Explicit Sensor Network Localization using Semidefinite Programming and Clique Reductions

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Outline

1. Preliminaries
   - SNL <–> GR <–> EDM <–> SDP
   - Further Notation/Preliminaries; Facial Structure of Cones
2. Clique/Facial Reduction (Exploit degeneracy)
   - Single Clique Reduction
   - Two Clique Reduction and EDM DELAYED Completion
   - Completing SNL; DELAYED use of Anchor Locations
3. Algorithm: Facial Reduct. via Subsp. Inters./DELAYED Compl.
   - Clique Unions and Node Absorptions
   - Results (low CPU time; high accuracy)
4. Noisy Data

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Sensor Network Localization, SNL, Problem

SNL - a Fundamental Problem of Distance Geometry; (easy to describe)

- n ad hoc wireless sensors (nodes) to locate in $\mathbb{R}^r$, (r is embedding dimension; sensors $p_i \in \mathbb{R}^r, i \in V := 1, \ldots, n$)
- m of the sensors are anchors, $p_i, i = n - m + 1, \ldots, n$) (positions known, using e.g. GPS)
- pairwise distances known within radio range $R > 0$, $D_{ij} = \|p_i - p_j\|^2, ij \in E$
-

$$P = \begin{bmatrix} p_1^T \\ \vdots \\ p_n^T \end{bmatrix} = \begin{bmatrix} X \\ A \end{bmatrix} \in \mathbb{R}^{n \times r}$$

Applications of Wireless Ad-Hoc Sensor Networks

Fully Connected, Inexpensive Wireless Networks

- Tracking Humans/Animals/Equipment/Weather
- geographic routing; data aggregation; topological control.
- monitoring: large buildings environmental sensors; natural habitat; soil humidity; earthquakes and volcanos; forest fires; weather and ocean currents.
- military, tracking of goods, vehicle positions, surveillance, (where open-air positioning is not feasible), random deployment in inaccessible terrains or disaster relief operations

Underlying Graph Realization/Partial EDM

Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega)$

- node set $\mathcal{V} = \{1, \ldots, n\}$
- edge set $(i, j) \in \mathcal{E}$; $\omega_{ij} = \|p_i - p_j\|^2$ known approximately
- The anchors form a clique (complete subgraph)
- Realization of $\mathcal{G}$ in $\Re^r$: a mapping of node $v_i \to p_i \in \Re^r$ with squared distances given by ω.

Corresponding Partial Euclidean Distance Matrix, EDM

$$D_{ij} = \begin{cases} d_{ij}^2 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise,} \end{cases}$$

$d_{ij}^2 = \omega_{ij}$ are known squared Euclidean distances between sensors p_i, p_j; anchors correspond to a clique.

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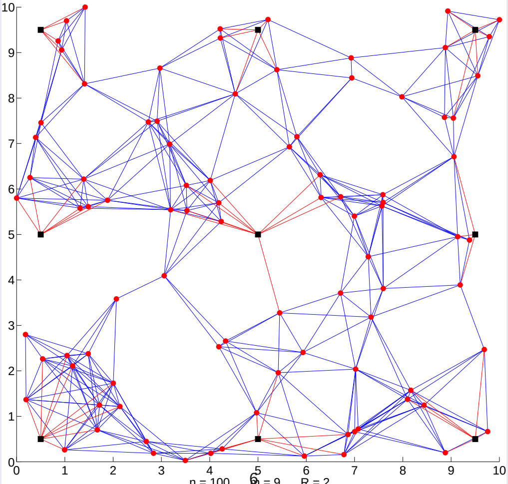
Sensor Localization Problem/Partial EDM

Sensors ○ and Anchors ■



Connections to Semidefinite Programming (SDP)

$\mathcal{S}^k_+$, Cone of (symmetric) SDP matrices in $\mathcal{S}^k$; $x^T Ax \geq 0$

inner product $\langle A, B\rangle = \operatorname{trace} AB$
Löwner (psd) partial order $A \succeq B, A \succ B$

