Coordinate shadows of semi-definite and Euclidean distance matrices

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Background/Notation

Consider undirected graph: $G = (V, E), |V| = n$, L self-loops

- classical semi-definite (PSD) completion problem: given data vector a indexed by E does there exist $n \times n$ positive semi-definite matrix X completing a (meaning $X_{ij} = a_{ij}$ for all $ij \in E$)
- Euclidean distance (EDM) completion problem: given such data vector a, does there exist a Euclidean distance matrix, EDM, $\mathcal{E}^n$, completing it.
- survey, many applications, and parallel results: [13, 1, 14, 15, 12].

- $X \in \mathcal{S}^n$ psd if $v^T X v \geq 0, \forall v$.
- $D \in \mathcal{E}^n \subset \mathcal{S}^n$ if there exist n points $p_i \in \mathbb{R}^k$ (for $i = 1, \ldots, n$) satisfying $D_{ij} = \|p_i - p_j\|^2, \forall i, j$.

Here: projections of PSD cone $\mathcal{S}^n_+$ and EDM cone $\mathcal{E}^n$

Projections onto matrix entries indexed by edge set E

"coordinate shadows", denoted by $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$
precisely the sets of data vectors that render corresponding completion problems feasible.

("spectrahedral shadows" e.g., [7, 9, 10, 2].)

Two goals

1. Highlight Geometry of $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$
2. geometry leads to simplified and transparent analysis of the Krislock-W. [11] EDM completion algorithm

We start with a basic question:

Under what conditions are coordinate shadows $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$ closed?

Part of: deciding if linear image of a general closed convex set is itself closed Pataki [17] (fundamental closure result is used in our proofs)
(fundamental connection to constraint qualifications, strong duality in convex opt., e.g., [18, 6, 5, 17])

Will show:

- surprisingly $\mathcal{P}(\mathcal{E}^n)$ is always closed
- $\mathcal{P}(\mathcal{S}^n_+)$ is closed iff
  the set vertices attached to self-loops $L = \{i \in V : ii \in E\}$ is disconnected from its complement L^c

Algorithmic significance of coordinate shadows

Feasible region of PSD completion problem

$$F_G := \{X \in \mathcal{S}^n_+ \ : X_{ij} = a_{ij} \text{ for } ij \in E\}.$$

Necessary conditions for $F_G \neq \emptyset$: data vector $a \in \mathbb{R}^E$ must be a partial PSD matrix (all its principal submatrices are positive semi-definite)
to guarantee sufficiency of $a \in \mathcal{P}(\mathcal{S}^n_+)$: need restriction of G to L is chordal and L is disconnected from L^c

Failure of Slater/pos. def. completion

Krislock-W.: even if $F_G \neq \emptyset$, Slater condition often fails;
i.e., small perturbations to any specified principal submatrix of a having deficient rank can yield the semi-definite completion problem infeasible.
i.e., the partial matrix a lies on the boundary of $\mathcal{P}(\mathcal{S}^n_+)$;
we can exploit this!

Analogous results for EDM

$$\{X \in \mathcal{E}^n : X_{ij} = a_{ij} \text{ for } ij \in E\}$$

rank of each principal submatrix of $a \in \mathbb{R}^E$ is replaced by its embedding dimension.

Preprocesss in Krislock-W./Combinatorial description

- utilizes cliques in graph G to systematically decrease size of EDM completion problem;
  found to be very efficient;
- In current work: use geometric argument with boundary of $\mathcal{P}(\mathcal{E}^n)$ playing a key role;
  show: when G is chordal and all cliques are considered, the preprocessing technique discovers the minimal face of $\mathcal{E}^n$ (respectively $\mathcal{S}_+^n$) containing the feasible region, i.e., a purely combinatorial description.

Convex geometry

- convex subset $F \subseteq C$ is a face of convex cone C, denoted $F \trianglelefteq C$, if F contains any line segment in C whose relative interior intersects F.
- The minimal face containing $S \subseteq C$ is the intersection of faces containing S; denoted $\mathrm{face}(S)$.
- $C^* = \{y \in \mathbb{E} : \langle y, x \rangle \geq 0 \text{ for all } x \in C\}$, nonneg. polar cone
- The conjugate face: $F^\triangle := C^* \cap F^\perp$
- The EDM and PSD cones are facially exposed: $F = C \cap v^\perp$, for some v.

