Explicit Sensor Network Localization using Semidefinite Programming and Clique Reductions

Nathan Krislock, Henry Wolkowicz

Department of Combinatorics & Optimization
University of Waterloo

Southern Ontario Numerical Analysis Day
May 8, 2009

Outline

1. Sensor Network Localization (SNL)
   - Introduction
   - Euclidean Distance Matrices and Semidefinite Matrices
2. Clique Reductions of SNL
   - Clique Reductions
   - Completing the EDM
3. Algorithm
   - Clique Unions and Node Absorptions
   - Results

Outline

Outline

1. Sensor Network Localization (SNL)
   - Introduction
   - Euclidean Distance Matrices and Semidefinite Matrices
2. Clique Reductions of SNL
   - Clique Reductions
   - Completing the EDM
3. Algorithm
   - Clique Unions and Node Absorptions
   - Results

Outline

Introduction

Motivation

Many applications use wireless sensor networks:

- natural habitat monitoring, weather monitoring, disaster relief operations, . . .

The Sensor Network Localization (SNL) Problem

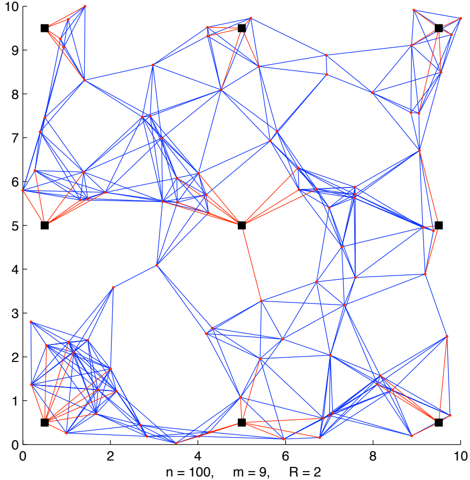
Given:

- Distances between sensors within a fixed radio range
- Positions of some fixed sensors (called anchors)

Goal:

- Determine locations of sensors

Introduction



Introduction

Notation

- $p_1, \ldots, p_{n-m} \in \mathbb{R}^r$ - unknown points (sensors)
- $a_1, \ldots, a_m \in \mathbb{R}^r$ - known points (anchors)
  - anchors also labeled $p_{n-m+1}, \ldots, p_n$

$$P = \begin{bmatrix} p_1^T \\ \vdots \\ p_n^T \end{bmatrix} = \begin{bmatrix} X \\ A \end{bmatrix} \in \mathbb{R}^{n \times r}$$

- r - embedding dimension (usually 2 or 3)
- Assumptions: $n >> m > r$
- $R > 0$ - radio range

Introduction

Graph Realization

- $G = (V, E, w)$ - underlying weighted graph
  - $V = \{1, \dots, n\}$
  - $(i, j) \in E$ if $w_{ij} = \|p_i - p_j\|$ is known
- Anchors form clique
- SNL problem $\equiv$ find realization of graph in $\mathbb{R}^r$

Euclidean Distance Matrix (EDM) Completion

- $D_p \in \mathcal{S}^n$ - partial EDM:

$$(D_p)_{ij} = \begin{cases} \|p_i - p_j\|^2 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- SNL problem $\equiv$ find EDM completion with embed. dim. $= r$

Introduction

Graph Realization

- $G = (V, E, w)$ - underlying weighted graph
  - $V = \{1, \ldots, n\}$
  - $(i, j) \in E$ if $w_{ij} = \|p_i - p_j\|$ is known
- Anchors form clique
- SNL problem $\equiv$ find realization of graph in $\mathbb{R}^r$

Euclidean Distance Matrix (EDM) Completion

- $D_p \in \mathcal{S}^n$ - partial EDM:

$$(D_p)_{ij} = \begin{cases} \|p_i - p_j\|^2 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- SNL problem $\equiv$ find EDM completion with embed. dim. $= r$

Outline

EDMs and Semidefinite Matrices

Linear Transformation $\mathcal{K}$

- If D is an EDM with embed. dim. r given by $P \in \mathbb{R}^{n \times r}$, then:

$$\begin{aligned} D_{ij} = \|p_i - p_j\|^2 &= p_i^T p_i + p_j^T p_j - 2 p_i^T p_j \\ &= \left(\operatorname{diag}(PP^T)e^T + e\operatorname{diag}(PP^T)^T - 2PP^T\right)_{ij} \\ &= \mathcal{K}(PP^T)_{ij} \end{aligned}$$

- Thus $D = \mathcal{K}(Y)$, where:

$$\boxed{\mathcal{K}(Y) := \operatorname{diag}(Y)e^T + e\operatorname{diag}(Y)^T - 2Y} \quad \text{and} \quad Y := PP^T$$

- $Y = PP^T$ is positive semidefinite ($Y \in \mathcal{S}_+^n$ or $Y \succeq 0$), $\operatorname{rank}(Y) = r$
- $\mathcal{K}$ maps the semidefinite cone, $\mathcal{S}_+^n$, onto the EDM cone, $\mathcal{E}^n$