$D = \mathcal{K}(B) \in \mathcal{E}^n$, $B = \mathcal{K}^\dagger(D) \in \mathcal{S}^n \cap \mathcal{S}_C$ (centered $Be = 0$)

$P^T = \begin{bmatrix} p_1 & p_2 & \dots & p_k \end{bmatrix} \in \mathcal{M}^{r\times k}$; $B := PP^T \in \mathcal{S}^k_+$;
$\operatorname{rank} B = r$; $D \in \mathcal{E}^k$ be corresponding EDM.

$$
\begin{aligned}
(\text{to } D \in \mathcal{E}^k) \quad D &= \left(\|p_i - p_j\|_2^2\right)_{i,j=1}^k \\
&= \left(p_i^T p_i + p_j^T p_j - 2p_i^T p_j\right)_{i,j=1}^k \\
&= \boxed{\operatorname{diag}(B)\, e^T + e \operatorname{diag}(B)^T - 2B} \\
&=: \mathcal{D}_e(B) - 2B \\
&=: \mathcal{K}(B) \quad (\text{from } B \in \mathcal{S}^k_+).
\end{aligned}
$$

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Current Techniques: SDP Relaxation

Nearest, Weighted, SDP Approx. (relax $\operatorname{rank} B$)

- $\min_{B \succeq 0, B \in \Omega} \|H \circ (\mathcal{K}(B) - D)\|$; $\operatorname{rank} B = r$; weights $H_{ij} = 1/\sqrt{D_{ij}}$ if $ij \in E$.
- with rank constraint: a non-convex, NP-hard program
- SDP relaxation is convex but: expensive/low accuracy/implicitly highly degenerate (cliques restrict ranks of feasible Bs)
- take advantage of degeneracy!: clique $\alpha, |\alpha| = k$ (corresp. $D(\alpha)$) with embed. dim. $= t \leq r < k \implies$ $\operatorname{rank} \mathcal{K}^\dagger(\bar{D}) = t \leq r \implies \operatorname{rank} B(\alpha) \leq \operatorname{rank} \mathcal{K}^\dagger(D(\alpha)) + 1$ $\implies \operatorname{rank} B = \operatorname{rank} \mathcal{K}^\dagger(D) \leq n - \boxed{(k - t - 1)} \implies$
  Slater's CQ (strict feasibility) fails

$\mathcal{K} : \mathcal{S}^n_+ \cap \mathcal{S}_C \to \mathcal{E}^n \subset \mathcal{S}^n \cap \mathcal{S}_H$

Linear Transformations: $\mathcal{D}_v(B), \mathcal{K}(B), \mathcal{T}(D)$

- allow: $\mathcal{D}_v(B) := \operatorname{diag}(B)\, v^T + v \operatorname{diag}(B)^T$;
  $\mathcal{D}_v(y) := yv^T + vy^T$
- adjoint $\mathcal{K}^*(D) = 2(\operatorname{Diag}(De) - D)$.
- $\mathcal{K}$ is $1-1$, onto between centered & hollow subspaces:
  $\mathcal{S}_C := \{B \in \mathcal{S}^n : Be = 0\}$;
  $\mathcal{S}_H := \{D \in \mathcal{S}^n : \operatorname{diag}(D) = 0\} = \mathcal{R}(\operatorname{offDiag})$
- $J := I - \frac{1}{n}ee^T$ (orthogonal projection onto $M := \{e\}^\perp$);
- $\mathcal{T}(D) := -\frac{1}{2} J \operatorname{offDiag}(D) J \qquad (= \mathcal{K}^\dagger(D))$