PSD matrices

Faces of PSD

$$F = \left\{ U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^T : A \in \mathcal{S}_+^r \right\},$$

for some orthogonal matrix U and some integer $r = 0, 1, \ldots, n$.

Conjugate face of face F

$$F^{\triangle} = \left\{ U \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} U^T : A \in \mathcal{S}_+^{n-r} \right\}.$$

convex set $Q \subset \mathcal{S}_+^n$; X any maximal rank matrix in Q

$$\text{face}(Q, \mathcal{S}_+^n) = \text{face}(X, \mathcal{S}_+^n)$$

EDM matrices (squared Euclidean distances)

$D \in \mathcal{S}^n$ is EDM, $\mathcal{E}^n$

if there exist n points $p_i \in \mathbb{R}^k$ (for $i = 1, \ldots, n$) satisfying

$$D_{ij} = \|p_i - p_j\|^2, \forall i, j.$$

Smallest integer k is embedding dimension of D, embdim D.

$\mathcal{E}^n$ is linearly isomorphic to $\mathcal{S}_+^{n-1}$

$\mathcal{K} : \mathcal{S}^n \to \mathcal{S}^n, \quad \mathcal{K}(X)_{ij} := X_{ii} + X_{jj} - 2X_{ij}$.

adjoint $\mathcal{K}^*(D) = 2(\mathrm{Diag}\,(De) - D) \quad (= 2Lapl$ of $D)$.

Following equations hold:

$\mathrm{rge}\,\mathcal{K} = \mathcal{S}_H, \mathcal{K}(\mathcal{S}_+^n) = \mathcal{E}^n \qquad \mathrm{rge}\,\mathcal{K}^* = \mathcal{S}_c$

where

$\mathcal{S}_c := \{X \in \mathcal{S}^n : Xe = 0\}; \qquad \mathcal{S}_H := \{D \in \mathcal{S}^n : \mathrm{diag}\,(D) = 0\}$,

are the centered and hollow matrices, respectively.

PSD completions

projection map $\mathcal{P} : \mathcal{S}^n \to \mathbb{R}^E$

$$\mathcal{P}(A) := (A_{ij})_{ij \in E}.$$

adjoint map: $\mathcal{P}^* : \mathbb{R}^E \to \mathcal{S}^n$

$$(\mathcal{P}^*(y))_{ij} = \begin{cases} y_{ij}, & \text{if } ij \in E \\ 0, & \text{otherwise,} \end{cases}$$

for indices $i \leq j$.

Connection to Laplacian (used in proofs)

Laplacian operator $\mathcal{L}\colon \mathbb{R}^E \to \mathcal{S}^n$

$$\mathcal{L}(a) := \frac{1}{2}(\mathcal{P} \circ \mathcal{K})^*(a) = \mathrm{Diag}\,(\mathcal{P}^*(a)e) - \mathcal{P}^*(a).$$

When G is connected:

$$\ker \mathcal{L}(a) = \mathrm{span}\{e\} \qquad \text{and} \qquad \mathrm{face}(\mathcal{L}(a), \mathcal{S}^n_+) = \mathcal{S}_c \cap \mathcal{S}^n_+ .$$

$\mathcal{P}(\mathcal{S}^n_+)$ - set of all *partial* PSD completable matrices

partial matrix $a \in \mathbb{R}^E$ is a partial PSD matrix if:

all principal submatrices, defined by a, are PSD matrices

***G* itself is a PSD completable graph**

if every partial PSD matrix $a \in \mathbb{R}^E$ is completable to a PSD matrix.

PD completions, partial PD matrices, and PD completable graphs are defined similarly.

Chordality

We call a graph chordal if any cycle of four or more nodes has a chord, i.e., an edge exists joining any two nodes that are not adjacent in the cycle.

Correction of Theorem in GJSW [8]

Theorem (PSD completable matrices & chordal graphs)

The following are true.