EDMs and Semidefinite Matrices

Linear Transformation $\mathcal{K}$

- If D is an EDM with embed. dim. r given by $P \in \mathbb{R}^{n \times r}$, then:

$$\begin{aligned} D_{ij} = \|p_i - p_j\|^2 &= p_i^T p_i + p_j^T p_j - 2 p_i^T p_j \\ &= \left(\operatorname{diag}(PP^T) e^T + e \operatorname{diag}(PP^T)^T - 2PP^T \right)_{ij} \\ &= \mathcal{K}(PP^T)_{ij} \end{aligned}$$

- Thus $D = \mathcal{K}(Y)$, where:

$$\boxed{\mathcal{K}(Y) := \operatorname{diag}(Y) e^T + e \operatorname{diag}(Y)^T - 2Y} \quad \text{and} \quad Y := PP^T$$

- $Y = PP^T$ is positive semidefinite ($Y \in \mathcal{S}_+^n$ or $Y \succeq 0$), $\operatorname{rank}(Y) = r$
- $\mathcal{K}$ maps the semidefinite cone, $\mathcal{S}_+^n$, onto the EDM cone, $\mathcal{E}^n$

EDMs and Semidefinite Matrices

Linear Transformation $\mathcal{K}$

- If D is an EDM with embed. dim. r given by $P \in \mathbb{R}^{n \times r}$, then:

$$
\begin{aligned}
D_{ij} = \|p_i - p_j\|^2 &= p_i^T p_i + p_j^T p_j - 2 p_i^T p_j \\
&= \left(\operatorname{diag}(PP^T) e^T + e \operatorname{diag}(PP^T)^T - 2PP^T \right)_{ij} \\
&= \mathcal{K}(PP^T)_{ij}
\end{aligned}
$$

- Thus $D = \mathcal{K}(Y)$, where:

$$
\boxed{\mathcal{K}(Y) := \operatorname{diag}(Y) e^T + e \operatorname{diag}(Y)^T - 2Y} \quad \text{and} \quad Y := PP^T
$$

- $Y = PP^T$ is positive semidefinite ($Y \in \mathcal{S}_+^n$ or $Y \succeq 0$), $\operatorname{rank}(Y) = r$
- $\mathcal{K}$ maps the semidefinite cone, $\mathcal{S}_+^n$, onto the EDM cone, $\mathcal{E}^n$

EDMs and Semidefinite Matrices

Properties of $\mathcal{K}$

- Define the centered and hollow subspaces

$$\boxed{\mathcal{S}_C := \{Y \in \mathcal{S}^n : Ye = 0\}} \quad \text{and} \quad \boxed{\mathcal{S}_H := \{D \in \mathcal{S}^n : \operatorname{diag}(D) = 0\}}$$

- $\boxed{\mathcal{K}(Y) = \operatorname{diag}(Y)e^T + e\operatorname{diag}(Y)^T - 2Y} \implies \operatorname{range}(\mathcal{K}) = \mathcal{S}_H$
- For $D \in \mathcal{S}_H$ we have $\boxed{\mathcal{K}^\dagger(D) = -\frac{1}{2}JDJ}$ where $J := I - \frac{1}{n}ee^T$ is the orthogonal projection onto $\{e\}^\perp$
- $\mathcal{K}$ and $\mathcal{K}^\dagger$ are one-to-one and onto:

$$\boxed{\mathcal{K}^\dagger(\mathcal{S}_H) = \mathcal{S}_C} \quad \text{and} \quad \boxed{\mathcal{K}(\mathcal{S}_C) = \mathcal{S}_H}$$

$$\boxed{\mathcal{K}^\dagger(\mathcal{E}^n) = \mathcal{S}_+^n \cap \mathcal{S}_C} \quad \text{and} \quad \boxed{\mathcal{K}(\mathcal{S}_+^n \cap \mathcal{S}_C) = \mathcal{E}^n}$$

EDMs and Semidefinite Matrices

Properties of $\mathcal{K}$

- Define the centered and hollow subspaces

$$\boxed{\mathcal{S}_C := \{Y \in \mathcal{S}^n : Ye = 0\}} \quad \text{and} \quad \boxed{\mathcal{S}_H := \{D \in \mathcal{S}^n : \operatorname{diag}(D) = 0\}}$$

- $\boxed{\mathcal{K}(Y) = \operatorname{diag}(Y)e^T + e\operatorname{diag}(Y)^T - 2Y} \quad \Longrightarrow \quad \operatorname{range}(\mathcal{K}) = \mathcal{S}_H$
- For $D \in \mathcal{S}_H$ we have $\boxed{\mathcal{K}^\dagger(D) = -\frac{1}{2}JDJ}$ where $J := I - \frac{1}{n}ee^T$ is the orthogonal projection onto $\{e\}^\perp$
- $\mathcal{K}$ and $\mathcal{K}^\dagger$ are one-to-one and onto:

$$\boxed{\mathcal{K}^\dagger(\mathcal{S}_H) = \mathcal{S}_C} \quad \text{and} \quad \boxed{\mathcal{K}(\mathcal{S}_C) = \mathcal{S}_H}$$

$$\boxed{\mathcal{K}^\dagger(\mathcal{E}^n) = \mathcal{S}^n_+ \cap \mathcal{S}_C} \quad \text{and} \quad \boxed{\mathcal{K}(\mathcal{S}^n_+ \cap \mathcal{S}_C) = \mathcal{E}^n}$$