Properties of Linear Transformations

$\mathcal{K}, \mathcal{T}, \mathrm{Diag}, \mathcal{D}_e$

$$\mathcal{R}(\mathcal{K}) = \mathcal{S}_H; \qquad \underline{\mathcal{N}(\mathcal{K}) = \mathcal{R}(\mathcal{D}_e)};$$

$$\mathcal{R}(\mathcal{K}^*) = \mathcal{R}(\mathcal{T}) = \mathcal{S}_C; \qquad \mathcal{N}(\mathcal{K}^*) = \mathcal{N}(\mathcal{T}) = \mathcal{R}(\mathrm{Diag});$$

$$\mathcal{S}^n = \mathcal{S}_H \oplus \mathcal{R}(\mathrm{Diag}) = \mathcal{S}_C \oplus \mathcal{R}(\mathcal{D}_e).$$

$$\mathcal{T}(\mathcal{E}^n) = \mathcal{S}^n_+ \cap \mathcal{S}_C \quad \text{and} \quad \mathcal{K}(\mathcal{S}^n_+ \cap \mathcal{S}_C) = \mathcal{E}^n.$$

Semidefinite Cone, Faces

Faces of cone K

- $F \subseteq K$ is a face of K, denoted $\boxed{F \trianglelefteq K}$, if $\left(x, y \in K, \ \frac{1}{2}(x+y) \in F\right) \implies \left(\operatorname{cone}\{x, y\} \subseteq F\right)$.
- $\boxed{F \triangleleft K}$, if $F \trianglelefteq K$, $F \neq K$; F is $\boxed{\text{proper face}}$ if $\{0\} \neq F \triangleleft K$.
- $F \trianglelefteq K$ is $\boxed{\text{exposed}}$ if: intersection of K with a hyperplane.
- $\boxed{\operatorname{face}(S)}$ denotes smallest face of K that contains set S.

$\mathcal{S}^k_+$ is a Facially Exposed Cone

All faces are exposed.

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Facial Structure of SDP Cone; Equivalent SUBSPACES

Face $F \unlhd \mathcal{S}^k_+$ Equivalence to $\mathcal{R}(U)$ Subspace of $\mathbb{R}^k$

$F \unlhd \mathcal{S}^k_+$ determined by range of any $S \in \operatorname{relint} F$,
i.e. let $S = U\Gamma U^T$ be compact spectral decomposition; $\Gamma \in \mathcal{S}^t_{++}$
is diagonal matrix of pos. eigenvalues; $\boxed{F = U\mathcal{S}^t_+ U^T}$
(F associated with $\mathcal{R}(U)$)

$$\dim F = t(t+1)/2.$$

face F representation by subspace $\mathcal{L}$

(subspace) $\mathcal{L} = \mathcal{R}(T)$, T is $k \times t$ full column, then:

$$F := T\mathcal{S}^t_+ T^T \unlhd \mathcal{S}^k_+$$

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Further Notation

Matrix with Fixed Principal Submatrix

For $Y \in \mathcal{S}^n$, $\alpha \subseteq \{1,\ldots,n\}$: $Y[\alpha]$ denotes principal submatrix formed from rows & cols with indices α.

Sets with Fixed Principal Submatrices

If $|\alpha| = k$ and $\bar{Y} \in \mathcal{S}^k$, then:

- $\mathcal{S}^n(\alpha, \bar{Y}) := \{Y \in \mathcal{S}^n : Y[\alpha] = \bar{Y}\}$,
- $\mathcal{S}^n_+(\alpha, \bar{Y}) := \{Y \in \mathcal{S}^n_+ : Y[\alpha] = \bar{Y}\}$
  i.e. the subset of matrices $Y \in \mathcal{S}^n$ ($Y \in \mathcal{S}^n_+$) with principal submatrix $Y[\alpha]$ fixed to $\bar{Y}$.

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Single Clique/Facial Reduction

$\bar{D} \in \mathcal{E}^k$, $\alpha \subseteq 1\!:\!n$, $|\alpha| = k$

Define $\mathcal{E}^n(\alpha, \bar{D}) := \left\{ D \in \mathcal{E}^n : D[\alpha] = \bar{D} \right\}$.