1. *The graph G is PD completable if and only if the graph induced by G on L is chordal.*
2. *Supposing equality $L = V$ holds, the graph G is PSD completable if and only if G is chordal.*

Without $L = V$: $\begin{bmatrix} 0 & 1 \\ 1 & ? \end{bmatrix}$ chordal/not psd completable

EDM completable, $\mathcal{P}(\mathcal{E}^n) \quad (= \mathcal{L}^*(\mathcal{S}^n_+))$

- $L = \emptyset$ (the diagonal of an EDM is always fixed at zero)
- a completion $A \in \mathcal{S}^n$ of a partial matrix $a \in \mathbb{R}^E$ is an EDM completion if A is an EDM.
- a partial matrix $a \in \mathbb{R}^E$ is a partial EDM if any existing principal submatrix, defined by a, is an EDM.
- G is an EDM completable graph if any partial EDM is completable to an EDM.

Theorem (Bakonyi-Johnson [3], EDM complet. & chord. gr.)

The graph G is EDM completable if and only if G is chordal.

Theorem (Main result 1: Closedness of projected PSD cone)

projected set $\mathcal{P}(\mathcal{S}^n_+)$ is closed iff

vertices in L are disconnected from those in complement L^c

Moreover, if latter condition fails, then:
for any edge $i^*j^* \in E$ joining a vertex in L with a vertex in L^c, any partial matrix $a \in \mathbb{R}^E$ satisfying

$$a_{i^*j^*} \neq 0 \text{ and } a_{ij} = 0 \text{ for all } ij \in E \cap (L \times L),$$

lies in $\left(\operatorname{cl} \mathcal{P}(\mathcal{S}^n_+)\right) \setminus \mathcal{P}(\mathcal{S}^n_+)$.

Corollary (PSD completability, chordal graphs, and connectivity)

The graph G is PSD completable if and only if the graph induced by G on L is chordal and L is disconnected from L^c.

Theorem (Main result 2: Closedness of projected EDM cone)

The projected image $\mathcal{P}(\mathcal{E}^n)$ is always closed.

Boundaries/projected sets/facial reduction

Conic system

$F := \{X \in C : \mathcal{M}(X) = b\}$,
C closed convex cone; $\mathcal{M} : \mathbb{E} \to \mathbb{Y}$ surjective linear transformation; $\mathbb{E}, \mathbb{Y}$ Euclidean spaces;

Slater condition

if there exists $X \in \operatorname{int} C$ satisfying system $\mathcal{M}(X) = b$.
Equivalently, (since $\mathcal{M}$ is surjective/open mapping)

$$b \in \operatorname{int} \mathcal{M}(C).$$

Theorem (Facial reduction)

For any vector v exposing $\operatorname{face}(b, \mathcal{M}(C))$, the vector $\mathcal{M}^* v$ exposes $\operatorname{face}(F, C)$.

Restrict conic system to linear span of $\operatorname{face}(F, C)$,
where F is minimal face;
then (strict feasibility) Slater's holds

Exploit structure/efficient facial reduction

Consider subproblems using indices $I \subseteq E$

For example I describes a clique in G.
Krislock-W. algorithm:

- Use cliques to facially reduce the problem;
- if two cliques intersect 'rigidly' then take the intersection of faces to find the union of the cliques, i.e., this completes all distances in the union of the cliques

Theorem (Clique facial reduction for PSD completions)

Let $\chi \subseteq L$ be any k-clique in the graph G. Let $a \in \mathbb{R}^E$ be a partial PSD matrix and define

$$F_\chi := \{X \in \mathcal{S}^n_+ \,:\, X_{ij} = a_{ij}, \forall ij \in E(\chi)\}$$

where $E(\chi)$ denotes edge set in subgraph induced by G on χ. Then for any matrix v_χ exposing $\mathrm{face}(a_\chi, \mathcal{S}^\chi_+)$, the matrix

$$\mathcal{P}^*_\chi v_\chi \quad \textit{exposes} \quad \mathrm{face}(F_\chi, \mathcal{S}^n_+).$$

Find minimal face using only cliques?

Example (Slater condition & nonchordal graphs)

$G = (V, E)$ cycle, $V = \{1, 2, 3, 4\}$, all loops,
$E = \{12, 23, 34, 14\} \cup \{11, 22, 33, 44\}$.

$$C(\epsilon), \epsilon \geq 0 : \qquad \begin{bmatrix} 1+\epsilon & 1 & ? & -1 \\ 1 & 1+\epsilon & 1 & ? \\ ? & 1 & 1+\epsilon & 1 \\ -1 & ? & 1 & 1+\epsilon \end{bmatrix}.$$

Note all specified principal submatrices are positive definite; all faces arising from cliques are trivial.
$a(\epsilon) \in \mathbb{R}^E$ partial matrices. [8, Lemma 6] implies there exists a unique positive semidefinite matrix A satisfying $A_{ij} = 1, \forall |i - j| \leq 1$, namely the matrix of all 1's. Hence $C(0)$ is infeasible, i.e., $a(0)$ lies outside of $\mathcal{P}(\mathcal{S}^4_+)$.