Properties of $\mathcal{K}$

- Define the centered and hollow subspaces

$$\mathcal{S}_C := \{Y \in \mathcal{S}^n : Ye = 0\} \quad \text{and} \quad \mathcal{S}_H := \{D \in \mathcal{S}^n : \operatorname{diag}(D) = 0\}$$

- $\mathcal{K}(Y) = \operatorname{diag}(Y)e^T + e\operatorname{diag}(Y)^T - 2Y \quad \Longrightarrow \quad \operatorname{range}(\mathcal{K}) = \mathcal{S}_H$
- For $D \in \mathcal{S}_H$ we have $\mathcal{K}^\dagger(D) = -\frac{1}{2}JDJ$ where $J := I - \frac{1}{n}ee^T$ is the orthogonal projection onto $\{e\}^\perp$
- $\mathcal{K}$ and $\mathcal{K}^\dagger$ are one-to-one and onto:

$$\mathcal{K}^\dagger(\mathcal{S}_H) = \mathcal{S}_C \quad \text{and} \quad \mathcal{K}(\mathcal{S}_C) = \mathcal{S}_H$$

$$\mathcal{K}^\dagger(\mathcal{E}^n) = \mathcal{S}_+^n \cap \mathcal{S}_C \quad \text{and} \quad \mathcal{K}(\mathcal{S}_+^n \cap \mathcal{S}_C) = \mathcal{E}^n$$

EDMs and Semidefinite Matrices

Properties of $\mathcal{K}$

- Define the centered and hollow subspaces

$$\boxed{\mathcal{S}_C := \{Y \in \mathcal{S}^n : Ye = 0\}} \quad \text{and} \quad \boxed{\mathcal{S}_H := \{D \in \mathcal{S}^n : \operatorname{diag}(D) = 0\}}$$

- $\boxed{\mathcal{K}(Y) = \operatorname{diag}(Y)e^T + e\operatorname{diag}(Y)^T - 2Y} \quad \Longrightarrow \quad \operatorname{range}(\mathcal{K}) = \mathcal{S}_H$
- For $D \in \mathcal{S}_H$ we have $\boxed{\mathcal{K}^\dagger(D) = -\frac{1}{2}JDJ}$ where $J := I - \frac{1}{n}ee^T$ is the orthogonal projection onto $\{e\}^\perp$
- $\mathcal{K}$ and $\mathcal{K}^\dagger$ are one-to-one and onto:

$$\boxed{\mathcal{K}^\dagger(\mathcal{S}_H) = \mathcal{S}_C} \quad \text{and} \quad \boxed{\mathcal{K}(\mathcal{S}_C) = \mathcal{S}_H}$$

$$\boxed{\mathcal{K}^\dagger(\mathcal{E}^n) = \mathcal{S}_+^n \cap \mathcal{S}_C} \quad \text{and} \quad \boxed{\mathcal{K}(\mathcal{S}_+^n \cap \mathcal{S}_C) = \mathcal{E}^n}$$

EDMs and Semidefinite Matrices

Vector Formulation

Find $p_1, \ldots, p_n \in \mathbb{R}^r$ such that $\left\{ \begin{array}{ll} \|p_i - p_j\|^2 = (D_p)_{ij}, & \forall (i,j) \in \mathcal{E} \\ \|p_i - p_j\|^2 \geq R^2, & \forall (i,j) \notin \mathcal{E} \end{array} \right\}$

Matrix Formulation

Find $P \in \mathbb{R}^{n \times r}$ such that $\left\{ \begin{array}{l} W \circ \mathcal{K}(Y) = D_p \\ H \circ \mathcal{K}(Y) \geq R^2 \end{array} \right\}$, where $Y = PP^T$

Semidefinite Programming (SDP) Relaxation

Find $Y \in \mathcal{S}_+^n \cap \mathcal{S}_C$ such that $\left\{ \begin{array}{l} W \circ \mathcal{K}(Y) = D_p \\ H \circ \mathcal{K}(Y) \geq R^2 \end{array} \right\}$

- Vector/Matrix Formulation is non-convex and NP-HARD
- SDP Relaxation is convex, but degenerate (strict feasibility fails)

EDMs and Semidefinite Matrices

Vector Formulation

Find $p_1, \ldots, p_n \in \mathbb{R}^r$ such that $\left\{ \begin{array}{ll} \|p_i - p_j\|^2 = (D_p)_{ij}, & \forall (i,j) \in \mathcal{E} \\ \|p_i - p_j\|^2 \geq R^2, & \forall (i,j) \notin \mathcal{E} \end{array} \right\}$