Given $\bar{D}$; find a corresponding $B \succeq 0$; find the corresponding face; find the corresponding subspace.

if $\alpha = 1:k$; embed. dim of $\bar{D}$ is $t \leq r$

$$D = \begin{bmatrix} \bar{D} & \cdot \\ \cdot & \cdot \end{bmatrix},$$

BASIC THEOREM for Single Clique/Facial Reduction

THEOREM 1: Single Clique Reduction

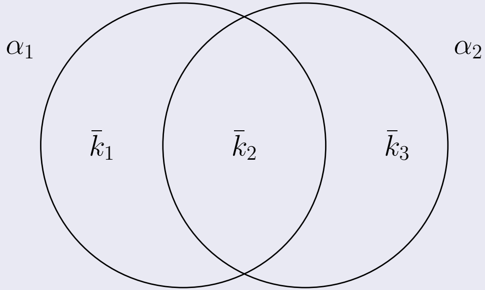
Let: $\bar{D} := D[1:k] \in \mathcal{E}^k$, $k < n$, with embedding dimension $t \leq r$; $B := \mathcal{K}^\dagger(\bar{D}) = \bar{U}_B S \bar{U}_B^T$, $\bar{U}_B \in \mathcal{M}^{k \times t}$, $\bar{U}_B^T \bar{U}_B = I_t$, $S \in \mathcal{S}^t_{++}$.
Furthermore, let $U_B := \begin{bmatrix} \bar{U}_B & \frac{1}{\sqrt{k}} e \end{bmatrix} \in \mathcal{M}^{k \times (t+1)}$,
$U := \begin{bmatrix} U_B & 0 \\ 0 & I_{n-k} \end{bmatrix}$, and let $\begin{bmatrix} V & \frac{U^T e}{\|U^T e\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal. Then:

$$\begin{aligned} \operatorname{face} \mathcal{K}^\dagger\left(\mathcal{E}^n(1:k, \bar{D})\right) &= \left(U \mathcal{S}_+^{n-k+t+1} U^T\right) \cap \mathcal{S}_C \\ &= (UV) \mathcal{S}_+^{n-k+t} (UV)^T \end{aligned}$$

Note that we add $\frac{1}{\sqrt{k}} e$ to represent $\mathcal{N}(\mathcal{K})$; then we use V to eliminate e to recover a centered face.

Positive Integers/Sets for Intersecting Cliques

$\alpha_1 := 1:(\bar{k}_1 + \bar{k}_2); \quad \alpha_2 := (\bar{k}_1 + 1):(\bar{k}_1 + \bar{k}_2 + \bar{k}_3)$



For each clique $|\alpha| = k$, we get a corresponding face/subspace ($k \times r$ matrix) representation. We now see how to handle two cliques, α_1, α_2, that intersect.

Two (Intersecting) Clique Reduction/Subsp. Repres.

THEOREM 2: Clique Intersection Using Subspace Intersection

$\{ \ \alpha_1, \alpha_2 \subseteq 1\!:\!n; \quad k := |\alpha_1 \cup \alpha_2|$

For $i = 1, 2$: $\bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$, embedding dimension t_i;

$B_i := \mathcal{K}^{\dagger}(\bar{D}_i) = \bar{U}_i S_i \bar{U}_i^T$, $\bar{U}_i \in \mathcal{M}^{k_i \times t_i}$, $\bar{U}_i^T \bar{U}_i = I_{t_i}$, $S_i \in \mathcal{S}_{++}^{t_i}$;

$U_i := \begin{bmatrix} \bar{U}_i & \frac{1}{\sqrt{k_i}} e \end{bmatrix} \in \mathcal{M}^{k_i \times (t_i+1)}$; and $\bar{U} \in \mathcal{M}^{k \times (t+1)}$ satisfies

$$\mathcal{R}(\bar{U}) = \mathcal{R}\left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{\bar{k}_3} \end{bmatrix} \right) \cap \mathcal{R}\left(\begin{bmatrix} I_{\bar{k}_1} & 0 \\ 0 & U_2 \end{bmatrix} \right), \text{ with } \bar{U}^T \bar{U} = I_{t+1}$$

cont. . .

