Explicit Sensor Network Localization using Semidefinite Programming and Facial Reduction

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Outline

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1. Preliminaries
   - SNL <–> GR <–> EDM <–> SDP
   - Further Notation/Preliminaries; Facial Structure of Cones
2. Clique/Facial Reduction (Exploit degeneracy)
   - Basic Single Clique Reduction
   - Two Clique Reduction and EDM DELAYED Completion
   - Completing SNL; DELAYED use of Anchor Locations
3. Algorithm: Facial Reduct. via Subsp. Inters./DELAYED Compl.
   - Clique Unions and Node Absorptions
   - Results (low CPU time; high accuracy)
4. Noisy Data

Outline

Sensor Network Localization, SNL, Problem

SNL - a Fundamental Problem of Distance Geometry; easy to describe - dates back to Grasssmann 1886

- n ad hoc wireless sensors (nodes) to locate in $\mathbb{R}^r$,
  (r is embedding dimension;
  sensors $p_i \in \mathbb{R}^r, i \in V := 1, \ldots, n$)
- m of the sensors are anchors, $p_i, i = n - m + 1, \ldots, n$)
  (positions known, using e.g. GPS)
- pairwise distances $D_{ij} = \|p_i - p_j\|^2, ij \in E$, are known within radio range $R > 0$
- $$P = \begin{bmatrix} p_1^T \\ \vdots \\ p_n^T \end{bmatrix} = \begin{bmatrix} X \\ A \end{bmatrix} \in \mathbb{R}^{n \times r}$$

Applications

Horst Stormer (Nobel Prize, Physics, 1949), “21 Ideas for the 21st Century”, Business Week. 8/23-30, 1999

Untethered micro sensors will go anywhere and measure anything - traffic flow, water level, number of people walking by, temperature. This is developing into something like a nervous system for the earth, a skin for the earth. The world will evolve this way.

Tracking Humans/Animals/Equipment/Weather (smart dust)

- geographic routing; data aggregation; topological control; soil humidity; earthquakes and volcanos; weather and ocean currents.
- military; tracking of goods; vehicle positions; surveillance; random deployment in inaccessible terrains.

Conferences/Journals/Research Groups/Books/Theses/Codes

- Conference, MELT 2008
- International Journal of Sensor Networks
- Research groups include: CENS at UCLA, Berkeley WEBS,
- recent related theses and books include: [10, 16, 8, 7, 11, 12, 6, 14, 17]
- recent algorithms specific for SNL: [1, 2, 3, 4, 5, 9, 15, 18, 13]

Underlying Graph Realization/Partial EDM NP-Hard

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

- node set $\mathcal{V} = \{1, \ldots, n\}$
- edge set $(i,j) \in \mathcal{E}$; $\omega_{ij} = \|p_i - p_j\|^2$ known approximately
- The anchors form a clique (complete subgraph)
- Realization of $\mathcal{G}$ in $\Re^r$: a mapping of node $v_i \rightarrow p_i \in \Re^r$ with squared distances given by ω.

Corresponding Partial Euclidean Distance Matrix, EDM

$$D_{ij} = \begin{cases} d_{ij}^2 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise (unknown distance),} \end{cases}$$

$d_{ij}^2 = \omega_{ij}$ are known squared Euclidean distances between sensors p_i, p_j; anchors correspond to a clique.

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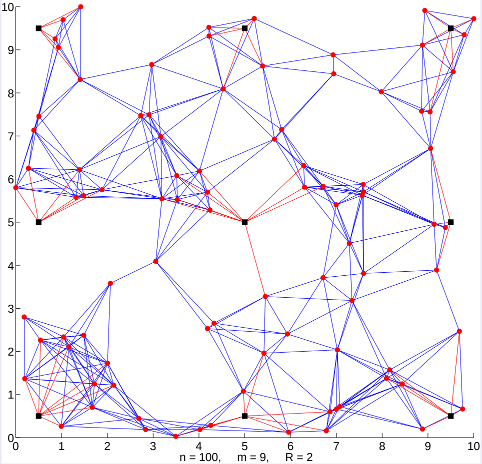
Sensor Localization Problem/Partial EDM

Sensors ○ and Anchors ■



Connections to Semidefinite Programming (SDP)

$\mathcal{S}^n_+$, Cone of (symmetric) SDP matrices in $\mathcal{S}^n$; $x^TAx \geq 0$

inner product $\langle A, B \rangle = \operatorname{trace} AB$
Löwner (psd) partial order $A \succeq B, A \succ B$