Matrix Formulation

Find $P \in \mathbb{R}^{n \times r}$ such that $\left\{ \begin{array}{l} W \circ \mathcal{K}(Y) = D_p \\ H \circ \mathcal{K}(Y) \geq R^2 \end{array} \right\}$, where $Y = PP^T$

Semidefinite Programming (SDP) Relaxation

Find $Y \in \mathcal{S}^n_+ \cap \mathcal{S}_C$ such that $\left\{ \begin{array}{l} W \circ \mathcal{K}(Y) = D_p \\ H \circ \mathcal{K}(Y) \geq R^2 \end{array} \right\}$

- Vector/Matrix Formulation is non-convex and NP-HARD
- SDP Relaxation is convex, but degenerate (strict feasibility fails)

EDMs and Semidefinite Matrices

Vector Formulation

Find $p_1, \ldots, p_n \in \mathbb{R}^r$ such that $\left\{ \begin{array}{ll} \|p_i - p_j\|^2 = (D_p)_{ij}, & \forall (i,j) \in \mathcal{E} \\ \|p_i - p_j\|^2 \geq R^2, & \forall (i,j) \notin \mathcal{E} \end{array} \right\}$

Matrix Formulation

Find $P \in \mathbb{R}^{n \times r}$ such that $\left\{ \begin{array}{l} W \circ \mathcal{K}(Y) = D_p \\ H \circ \mathcal{K}(Y) \geq R^2 \end{array} \right\}$, where $Y = PP^T$

Semidefinite Programming (SDP) Relaxation

Find $Y \in \mathcal{S}_+^n \cap \mathcal{S}_C$ such that $\left\{ \begin{array}{l} W \circ \mathcal{K}(Y) = D_p \\ H \circ \mathcal{K}(Y) \geq R^2 \end{array} \right\}$

- Vector/Matrix Formulation is non-convex and NP-HARD
- SDP Relaxation is convex, but degenerate (strict feasibility fails)

EDMs and Semidefinite Matrices

Vector Formulation

Find $p_1, \ldots, p_n \in \mathbb{R}^r$ such that $\left\{ \begin{array}{ll} \|p_i - p_j\|^2 = (D_p)_{ij}, & \forall (i,j) \in \mathcal{E} \\ \|p_i - p_j\|^2 \geq R^2, & \forall (i,j) \notin \mathcal{E} \end{array} \right\}$

Matrix Formulation

Find $P \in \mathbb{R}^{n \times r}$ such that $\left\{ \begin{array}{l} W \circ \mathcal{K}(Y) = D_p \\ H \circ \mathcal{K}(Y) \geq R^2 \end{array} \right\}$, where $Y = PP^T$

Semidefinite Programming (SDP) Relaxation

Find $Y \in \mathcal{S}^n_+ \cap \mathcal{S}_C$ such that $\left\{ \begin{array}{l} W \circ \mathcal{K}(Y) = D_p \\ H \circ \mathcal{K}(Y) \geq R^2 \end{array} \right\}$

- Vector/Matrix Formulation is non-convex and NP-HARD
- SDP Relaxation is convex, but degenerate (strict feasibility fails)

EDMs and Semidefinite Matrices

Faces of the Semidefinite Cone

- A cone $\mathcal{F} \subseteq \mathcal{S}^n_+$ is a face of $\mathcal{S}^n_+$ (denoted $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$) if

$$X, Y \in \mathcal{S}^n_+ \quad \text{and} \quad \frac{1}{2}(X+Y) \in \mathcal{F} \quad \implies \quad X, Y \in \mathcal{F}$$

- If $S \subseteq \mathcal{S}^n_+$, then $\mathrm{face}(S)$ is the smallest face of $\mathcal{S}^n_+$ containing S

Representing Faces of $\mathcal{S}^n_+$

If $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ and $X \in \mathrm{relint}(\mathcal{F})$ with $\mathrm{rank}(X) = t$, then

$$\mathcal{F} = U\mathcal{S}^t_+U^T$$

where $X = U\Lambda U^T$ is the compact eigenvalue decomp. with $U \in \mathbb{R}^{n \times t}$

EDMs and Semidefinite Matrices

Faces of the Semidefinite Cone

- A cone $\mathcal{F} \subseteq \mathcal{S}^n_+$ is a face of $\mathcal{S}^n_+$ (denoted $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$) if

$$X, Y \in \mathcal{S}^n_+ \quad \text{and} \quad \frac{1}{2}(X+Y) \in \mathcal{F} \quad \implies \quad X, Y \in \mathcal{F}$$

- If $S \subseteq \mathcal{S}^n_+$, then $\mathrm{face}(S)$ is the smallest face of $\mathcal{S}^n_+$ containing S