Two (Intersecting) Clique Reduction, cont. . .

THEOREM 2 Nonsing. Clique Inters. cont. . .

cont. . . with

$$\mathcal{R}(\bar{U}) = \mathcal{R}\left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{\bar{k}_3} \end{bmatrix}\right) \cap \mathcal{R}\left(\begin{bmatrix} I_{\bar{k}_1} & 0 \\ 0 & U_2 \end{bmatrix}\right), \text{ with } \bar{U}^T\bar{U} = I_{t+1};$$

let: $U := \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathcal{M}^{n\times(n-k+t+1)}$ and

$\begin{bmatrix} V & \frac{U^Te}{\|U^Te\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal. Then

$$\begin{aligned} \underline{\cap_{i=1}^2 \operatorname{face} \mathcal{K}^\dagger\left(\mathcal{E}^n(\alpha_i, \bar{D}_i)\right)} &= \left(U\mathcal{S}_+^{n-k+t+1}U^T\right) \cap \mathcal{S}_C \\ &= (UV)\mathcal{S}_+^{n-k+t}(UV)^T \end{aligned}$$

Expense/Work of (Two) Clique Reductions

Subspace Intersection for Two Intersecting Cliques

Suppose:

$$U_1 = \begin{bmatrix} U_1' & 0 \\ U_1'' & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} I & 0 \\ 0 & U_2'' \\ 0 & U_2' \end{bmatrix}$$

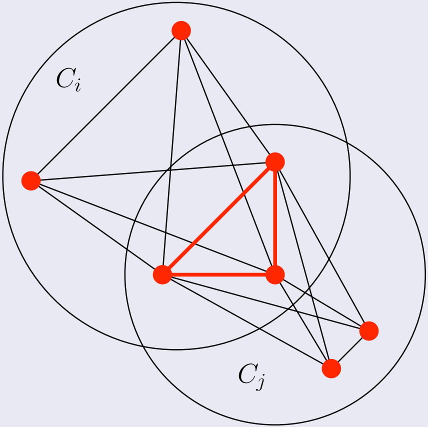
Then:

$$U := \begin{bmatrix} U_1' \\ U_1'' \\ U_2'(U_2'')^\dagger U_1'' \end{bmatrix} \quad \text{or} \quad U := \begin{bmatrix} U_1'(U_1'')^\dagger U_2'' \\ U_2'' \\ U_2' \end{bmatrix}$$

(Efficiently) satisfies:

$$\mathcal{R}(U) = \mathcal{R}(U_1) \cap \mathcal{R}(U_2)$$

Two (Intersecting) Clique Reduction Figure



Completion: missing distances can be recovered if desired.

Two (Intersecting) Clique Explicit Delayed Completion

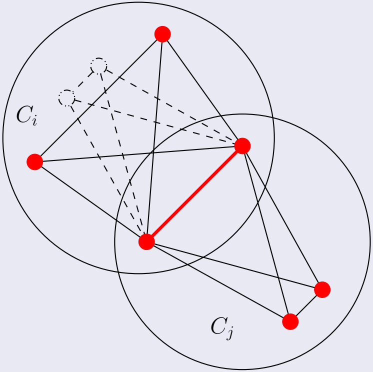
COR. Intersection with Embedding Dim. r/Completion

Hypotheses of Theorem 2 holds. Let $\bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$, for $i = 1, 2$, $\beta \subseteq \alpha_1 \cap \alpha_2, \gamma := \alpha_1 \cup \alpha_2, \bar{D} := D[\beta], B := \mathcal{K}^\dagger(\bar{D})$, $\quad \bar{U}_\beta := \bar{U}(\beta, :)$, where $\bar{U} \in \mathcal{M}^{k \times (t+1)}$ satisfies intersection equation of Theorem 2. Let $\begin{bmatrix} \bar{V} & \frac{\bar{U}^T e}{\|\bar{U}^T e\|} \end{bmatrix} \in \mathcal{M}^{t+1}$ be orthogonal. Let $\boxed{Z := (J\bar{U}_\beta \bar{V})^\dagger B ((J\bar{U}_\beta \bar{V})^\dagger)^T}$. If the embedding dimension for $\bar{D}$ is r, <u>THEN</u> $t = r$ in Theorem 2, and $Z \in \mathcal{S}^r_+$ is the unique solution of the equation $(J\bar{U}_\beta \bar{V}) Z (J\bar{U}_\beta \bar{V})^T = B$, and the exact completion is