$D = \mathcal{K}(B) \in \mathcal{E}^n$, $B = \mathcal{K}^\dagger(D) \in \mathcal{S}^n \cap \mathcal{S}_C$ (centered $Be = 0$)

$P^T = \begin{bmatrix} p_1 & p_2 & \dots & p_n \end{bmatrix} \in \mathcal{M}^{r \times n}$; $B := PP^T \in \mathcal{S}^n_+$;
$\operatorname{rank} B = r$; $D \in \mathcal{E}^n$ be corresponding EDM.

$$
\begin{array}{rcl}
(\text{to } D \in \mathcal{E}^n) \quad D & = & \left(\|p_i - p_j\|_2^2\right)_{i,j=1}^n \\
& = & \left(p_i^Tp_i + p_j^Tp_j - 2p_i^Tp_j\right)_{i,j=1}^n \\
& = & \boxed{\operatorname{diag}(B)\,e^T + e\,\operatorname{diag}(B)^T - 2B} \\
& =: & \mathcal{D}_e(B) - 2B \\
& =: & \mathcal{K}(B) \quad (\text{from } B \in \mathcal{S}^n_+).
\end{array}
$$

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(to $D \in \mathcal{E}^n$)

$$\begin{aligned} D &= \left(\|p_i - p_j\|_2^2\right)_{i,j=1}^n \\ &= \left(p_i^T p_i + p_j^T p_j - 2p_i^T p_j\right)_{i,j=1}^n \\ &= \boxed{\operatorname{diag}(B)e^T + e\operatorname{diag}(B)^T - 2B} \\ &=: \mathcal{D}_e(B) - 2B \\ &=: \mathcal{K}(B) \quad (\text{from } B \in \mathcal{S}^n_+). \end{aligned}$$

Current Techniques; SDP Relax.; Highly Degen.

Nearest, Weighted, SDP Approx. (relax $\operatorname{rank} B$)

- $\min_{B \succeq 0, B \in \Omega} \|H \circ (\mathcal{K}(B) - D)\|$; $\operatorname{rank} B = r$;
  typical weights: $H_{ij} = 1/\sqrt{D_{ij}}$, if $ij \in E$.
- with rank constraint: a non-convex, NP-hard program
- SDP relaxation is convex, BUT: expensive/low accuracy/implicitly highly degenerate (cliques restrict ranks of feasible Bs)

Instead: (Shall) Take Advantage of Degeneracy!

clique α, $|\alpha| = k$ (corresp. $D[\alpha]$) with embed. dim. $= t \leq r < k$
$\implies \operatorname{rank} \mathcal{K}^\dagger(D[\alpha]) = t \leq r \implies \operatorname{rank} B[\alpha] \leq \operatorname{rank} \mathcal{K}^\dagger(D[\alpha]) + 1$
$\implies \operatorname{rank} B = \operatorname{rank} \mathcal{K}^\dagger(D) \leq n - \boxed{(k - t - 1)} \implies$
Slater's CQ (strict feasibility) fails

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Linear Transformations: $\mathcal{D}_v(B), \mathcal{K}(B), \mathcal{T}(D)$

- allow: $\mathcal{D}_v(B) := \operatorname{diag}(B)\, v^T + v \operatorname{diag}(B)^T$;
  $\mathcal{D}_v(y) := yv^T + vy^T$
- adjoint $\mathcal{K}^*(D) = 2(\mathrm{Diag}\,(De) - D)$.
- $\mathcal{K}$ is $1-1$, onto between centered & hollow subspaces:
  $\mathcal{S}_C := \{B \in \mathcal{S}^n : Be = 0\}$;
  $\mathcal{S}_H := \{D \in \mathcal{S}^n : \operatorname{diag}(D) = 0\} = \mathcal{R}(\mathrm{offDiag}\,)$
- $J := I - \frac{1}{n}ee^T$ (orthogonal projection onto $M := \{e\}^\perp$);
- $\mathcal{T}(D) := -\frac{1}{2} J \mathrm{offDiag}\,(D) J \qquad (= \mathcal{K}^\dagger(D))$