Representing Faces of $\mathcal{S}^n_+$

If $\mathcal{F} \trianglelefteq \mathcal{S}^n_+$ and $X \in \mathrm{relint}(\mathcal{F})$ with $\mathrm{rank}(X) = t$, then

$$\mathcal{F} = U \mathcal{S}^t_+ U^T$$

where $X = U \Lambda U^T$ is the compact eigenvalue decomp. with $U \in \mathbb{R}^{n \times t}$

Outline

Clique Reductions

Theorem: Single Clique Reduction

Let:

- D_p be a partial EDM such that

$$D_p = \left[\begin{array}{c|c} \bar{D} & \cdot \\ \hline \cdot & \cdot \end{array}\right], \quad \text{for some } \bar{D} \in \mathcal{E}^k \text{ with embed. dim. } t \le r$$

- $F := \left\{ Y \in \mathcal{S}_+^n \cap \mathcal{S}_C : \mathcal{K}(Y[1:k]) = \bar{D} \right\}$ (contains SDP feas. set)
- $B := \mathcal{K}^\dagger(\bar{D})$ has eigenvectors $\bar{U} \in \mathbb{R}^{k \times t}$ (Note: $\operatorname{rank}(B) = t$)
- $U := \left[\begin{array}{cc|c} \bar{U} & \frac{1}{\sqrt{k}}e & 0 \\ \hline 0 & 0 & I_{n-k} \end{array}\right]$ and $\left[V \quad \frac{U^T e}{\|U^T e\|} \right]$ be orthogonal

Then: $\operatorname{face}(F) = \left(U \mathcal{S}_+^{n-k+t+1} U^T \right) \cap \mathcal{S}_C = (UV) \mathcal{S}_+^{n-k+t} (UV)^T$

Clique Reductions

Theorem: Single Clique Reduction

Let:

- D_p be a partial EDM such that

$$D_p = \left[\begin{array}{c|c} \bar{D} & \cdot \\ \hline \cdot & \cdot \end{array}\right], \quad \text{for some } \bar{D} \in \mathcal{E}^k \text{ with embed. dim. } t \leq r$$

- $F := \left\{ Y \in \mathcal{S}_+^n \cap \mathcal{S}_C : \mathcal{K}(Y[1:k]) = \bar{D} \right\}$ (contains SDP feas. set)
- $B := \mathcal{K}^\dagger(\bar{D})$ has eigenvectors $\bar{U} \in \mathbb{R}^{k \times t}$ (Note: $\operatorname{rank}(B) = t$)
- $U := \left[\begin{array}{cc|c} \bar{U} & \frac{1}{\sqrt{k}} e & 0 \\ \hline 0 & 0 & I_{n-k} \end{array}\right]$ and $\left[V \quad \frac{U^T e}{\|U^T e\|} \right]$ be orthogonal

Then: $\operatorname{face}(F) = \left(U \mathcal{S}_+^{n-k+t+1} U^T \right) \cap \mathcal{S}_C = (UV) \mathcal{S}_+^{n-k+t} (UV)^T$

Clique Reductions

Theorem: Single Clique Reduction

Let:

- D_p be a partial EDM such that

$$D_p = \left[\begin{array}{c|c} \bar{D} & \cdot \\ \hline \cdot & \cdot \end{array}\right], \quad \text{for some } \bar{D} \in \mathcal{E}^k \text{ with embed. dim. } t \leq r$$

- $F := \left\{ Y \in \mathcal{S}^n_+ \cap \mathcal{S}_C : \mathcal{K}(Y[1:k]) = \bar{D} \right\}$ (contains SDP feas. set)
- $B := \mathcal{K}^\dagger(\bar{D})$ has eigenvectors $\bar{U} \in \mathbb{R}^{k \times t}$ (Note: $\operatorname{rank}(B) = t$)
- $U := \left[\begin{array}{cc|c} \bar{U} & \frac{1}{\sqrt{k}} e & 0 \\ \hline 0 & 0 & I_{n-k} \end{array}\right]$ and $\left[V \quad \frac{U^T e}{\|U^T e\|} \right]$ be orthogonal

Then: $$\operatorname{face}(F) = \left(U \mathcal{S}^{n-k+t+1}_+ U^T \right) \cap \mathcal{S}_C = (UV) \mathcal{S}^{n-k+t}_+ (UV)^T$$

Clique Reductions

Corollary: Two Clique Reduction

Let $D \in \mathcal{E}^n$ with embed. dim. r. Let $\alpha_1, \alpha_2 \subseteq 1:n$ and $k := |\alpha_1 \cup \alpha_2|$.