Example (Slater condition & nonchordal graphs cont...)

i.e., $a(0)$ lies outside of $\mathcal{P}(\mathcal{S}^4_+)$.
But, for large ϵ, partial matrices $a(\epsilon)$ lie in $\mathcal{P}(\mathcal{S}^4_+)$ due to diagonal dominance.
$\mathcal{P}(\mathcal{S}^4_+)$ is closed; therefore, there exists $\hat{\epsilon} > 0$, $a(\hat{\epsilon}) \in \text{bnd}(\mathcal{P}(\mathcal{S}^4_+))$, i.e., Slater condition fails for the completion problem $C(\hat{\epsilon})$. In fact, by solving the SDP:

$$\begin{array}{ll} \min & \epsilon \\ \text{s.t.} & \begin{bmatrix} 1+\epsilon & 1 & \alpha & -1 \\ 1 & 1+\epsilon & 1 & \beta \\ \alpha & 1 & 1+\epsilon & 1 \\ -1 & \beta & 1 & 1+\epsilon \end{bmatrix} \succeq 0 \end{array}$$

we deduce that $\hat{\epsilon} = \sqrt{2} - 1, \hat{\alpha} = \hat{\beta} = 0$ (verify using duality)

Theorem (Main result 3: Finding min. face on chordal graphs)

Suppose that graph induced by G on L is chordal. Consider a partial PSD matrix $a \in \mathbb{R}^E$ and the region

$$F = \{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij}, \textit{ for all } ij \in E\}.$$

Then the equality

$$\text{face}(F, \mathcal{S}^n_+) = \bigcap_{\chi \in \Theta} \text{face}(F_\chi, \mathcal{S}^n_+) \qquad \textit{holds},$$

where Θ denotes the set of all cliques in the restriction of G to L, and for each $\chi \in \Theta$ we define the relaxation

$$F_\chi := \{X \in \mathcal{S}^n_+ : X_{ij} = a_{ij}, \textit{ for all } ij \in E(\chi)\}.$$

Theorem (Clique facial reduction for EDM completions)

Let χ be any k-clique in the graph G. Let $a \in \mathbb{R}^E$ be a partial Euclidean distance matrix and define

$$F_\chi \quad := \quad \{X \in \mathcal{S}^n_+ \cap \mathcal{S}_c : [\mathcal{K}(X)]_{ij} = a_{ij}, \forall ij \in E(\chi)\}$$

Then for any matrix v_χ exposing face $\left(\mathcal{K}^\dagger(a_\chi), \mathcal{S}^\chi_+ \cap \mathcal{S}_c\right)$, the matrix

$$\mathcal{P}^*_\chi v_\chi \quad \textit{exposes} \quad \text{face}(F, \mathcal{S}^n_+ \cap \mathcal{S}_c).$$

Main result 4!: clique facial reduction 'enough' for EDM

Theorem (Clique facial reduction for EDM completions)

Let χ be any k-clique in the graph G. Let $a \in \mathbb{R}^E$ be a partial Euclidean distance matrix and define

$$F_\chi \quad := \quad \{X \in \mathcal{S}^n_+ \cap \mathcal{S}_c : [\mathcal{K}(X)]_{ij} = a_{ij}, \forall ij \in E(\chi)\}$$

Then for any matrix v_χ exposing face $\left(\mathcal{K}^\dagger(a_\chi), \mathcal{S}^\chi_+ \cap \mathcal{S}_c\right)$*, the matrix*

$$\mathcal{P}^*_\chi v_\chi \quad \textit{exposes} \quad \text{face}(F, \mathcal{S}^n_+ \cap \mathcal{S}_c).$$