Properties of Linear Transformations

$\mathcal{K}, \mathcal{T}, \mathrm{Diag}, \mathcal{D}_e$

$$\mathcal{R}(\mathcal{K}) = \mathcal{S}_H; \qquad \underline{\mathcal{N}(\mathcal{K}) = \mathcal{R}(\mathcal{D}_e)};$$

$$\mathcal{R}(\mathcal{K}^*) = \mathcal{R}(\mathcal{T}) = \mathcal{S}_C; \qquad \mathcal{N}(\mathcal{K}^*) = \mathcal{N}(\mathcal{T}) = \mathcal{R}(\mathrm{Diag});$$

$$\mathcal{S}^n = \mathcal{S}_H \oplus \mathcal{R}(\mathrm{Diag}) = \mathcal{S}_C \oplus \mathcal{R}(\mathcal{D}_e).$$

$\mathcal{T}(\mathcal{E}^n) = \mathcal{S}^n_+ \cap \mathcal{S}_C$ and $\mathcal{K}(\mathcal{S}^n_+ \cap \mathcal{S}_C) = \mathcal{E}^n.$

Semidefinite Cone, Faces

Faces of cone K

- $F \subseteq K$ is a face of K, denoted $\boxed{F \trianglelefteq K}$, if
  $\left(x, y \in K, \ \frac{1}{2}(x+y) \in F\right) \implies \left(\operatorname{cone}\{x, y\} \subseteq F\right).$
- $\boxed{F \lhd K}$, if $F \trianglelefteq K, F \neq K$; F is $\boxed{\text{proper face}}$ if $\{0\} \neq F \lhd K$.
- $F \trianglelefteq K$ is $\boxed{\text{exposed}}$ if: intersection of K with a hyperplane.
- $\boxed{\operatorname{face}(S)}$ denotes smallest face of K that contains set S.

$\mathcal{S}^n_+$ is a Facially Exposed Cone

All faces are exposed.

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Facial Structure of SDP Cone; Equivalent SUBSPACES

Face $F \trianglelefteq \mathcal{S}^n_+$ Equivalence to $\mathcal{R}(U)$ Subspace of $\mathbb{R}^n$

$F \trianglelefteq \mathcal{S}^n_+$ determined by range of any $S \in \operatorname{relint} F$,
i.e. let $S = U\Gamma U^T$ be compact spectral decomposition; $\Gamma \in \mathcal{S}^t_{++}$
is diagonal matrix of pos. eigenvalues; $\boxed{F = U\mathcal{S}^t_+ U^T}$
(F associated with $\mathcal{R}(U)$)

$$\dim F = t(t+1)/2.$$

face F representation by subspace $\mathcal{L}$

(subspace) $\mathcal{L} = \mathcal{R}(T)$, T is $n \times t$ full column, then:

$$F := T\mathcal{S}^t_+ T^T \trianglelefteq \mathcal{S}^n_+$$

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Further Notation

Matrix with Fixed Principal Submatrix

For $Y \in \mathcal{S}^n$, $\alpha \subseteq \{1, \ldots, n\}$: $Y[\alpha]$ denotes principal submatrix formed from rows & cols with indices α.

Sets with Fixed Principal Submatrices

If $|\alpha| = k$ and $\bar{Y} \in \mathcal{S}^k$, then:

- $\mathcal{S}^n(\alpha, \bar{Y}) := \{Y \in \mathcal{S}^n : Y[\alpha] = \bar{Y}\}$,
- $\mathcal{S}^n_+(\alpha, \bar{Y}) := \{Y \in \mathcal{S}^n_+ : Y[\alpha] = \bar{Y}\}$
  i.e. the subset of matrices $Y \in \mathcal{S}^n$ ($Y \in \mathcal{S}^n_+$) with principal submatrix $Y[\alpha]$ fixed to $\bar{Y}$.

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Basic Single Clique/Facial Reduction

$\bar{D} \in \mathcal{E}^k$, $\alpha \subseteq 1{:}n$, $|\alpha| = k$

Define $\mathcal{E}^n(\alpha, \bar{D}) := \left\{D \in \mathcal{E}^n : D[\alpha] = \bar{D}\right\}$.