For $i = 1, 2$ let:

- $t_i :=$ embed. dim. of $D[\alpha_i] \in \mathcal{E}^{k_i}$
- $F_i := \left\{ Y \in \mathcal{S}_+^n \cap \mathcal{S}_C : \mathcal{K}(Y[\alpha_i]) = D[\alpha_i] \right\}$ (contains SDP feas. set)
- $\mathrm{face}(F_i) =: \left(U_i \mathcal{S}_+^{n-k_i+t_i+1} U_i^T \right) \cap \mathcal{S}_C$

Let:

- $U \in \mathbb{R}^{n \times t}$ full column rank s.t. $\mathrm{col}(U) = \mathrm{col}(U_1) \cap \mathrm{col}(U_2)$
- $\begin{bmatrix} V & \frac{U^T e}{\|U^T e\|} \end{bmatrix}$ be orthogonal

Then: $\boxed{\mathrm{face}(F_1 \cap F_2) = \left(U \mathcal{S}_+^{n-k+t+1} U^T \right) \cap \mathcal{S}_C = (UV) \mathcal{S}_+^{n-k+t} (UV)^T}$

Clique Reductions

Corollary: Two Clique Reduction

Let $D \in \mathcal{E}^n$ with embed. dim. r. Let $\alpha_1, \alpha_2 \subseteq 1\!:\!n$ and $k := |\alpha_1 \cup \alpha_2|$.
For $i = 1, 2$ let:

- $t_i :=$ embed. dim. of $D[\alpha_i] \in \mathcal{E}^{k_i}$
- $F_i := \left\{ Y \in \mathcal{S}_+^n \cap \mathcal{S}_C : \mathcal{K}(Y[\alpha_i]) = D[\alpha_i] \right\}$ (contains SDP feas. set)
- $\text{face}(F_i) =: \left(U_i \mathcal{S}_+^{n-k_i+t_i+1} U_i^T \right) \cap \mathcal{S}_C$

Let:

- $U \in \mathbb{R}^{n \times t}$ full column rank s.t. $\text{col}(U) = \text{col}(U_1) \cap \text{col}(U_2)$
- $\begin{bmatrix} V & \frac{U^T e}{\|U^T e\|} \end{bmatrix}$ be orthogonal

Then: $\boxed{\text{face}(F_1 \cap F_2) = \left(U \mathcal{S}_+^{n-k+t+1} U^T \right) \cap \mathcal{S}_C = (UV) \mathcal{S}_+^{n-k+t} (UV)^T}$

Clique Reductions

Corollary: Two Clique Reduction

Let $D \in \mathcal{E}^n$ with embed. dim. r. Let $\alpha_1, \alpha_2 \subseteq 1:n$ and $k := |\alpha_1 \cup \alpha_2|$.
For $i = 1, 2$ let:

- $t_i :=$ embed. dim. of $D[\alpha_i] \in \mathcal{E}^{k_i}$
- $F_i := \left\{ Y \in \mathcal{S}^n_+ \cap \mathcal{S}_C : \mathcal{K}(Y[\alpha_i]) = D[\alpha_i] \right\}$ (contains SDP feas. set)
- $\mathrm{face}(F_i) =: \left(U_i \mathcal{S}^{n-k_i+t_i+1}_+ U_i^T \right) \cap \mathcal{S}_C$

Let:

- $U \in \mathbb{R}^{n \times t}$ full column rank s.t. $\mathrm{col}(U) = \mathrm{col}(U_1) \cap \mathrm{col}(U_2)$
- $\begin{bmatrix} V & \frac{U^T e}{\|U^T e\|} \end{bmatrix}$ be orthogonal

Then: $\mathrm{face}(F_1 \cap F_2) = \left(U \mathcal{S}^{n-k+t+1}_+ U^T \right) \cap \mathcal{S}_C = (UV) \mathcal{S}^{n-k+t}_+ (UV)^T$

Clique Reductions

Corollary: Two Clique Reduction

Let $D \in \mathcal{E}^n$ with embed. dim. r. Let $\alpha_1, \alpha_2 \subseteq 1:n$ and $k := |\alpha_1 \cup \alpha_2|$.
For $i = 1, 2$ let:

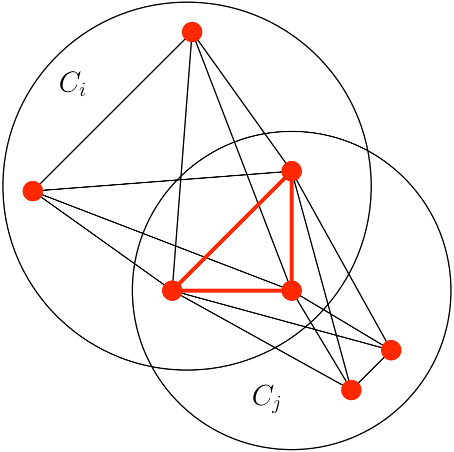
- $t_i :=$ embed. dim. of $D[\alpha_i] \in \mathcal{E}^{k_i}$
- $F_i := \left\{ Y \in \mathcal{S}_+^n \cap \mathcal{S}_C : \mathcal{K}(Y[\alpha_i]) = D[\alpha_i] \right\}$ (contains SDP feas. set)
- $\mathrm{face}(F_i) =: \left(U_i \mathcal{S}_+^{n-k_i+t_i+1} U_i^T \right) \cap \mathcal{S}_C$