Summary

- studied the geometry of projections/coordinate-shadows $\mathcal{P}(\mathcal{S}^n_+)$ and $\mathcal{P}(\mathcal{E}^n)$
- Surprisingly $\mathcal{P}(\mathcal{E}^n)$ is always closed; while $\mathcal{P}(\mathcal{S}^n_+)$ closure depends on subgraph/loops/connectedness
- Can exploit the structure of the boundaries
- facial reduction; using cliques is enough for EDM completions in chordal case
- Results are based on May 2014 Research Report:
  "Coordinate shadows of semi-definite and Euclidean distance matrices"
  Dmitriy Drusvyatskiy, Gabor Pataki, Henry Wolkowicz
  http://www.optimization-online.org/DB_HTML/2014/05/4349.html

References I

- A.Y. Alfakih and H. Wolkowicz, *Matrix completion problems*, Handbook of semidefinite programming, Internat. Ser. Oper. Res. Management Sci., vol. 27, Kluwer Acad. Publ., Boston, MA, 2000, pp. 533–545. MR MR1778240
- A. Auslender, *Closedness criteria for the image of a closed set by a linear operator*, Numer. Funct. Anal. Optim. **17** (1996), no. 5-6, 503–515. MR 1404832 (97j:47091)
- M. Bakonyi and C.R. Johnson, *The Euclidean distance matrix completion problem*, SIAM J. Matrix Anal. Appl. **16** (1995), no. 2, 646–654. MR MR1321802 (96a:15025)
- J.M. Borwein and H. Wolkowicz, *Facial reduction for a cone-convex programming problem*, J. Austral. Math. Soc. Ser. A **30** (1980/81), no. 3, 369–380. MR 614086 (83b:90121)

- R.J. Duffin, *Infinite programs*, Linear Equalities and Related Systems (A.W. Tucker, ed.), Princeton University Press, Princeton, NJ, 1956, pp. 157–170.

- R.J. Duffin, R. G. JEROSLOW, and L. A. KARLOVITZ, *Duality in semi-infinite linear programming*, Semi-infinite programming and applications (Austin, Tex., 1981), Lecture Notes in Econom. and Math. Systems, vol. 215, Springer, Berlin, 1983, pp. 50–62.

- J. Gouveia, P.A. Parrilo, and R.R. Thomas, *Lifts of convex sets and cone factorizations*, Math. Oper. Res. **38** (2013), no. 2, 248–264. MR 3062007

- B. Grone, C.R. Johnson, E. Marques de Sa, and H. Wolkowicz, *Positive definite completions of partial Hermitian matrices*, Linear Algebra Appl. **58** (1984), 109–124. MR 85d:05169

References III

- J.W. Helton and J. Nie, *Sufficient and necessary conditions for semidefinite representability of convex hulls and sets*, SIAM J. Optim. **20** (2009), no. 2, 759–791. MR 2515796 (2010i:14105)
- ______, *Semidefinite representation of convex sets*, Math. Program. **122** (2010), no. 1, Ser. A, 21–64. MR 2533752 (2010j:90103)
- N. Krislock and H. Wolkowicz, *Explicit sensor network localization using semidefinite representations and facial reductions*, SIAM J. Optim. **20** (2010), no. 5, 2679–2708. MR 2678410 (2011g:90208)
- M. Laurent, *A connection between positive semidefinite and Euclidean distance matrix completion problems*, Linear Algebra Appl. **273** (1998), 9–22. MR MR1491595 (98m:15039)

- ______, *A tour d'horizon on positive semidefinite and Euclidean distance matrix completion problems*, Topics in semidefinite and interior-point methods (Toronto, ON, 1996), Fields Inst. Commun., vol. 18, Amer. Math. Soc., Providence, RI, 1998, pp. 51–76. MR MR1607310 (99c:05135)

- ______, *Matrix completion problems*, Encyclopedia of Optimization, Springer US, 2001, pp. 1311–1319.

- T. Netzer, *Spectrahedra and their shadows*, Habilitationsschrift, Universität Leipzig, 2012.

- G. Pataki, *On the closedness of the linear image of a closed convex cone*, Math. Oper. Res. **32** (2007), no. 2, 395–412. MR MR2324434

- G. Pataki, *Bad semidefinite programs: they all look the same*, Tech. report, Department of Operations Research, University of North Carolina, Chapel Hill, 2011.

R. T. Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. MR 0274683 (43 #445)

Thanks for your attention!

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