Given $\bar{D}$; find a corresponding $B \succeq 0$; find the corresponding face; find the corresponding subspace.

if $\alpha = 1:k$; embed. dim of $\bar{D}$ is $t \leq r$

$$D = \begin{bmatrix} \bar{D} & \cdot \\ \cdot & \cdot \end{bmatrix},$$

BASIC THEOREM for Single Clique/Facial Reduction

THEOREM 1: Single Clique/Facial Reduction

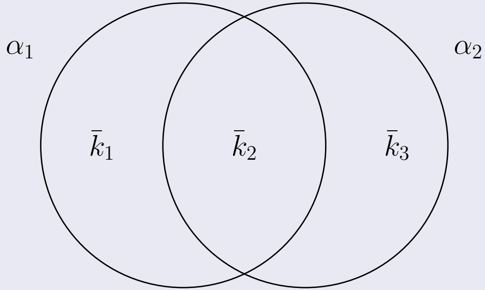
Let: $\bar{D} := D[1:k] \in \mathcal{E}^k$, $k < n$, with embedding dimension $t \leq r$; $B := \mathcal{K}^\dagger(\bar{D}) = \bar{U}_B S \bar{U}_B^T$, $\bar{U}_B \in \mathcal{M}^{k\times t}$, $\bar{U}_B^T \bar{U}_B = I_t$, $S \in \mathcal{S}^t_{++}$.
Furthermore, let $U_B := \begin{bmatrix} \bar{U}_B & \frac{1}{\sqrt{k}} e \end{bmatrix} \in \mathcal{M}^{k\times(t+1)}$,
$U := \begin{bmatrix} U_B & 0 \\ 0 & I_{n-k} \end{bmatrix}$, and let $\begin{bmatrix} V & \frac{U^T e}{\|U^T e\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal. Then:

$$\begin{aligned} \text{face}\, \mathcal{K}^\dagger\left(\mathcal{E}^n(1:k, \bar{D})\right) &= \left(U \mathcal{S}^{n-k+t+1}_+ U^T\right) \cap \mathcal{S}_C \\ &= (UV)\mathcal{S}^{n-k+t}_+ (UV)^T \end{aligned}$$

Note that we add $\frac{1}{\sqrt{k}} e$ to represent $\mathcal{N}(\mathcal{K})$; then we use V to eliminate e to recover a centered face.

Sets for Intersecting Cliques/Faces

$\alpha_1 := 1{:}(\bar{k}_1 + \bar{k}_2); \quad \alpha_2 := (\bar{k}_1 + 1){:}(\bar{k}_1 + \bar{k}_2 + \bar{k}_3)$



For each clique $|\alpha| = k$, we get a corresponding face/subspace ($k \times r$ matrix) representation. We now see how to handle two cliques, α_1, α_2, that intersect.

Two (Intersecting) Clique Reduction/Subsp. Repres.

THEOREM 2: Clique/Facial Intersection Using Subspace Intersection

$\{ \alpha_1, \alpha_2 \subseteq 1\!:\!n; \quad k := |\alpha_1 \cup \alpha_2|$

For $i = 1,2$: $\bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$, embedding dimension t_i;

$B_i := \mathcal{K}^\dagger(\bar{D}_i) = \bar{U}_i S_i \bar{U}_i^T$, $\bar{U}_i \in \mathcal{M}^{k_i \times t_i}$, $\bar{U}_i^T \bar{U}_i = I_{t_i}$, $S_i \in \mathcal{S}^{t_i}_{++}$;

$U_i := \begin{bmatrix} \bar{U}_i & \frac{1}{\sqrt{k_i}} e \end{bmatrix} \in \mathcal{M}^{k_i \times (t_i+1)}$; and $\bar{U} \in \mathcal{M}^{k \times (t+1)}$ satisfies

$$\mathcal{R}(\bar{U}) = \mathcal{R}\left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{\bar{k}_3} \end{bmatrix} \right) \cap \mathcal{R}\left(\begin{bmatrix} I_{\bar{k}_1} & 0 \\ 0 & U_2 \end{bmatrix} \right), \text{ with } \bar{U}^T \bar{U} = I_{t+1}$$

cont. . .

Two (Intersecting) Clique Reduction, cont...

THEOREM 2 Nonsing. Clique/Facial Inters. cont...

cont... with

$$\mathcal{R}(\bar{U}) = \mathcal{R}\left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{\bar{k}_3} \end{bmatrix}\right) \cap \mathcal{R}\left(\begin{bmatrix} I_{\bar{k}_1} & 0 \\ 0 & U_2 \end{bmatrix}\right), \text{ with } \bar{U}^T\bar{U} = I_{t+1};$$

let: $U := \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathcal{M}^{n\times(n-k+t+1)}$ and $\begin{bmatrix} V & \frac{U^Te}{\|U^Te\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal. Then

$$\bigcap_{i=1}^{2} \text{face}\,\mathcal{K}^{\dagger}\left(\mathcal{E}^n(\alpha_i, \bar{D}_i)\right) = \left(U\mathcal{S}_+^{n-k+t+1}U^T\right) \cap \mathcal{S}_C = (UV)\mathcal{S}_+^{n-k+t}(UV)^T$$