Let:

- $U \in \mathbb{R}^{n \times t}$ full column rank s.t. $\mathrm{col}(U) = \mathrm{col}(U_1) \cap \mathrm{col}(U_2)$
- $\begin{bmatrix} V & \frac{U^T e}{\|U^T e\|} \end{bmatrix}$ be orthogonal

Then: $\boxed{\mathrm{face}(F_1 \cap F_2) = \left(U \mathcal{S}_+^{n-k+t+1} U^T \right) \cap \mathcal{S}_C = (UV)\mathcal{S}_+^{n-k+t}(UV)^T}$

Clique Reductions



Clique Reductions

Subspace Intersection for Two Intersecting Cliques

Suppose:

$$U_1 = \begin{bmatrix} U_1' & 0 \\ U_1'' & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} I & 0 \\ 0 & U_2'' \\ 0 & U_2' \end{bmatrix}$$

Then:

$$U := \begin{bmatrix} U_1' \\ U_1'' \\ U_2'(U_2'')^\dagger U_1'' \end{bmatrix} \quad \text{or} \quad U := \begin{bmatrix} U_1'(U_1'')^\dagger U_2'' \\ U_2'' \\ U_2' \end{bmatrix}$$

Satisfies:

$$\mathrm{col}(U) = \mathrm{col}(U_1) \cap \mathrm{col}(U_2)$$

Outline

Completing the EDM

Completing the EDM and Finding Positions

Let:

- $D \in \mathcal{E}^n$ with embed. dim. r
- $D_p := W \circ D$ be a partial EDM (for some 0–1 matrix W)
- $F := \left\{ Y \in \mathcal{S}^n_+ \cap \mathcal{S}_C : W \circ \mathcal{K}(Y) = D_p \right\}$
- $\text{face}(F) =: (UV)\mathcal{S}^r_+(UV)^T$

If $D_p[\beta]$ is complete with embed. dim. r then:

- $(JU[\beta,:]V)Z(JU[\beta,:]V)^T = \mathcal{K}^\dagger(D_p[\beta])$ has a unique solution Z
- $D = \mathcal{K}\left(PP^T\right)$ where $\boxed{P := UVZ^{\frac{1}{2}} \in \mathbb{R}^{n \times r}}$

Completing the EDM

Completing the EDM and Finding Positions

Let:

- $D \in \mathcal{E}^n$ with embed. dim. r
- $D_p := W \circ D$ be a partial EDM (for some 0–1 matrix W)
- $F := \left\{ Y \in \mathcal{S}^n_+ \cap \mathcal{S}_C : W \circ \mathcal{K}(Y) = D_p \right\}$
- $\text{face}(F) =: (UV)\mathcal{S}^r_+(UV)^T$

If $D_p[\beta]$ is complete with embed. dim. r then:

- $(JU[\beta,:]V)Z(JU[\beta,:]V)^T = \mathcal{K}^\dagger(D_p[\beta])$ has a unique solution Z
- $D = \mathcal{K}\left(PP^T\right)$ where $\boxed{P := UVZ^{\frac{1}{2}} \in \mathbb{R}^{n \times r}}$

Completing the EDM

Rotate to Align the Anchor Positions

- Given $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{R}^{n \times r}$ such that $D = \mathcal{K}\left(PP^T\right)$
- Solve the orthogonal Procrustes problem:

$$\boxed{\begin{array}{ll} \min & \|A - P_2 Q\| \\ \text{s.t.} & Q^T Q = I \end{array}}$$

  using SVD (Golub/Van Loan, Algorithm 12.4.1)
- Set $X := P_1 Q$

Outline

Algorithm

Initialize

$$C_i := \left\{ j : (D_p)_{ij} < (R/2)^2 \right\}, \quad \text{for } i = 1, \ldots, n$$

Iterate

- For $|C_i \cap C_j| \geq r + 1$, do Rigid Clique Union
- For $|C_i \cap \mathcal{N}(j)| \geq r + 1$, do Rigid Node Absorption
- For $|C_i \cap C_j| = r$, do Non-Rigid Clique Union (lower bounds)
- For $|C_i \cap \mathcal{N}(j)| = r$, do Non-Rigid Node Absorption (lower bounds)

Finalize

When there is a clique containing all the anchors, use the computed facial representation and the positions of the anchors to solve for X

Algorithm

Initialize

$$C_i := \left\{ j : (D_p)_{ij} < (R/2)^2 \right\}, \quad \text{for } i = 1, \ldots, n$$

Iterate

- For $|C_i \cap C_j| \geq r + 1$, do Rigid Clique Union
- For $|C_i \cap \mathcal{N}(j)| \geq r + 1$, do Rigid Node Absorption
- For $|C_i \cap C_j| = r$, do Non-Rigid Clique Union (lower bounds)
- For $|C_i \cap \mathcal{N}(j)| = r$, do Non-Rigid Node Absorption (lower bounds)

Finalize

When there is a clique containing all the anchors, use the computed facial representation and the positions of the anchors to solve for X

