (why?); therefore, there exists $\hat{\epsilon} > 0$, $a(\hat{\epsilon}) \in \text{bnd}(\mathcal{P}(\mathcal{S}^4_+))$, i.e., Slater condition fails for the completion problem $C(\hat{\epsilon})$. In fact, by solving the SDP:

$$\begin{array}{ll} \min & \epsilon \\ \text{s.t.} & \begin{bmatrix} 1+\epsilon & 1 & \alpha & -1 \\ 1 & 1+\epsilon & 1 & \beta \\ \alpha & 1 & 1+\epsilon & 1 \\ -1 & \beta & 1 & 1+\epsilon \end{bmatrix} \succeq 0 \end{array}$$

we deduce that $\hat{\epsilon} = \sqrt{2} - 1, \hat{\alpha} = \hat{\beta} = 0$ (verify using duality)

Main result 3!: clique facial reduction 'enough' for EDM

Theorem (Clique facial reduction for EDM is sufficient)

Suppose that G *is chordal, and consider a partial Euclidean distance matrix* $a \in \mathbb{R}^E$ *and the region*

$$F := \{X \in \mathcal{S}_c \cap \mathcal{S}^n_+ \ : [\mathcal{K}(X)]_{ij} = a_{ij} \textit{ for all } ij \in E\}.$$

Let Θ *denote the set of all cliques in* G*, and for each* $\chi \in \Theta$ *define*

$$F_\chi := \{X \in \mathcal{S}_c \cap \mathcal{S}^n_+ \ : [\mathcal{K}(X)]_{ij} = a_{ij} \textit{ for all } ij \in E(\chi)\}.$$

Then the equality

$$\operatorname{face}(F, \mathcal{S}_c \cap \mathcal{S}^n_+) = \bigcap_{\chi \in \Theta} \operatorname{face}(F_\chi, \mathcal{S}_c \cap \mathcal{S}^n_+) \qquad \textit{holds}.$$

Summary

- studied the geometry of projections/coordinate-shadows $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$
- Surprisingly $\mathcal{P}(\mathcal{E}^n)$ is always closed; while $\mathcal{P}(\mathcal{S}^n_+)$ closure depends on subgraph/loops/connectedness
- Can exploit the structure of the boundaries
- facial reduction; using cliques is enough for EDM completions in chordal case
- Results are based on May 2014 Research Report:
  "Coordinate shadows of semi-definite and Euclidean distance matrices"
  Dmitriy Drusvyatskiy, Gabor Pataki, Henry Wolkowicz
  http://www.optimization-online.org/DB_HTML/2014/05/4349.html

Thanks for your attention!

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