Expense/Work of (Two) Clique/Facial Reductions

Subspace Intersection for Two Intersecting Cliques/Faces

Suppose:

$$U_1 = \begin{bmatrix} U_1' & 0 \\ U_1'' & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} I & 0 \\ 0 & U_2'' \\ 0 & U_2' \end{bmatrix}$$

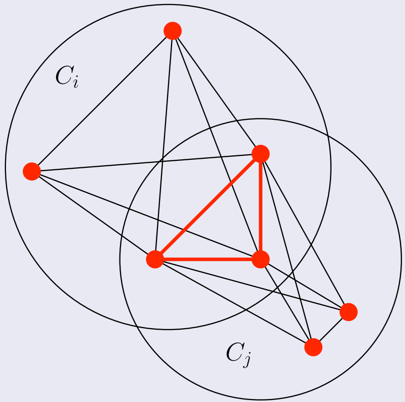
Then:

$$U := \begin{bmatrix} U_1' \\ U_1'' \\ U_2'(U_2'')^\dagger U_1'' \end{bmatrix} \quad \text{or} \quad U := \begin{bmatrix} U_1'(U_1'')^\dagger U_2'' \\ U_2'' \\ U_2' \end{bmatrix}$$

(Efficiently) satisfies:

$$\mathcal{R}(U) = \mathcal{R}(U_1) \cap \mathcal{R}(U_2)$$

Two (Intersecting) Clique Reduction Figure



Completion: missing distances can be recovered if desired.

Two (Intersecting) Clique Explicit Delayed Completion

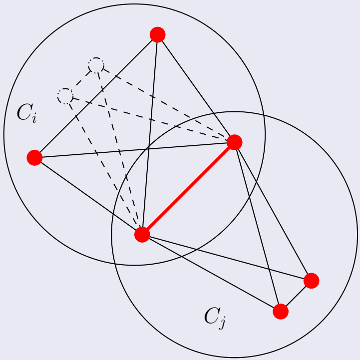
COR. Intersection with Embedding Dim. r/Completion

Hypotheses of Theorem 2 holds. Let $\bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$, for $i = 1, 2$, $\beta \subseteq \alpha_1 \cap \alpha_2, \gamma := \alpha_1 \cup \alpha_2, \bar{D} := D[\beta], B := \mathcal{K}^\dagger(\bar{D}), \quad \bar{U}_\beta := \bar{U}(\beta, :)$, where $\bar{U} \in \mathcal{M}^{k \times (t+1)}$ satisfies intersection equation of Theorem 2. Let $\begin{bmatrix} \bar{V} & \frac{\bar{U}^T e}{\|\bar{U}^T e\|} \end{bmatrix} \in \mathcal{M}^{t+1}$ be orthogonal. Let $\boxed{Z := (J\bar{U}_\beta \bar{V})^\dagger B ((J\bar{U}_\beta \bar{V})^\dagger)^T}$. If the embedding dimension for $\bar{D}$ is r, THEN $t = r$ in Theorem 2, and $Z \in \mathcal{S}^r_+$ is the unique solution of the equation $(J\bar{U}_\beta \bar{V}) Z (J\bar{U}_\beta \bar{V})^T = B$, and the exact completion is

$$\boxed{D[\gamma] = \mathcal{K}\left(PP^T\right)} \quad \text{where} \quad \boxed{P := UVZ^{\frac{1}{2}} \in \mathbb{R}^{|\gamma| \times r}}$$

2 (Inters.) Clique Red. Figure/Singular Case

Two (Intersecting) Clique Reduction Figure/Singular Case



Use R as lower bound in singular/nonrigid case.

Two (Inters.) Clique Explicit Compl.; Sing. Case

COR. Clique-Sing.; Intersect. Embedding Dim. $r-1$

Hypotheses of previous COR holds. For $i=1,2$, let $\beta \subset \delta_i \subseteq \alpha_i$, $A_i := J\bar{U}_{\delta_i}\bar{V}$, where $\bar{U}_{\delta_i} := \bar{U}(\delta_i,:)$, and $B_i := \mathcal{K}^\dagger(D[\delta_i])$. Let $\bar{Z} \in \mathcal{S}^t$ be a particular solution of the linear systems

$$\begin{array}{rcl} A_1ZA_1^T & = & B_1 \\ A_2ZA_2^T & = & B_2. \end{array} .$$

If the embedding dimension of $D[\delta_i]$ is r, for $i=1,2$, but the embedding dimension of $\bar{D} := D[\beta]$ is $r-1$, then the following holds. cont. . .