Algorithm

Initialize

$$C_i := \left\{ j : (D_p)_{ij} < (R/2)^2 \right\}, \quad \text{for } i = 1, \ldots, n$$

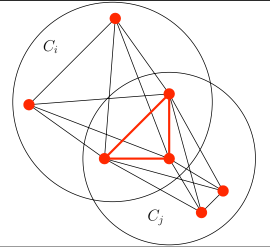
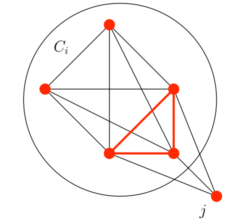
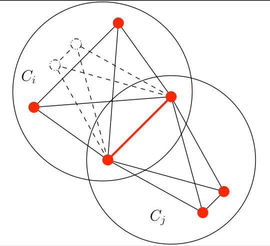
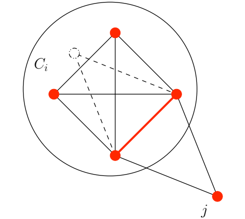
Iterate

- For $|C_i \cap C_j| \geq r + 1$, do Rigid Clique Union
- For $|C_i \cap \mathcal{N}(j)| \geq r + 1$, do Rigid Node Absorption
- For $|C_i \cap C_j| = r$, do Non-Rigid Clique Union (lower bounds)
- For $|C_i \cap \mathcal{N}(j)| = r$, do Non-Rigid Node Absorption (lower bounds)

Finalize

When there is a clique containing all the anchors, use the computed facial representation and the positions of the anchors to solve for X

Algorithm

	Clique Union	Node Absorption
Rigid		
Non-rigid		

Outline

Results

Rigid Clique Union

n / R	0.7	0.6	0.5	0.4
2000	1	7	91	362
4000	1	1	1	16
6000	1	1	1	1
8000	1	1	1	1
10000	1	1	1	1

Remaining Cliques

n / R	0.7	0.6	0.5	0.4
2000	4.8	4.6	4.2	4.1
4000	9.2	9.4	9.1	9.2
6000	16.0	14.7	15.3	14.9
8000	22.9	22.5	20.9	21.0
10000	38.3	32.7	29.1	30.7

CPU Seconds

n / R	0.7	0.6	0.5	0.4
2000	−10.1	−10.8	−	−
4000	−10.9	−11.0	−10.5	−9.6
6000	−11.6	−10.7	−10.6	−10.0
8000	−11.1	−11.0	−10.7	−9.2
10000	−11.0	−11.0	−10.2	−10.4

Max log(Error)

Rigid Clique Union and Node Absorption

n / R	0.7	0.6	0.5	0.4
2000	1	1	2	78
4000	1	1	1	1
6000	1	1	1	1
8000	1	1	1	1
10000	1	1	1	1

Remaining Cliques

n / R	0.7	0.6	0.5	0.4
2000	4.9	4.9	6.1	13.2
4000	9.2	9.5	9.1	9.8
6000	16.1	15.1	15.1	14.8
8000	22.7	22.4	21.0	21.3
10000	32.5	32.4	28.8	30.6

CPU Seconds

n / R	0.7	0.6	0.5	0.4
2000	−10.1	−10.8	−9.8	−8.8
4000	−10.9	−11.0	−10.5	−9.6
6000	−11.6	−10.7	−10.6	−10.0
8000	−11.1	−11.0	−10.7	−9.2
10000	−11.0	−11.0	−10.2	−10.4

Max log(Error)

Summary

- SDP relaxation of SNL is highly degenerate: The feasible set of this SDP is restricted to a low dimensional face of the SDP cone, causing the Slater constraint qualification (strict feasibility) to fail
- We take advantage of this degeneracy by finding explicit representations of the faces of the SDP cone corresponding to unions of intersecting cliques
- Without using an SDP-solver (eg. SeDuMi or SDPT3), we quickly compute the exact solution to the SDP relaxation (except for round-off error from computing eigenvectors, etc.)

Summary

- SDP relaxation of SNL is highly degenerate: The feasible set of this SDP is restricted to a low dimensional face of the SDP cone, causing the Slater constraint qualification (strict feasibility) to fail
- We take advantage of this degeneracy by finding explicit representations of the faces of the SDP cone corresponding to unions of intersecting cliques
- Without using an SDP-solver (eg. SeDuMi or SDPT3), we quickly compute the exact solution to the SDP relaxation (except for round-off error from computing eigenvectors, etc.)

Summary

- SDP relaxation of SNL is highly degenerate: The feasible set of this SDP is restricted to a low dimensional face of the SDP cone, causing the Slater constraint qualification (strict feasibility) to fail
- We take advantage of this degeneracy by finding explicit representations of the faces of the SDP cone corresponding to unions of intersecting cliques
- Without using an SDP-solver (eg. SeDuMi or SDPT3), we quickly compute the exact solution to the SDP relaxation (except for round-off error from computing eigenvectors, etc.)