2 (Inters.) Clique Expl. Compl.; Degen. cont. . .

COR. Clique-Degen. cont. . .

The following holds:

1. $\dim \mathcal{N}(A_i) = 1$, for $i = 1, 2$.
2. For $i = 1, 2$, let $n_i \in \mathcal{N}(A_i)$, $\|n_i\|_2 = 1$, and $\Delta Z := n_1 n_2^T + n_2 n_1^T$. Then, Z is a solution of the linear systems if and only if
   $Z = \bar{Z} + \tau \Delta Z,$ for some $\tau \in \mathcal{R}$
3. There are at most two nonzero solutions, τ_1 and τ_2, for the generalized eigenvalue problem $-\Delta Z v = \tau \bar{Z} v$, $v \neq 0$. Set $Z_i := \bar{Z} + \frac{1}{\tau_i} \Delta Z$, for $i = 1, 2$. Then the exact completion is one of $D[\gamma] \in \{\mathcal{K}(\bar{U}\bar{V}Z_i\bar{V}^T\bar{U}^T) : i = 1, 2\}$

Completing SNL (Delayed use of Anchor Locations)

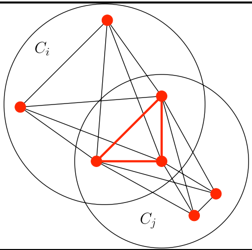
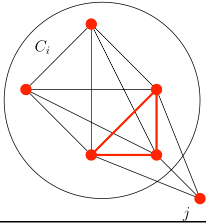
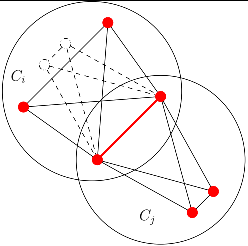
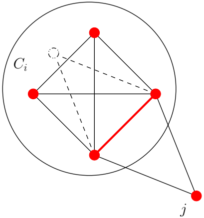
Rotate to Align the Anchor Positions

- Given $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{R}^{n \times r}$ such that $D = \mathcal{K}(PP^T)$
- Solve the orthogonal Procrustes problem:

$$\begin{array}{ll} \min & \|A - P_2 Q\| \\ \text{s.t.} & Q^T Q = I \end{array}$$

  $P_2^T A = U \Sigma V^T$ SVD decomposition; set $Q = UV^T$; (Golub/Van Loan, Algorithm 12.4.1)
- Set $X := P_1 Q$

Algorithm: Four Cases

	Clique Union	Node Absorption
Rigid		
Non-rigid		

ALGOR: clique union; facial reduct.; delay compl.

Initialize: Find initial set of cliques.

$C_i := \{j : (D_p)_{ij} < (R/2)^2\}$, for $i = 1, \ldots, n$

Iterate

- For $|C_i \cap C_j| \geq r + 1$, do Rigid Clique Union
- For $|C_i \cap \mathcal{N}(j)| \geq r + 1$, do Rigid Node Absorption
- For $|C_i \cap C_j| = r$, do Non-Rigid Clique Union (lower bnds)
- For $|C_i \cap \mathcal{N}(j)| = r$, do Non-Rigid Node Absorp. (lower bnds)

Finalize

When $\exists$ a clique containing all anchors, use computed facial representation and positions of anchors to solve for X

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Results - Data for Random Noisless Problems

- 2.16 GHz Intel Core 2 Duo, 2 GB of RAM
- Dimension $r = 2$
- Square region: $[0,1] \times [0,1]$
- $m = 9$ anchors
- Using only Rigid Clique Union and Rigid Node Absorption
- Error measure: Root Mean Square Deviation

$$\text{RMSD} = \left(\frac{1}{n} \sum_{i=1}^{n} \|p_i - p_i^{\text{true}}\|^2 \right)^{1/2}$$

Results - Large n (SDP size $O(n^2)$)

n # of Sensors Located

n # sensors \ R	0.07	0.06	0.05	0.04
2000	2000	2000	1956	1374
6000	6000	6000	6000	6000
10000	10000	10000	10000	10000

CPU Seconds

# sensors \ R	0.07	0.06	0.05	0.04
2000	1	1	1	3
6000	5	5	4	4
10000	10	10	9	8

RMSD (over located sensors)

n # sensors \ R	0.07	0.06	0.05	0.04
2000	$4e-16$	$5e-16$	$6e-16$	$3e-16$
6000	$4e-16$	$4e-16$	$3e-16$	$3e-16$
10000	$3e-16$	$5e-16$	$4e-16$	$4e-16$

Results - N Huge SDPs Solved

Large-Scale Problems

# sensors	# anchors	radio range	RMSD	Time
20000	9	.025	$5e-16$	25s
40000	9	.02	$8e-16$	1m 23s
60000	9	.015	$5e-16$	3m 13s
100000	9	.01	$6e-16$	9m 8s

Size of SDPs Solved: $N = \binom{n}{2}$ (# vrbls)

$\mathcal{E}_n(\text{density of } \mathcal{G}) = \pi R^2$; $M = \mathcal{E}_n(|E|) = \pi R^2 N$ (# constraints)

Size of SDP Problems:

$M = [3,078,915 \quad 12,315,351 \quad 27,709,309 \quad 76,969,790]$

$N = 10^9\,[0.2000 \quad 0.8000 \quad 1.8000 \quad 5.0000]$

Locally Recover Exact EDMs

Nearest EDM

- Given clique α; corresp. EDM $D_\epsilon = D + N_\epsilon$, N_ϵ noise
- we need to find the smallest face containing $\mathcal{E}^n(\alpha, D)$.
- $$\left\{ \begin{array}{ll} \min & \|\mathcal{K}(X) - D_\epsilon\| \\ \text{s.t.} & \operatorname{rank}(X) = r, Xe = 0, X \succeq 0 \\ & X \succeq 0. \end{array} \right.$$
- Eliminate the constraints: $Ve = 0, V^T V = I$, $\mathcal{K}_V(X) := \mathcal{K}(VXV^T)$:

$$\begin{array}{rcll} U_r^* & \in & \operatorname{argmin} & \frac{1}{2}\left\|\mathcal{K}_V(UU^T) - D_\epsilon\right\|_F^2 \\ & & \text{s.t.} & U \in M^{(n-1)r}. \end{array}$$

The nearest EDM is $D^* = \mathcal{K}_V(U_r^*(U_r^*)^T)$.

Solve Overdetermined Nonlin. Least Squares Prob.

Newton (expensive) or Gauss-Newton (less accurate)

$$F(U) := \text{us2vec}\left(\mathcal{K}_V(UU^T) - D_\epsilon\right), \quad \min_U f(U) := \frac{1}{2}\|F(U)\|^2$$

Derivatives: gradient and Hessian

$$\nabla f(U)(\Delta U) = \langle 2\left(\mathcal{K}_V^*\left[\mathcal{K}_V(UU^T) - D_\epsilon\right]\right)U, \Delta U\rangle$$

$$\nabla^2 f(U) = 2\text{vec}\left(\mathcal{L}_U^*\mathcal{K}_V^*\mathcal{K}_V\mathcal{S}_\Sigma\mathcal{L}_U + \mathcal{K}_V^*\left(\mathcal{K}_V(UU^T) - D_\epsilon\right)\right)\text{Mat}$$

where $\mathcal{L}_U(\cdot) = \cdot U^T$; $\mathcal{S}_\Sigma(U) = \frac{1}{2}(U + U^T)$

random noisy probs; $r = 2, m = 9, nf = 1e-6$

- Using only Rigid Clique Union, preliminary results:

remaining cliques

n / *R*	1.0	0.9	0.8	0.7	0.6
1000	1.00	5.00	11.00	40.00	124.00
2000	1.00	1.00	1.00	1.00	7.00
3000	1.00	1.00	1.00	1.00	1.00
4000	1.00	1.00	1.00	1.00	1.00
5000	1.00	1.00	1.00	1.00	1.00

cpu seconds

n / *R*	1.0	0.9	0.8	0.7	0.6
1000	9.43	6.98	5.57	5.04	4.05
2000	12.46	12.18	12.43	11.18	9.89
3000	18.08	18.50	19.07	18.33	16.33
4000	25.18	24.01	24.02	23.80	22.12
5000	38.13	31.66	30.26	30.32	29.88

max-log-error

n / *R*	1.0	0.9	0.8	0.7	0.6
1000	−3.28	−4.19	−2.92	*Inf*	*Inf*
2000	−3.63	−3.81	−3.82	−2.39	−3.73
3000	−3.51	−3.98	−3.25	−3.90	−3.28
4000	−4.15	−4.05	−3.52	−3.04	−3.33
5000	−4.80	−4.38	−3.89	−4.13	−3.40

Summary

- SDP relaxation of SNL is highly (implicitly) degenerate: The feasible set of this SDP is restricted to a low dim. face of the SDP cone, causing the Slater constraint qualification (strict feasibility) to fail
- We take advantage of this degeneracy by finding explicit representations of intersections of faces of the SDP cone corresponding to unions of intersecting cliques
- Without using an SDP-solver (eg. SeDuMi or SDPT3), we quickly compute the exact solution to the SDP relaxation

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- P. BISWAS, T.-C. LIAN, T.-C. WANG, and Y. YE, *Semidefinite programming based algorithms for sensor network localization*, ACM Trans. Sen. Netw. **2** (2006), no. 2, 188–220.
- P. BISWAS, T.-C. LIANG, Y. YE, K-C. TOH, and T.-C. WANG, *Semidefinite programming approaches for sensor network localization with noisy distance measurements*, IEEE Transactions onAutomation Science and Engineering **3** (2006), no. 4, 360–371.
- P. BISWAS and Y. YE, *Semidefinite programming for ad hoc wireless sensor network localization*, IPSN '04: Proceedings of the 3rd international symposium on Information processing in sensor networks (New York, NY, USA), ACM, 2004, pp. 46–54.

- P. BISWAS and Y. YE, *A distributed method for solving semidefinite programs arising from ad hoc wireless sensor network localization*, Multiscale optimization methods and applications, Nonconvex Optim. Appl., vol. 82, Springer, New York, 2006, pp. 69–84. MR MR2191577
- M.W. CARTER, H.H. JIN, M.A. SAUNDERS, and Y. YE, *SpaseLoc: an adaptive subproblem algorithm for scalable wireless sensor network localization*, SIAM J. Optim. **17** (2006), no. 4, 1102–1128. MR MR2274505 (2007j:90005)
- A. CASSIOLI, *Global optimization of highly multimodal problems*, Ph.D. thesis, Universita di Firenze, Dipartimento di sistemi e informatica, Via di S.Marta 3, 50139 Firenze, Italy, 2008.
- K. CHAKRABARTY and S.S. IYENGAR, Springer, London, 2005.

- J. DATTORRO, *Convex optimization & euclidean distance geometry*, Meboo Publishing, USA, 2005.
- Y. DING, N. KRISLOCK, J. QIAN, and H. WOLKOWICZ, *Sensor network localization, Euclidean distance matrix completions, and graph realization*, Optimization and Engineering **to appear** (2006), no. CORR 2006-23, to appear.
- B. HENDRICKSON, *The molecule problem: Determining conformation from pairwise distances*, Ph.D. thesis, Cornell University, 1990.
- H. JIN, *Scalable sensor localization algorithms for wireless sensor networks*, Ph.D. thesis, Toronto University, Toronto, Ontario, Canada, 2005.

- D.S. KIM, *Sensor network localization based on natural phenomena*, Ph.D. thesis, Dept, Electr. Eng. and Comp. Sc., MIT, 2006.
- N. KRISLOCK and H. WOLKOWICZ, *Explicit sensor network localization using semidefinite representations and clique reductions*, Tech. Report CORR 2009-04, University of Waterloo, Waterloo, Ontario, 2009, Available at URL: `www.optimization-online.org/DB_HTML/2009/05/2297.html`
- S. NAWAZ, *Anchor free localization for ad-hoc wireless sensor networks*, Ph.D. thesis, University of New South Wales, 2008.
- T.K. PONG and P. TSENG, *(Robust) edge-based semidefinite programming relaxation of sensor network localization*, Tech. Report Jan-09, University of Washington, Seattle, WA, 2009.

K. ROMER, *Time synchronization and localization in sensor networks*, Ph.D. thesis, ETH Zurich, 2005.

S. URABL, *Cooperative localization in wireless sensor networks*, Master's thesis, University of Klagenfurt, Klagenfurt, Austria, 2009.

Z. WANG, S. ZHENG, S. BOYD, and Y. YE, *Further relaxations of the semidefinite programming approach to sensor network localization*, SIAM J. Optim. **19** (2008), no. 2, 655–673. MR MR2425034

Thanks for your attention!

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