$$\boxed{D[\gamma] = \mathcal{K}\left(PP^T\right)} \quad \text{where} \quad \boxed{P := UVZ^{\frac{1}{2}} \in \mathbb{R}^{|\gamma| \times r}}$$

2 (Inters.) Clique Red. Figure/Singular Case

Two (Intersecting) Clique Reduction Figure/Singular Case



Use R as lower bound in singular/nonrigid case.

Two (Inters.) Clique Explicit Compl.; Sing. Case

COR. Clique-Sing.; Intersect. Embedding Dim. $r-1$

Hypotheses of previous COR holds. For $i=1,2$, let $\beta \subset \delta_i \subseteq \alpha_i$, $A_i := J\bar{U}_{\delta_i}\bar{V}$, where $\bar{U}_{\delta_i} := \bar{U}(\delta_i,:)$, and $B_i := \mathcal{K}^\dagger(D[\delta_i])$. Let $\bar{Z} \in \mathcal{S}^t$ be a particular solution of the linear systems

$$\begin{array}{rcl} A_1 Z A_1^T & = & B_1 \\ A_2 Z A_2^T & = & B_2. \end{array}.$$

If the embedding dimension of $D[\delta_i]$ is r, for $i=1,2$, but the embedding dimension of $\bar{D} := D[\beta]$ is $r-1$, then the following holds. cont. . .

2 (Inters.) Clique Expl. Compl.; Degen. cont. . .

COR. Clique-Degen. cont. . .

The following holds:

1. $\dim \mathcal{N}(A_i) = 1$, for $i = 1, 2$.
2. For $i = 1, 2$, let $n_i \in \mathcal{N}(A_i)$, $\|n_i\|_2 = 1$, and $\Delta Z := n_1 n_2^T + n_2 n_1^T$. Then, Z is a solution of the linear systems if and only if
$Z = \bar{Z} + \tau \Delta Z,$ for some $\tau \in \mathcal{R}$
3. There are at most two nonzero solutions, τ_1 and τ_2, for the generalized eigenvalue problem $-\Delta Z v = \tau \bar{Z} v$, $v \neq 0$. Set $Z_i := \bar{Z} + \frac{1}{\tau_i} \Delta Z$, for $i = 1, 2$. Then the exact completion is one of $D[\gamma] \in \{\mathcal{K}(\bar{U}\bar{V}Z_i\bar{V}^T\bar{U}^T) : i = 1, 2\}$

Completing SNL (Delayed use of Anchor Locations)

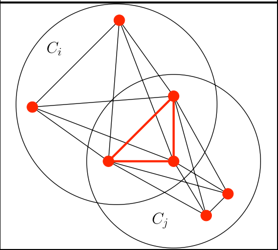
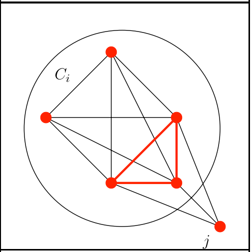
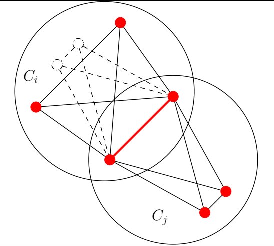
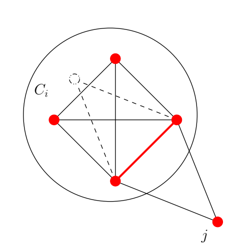
Rotate to Align the Anchor Positions

- Given $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{R}^{n \times r}$ such that $D = \mathcal{K}(PP^T)$
- Solve the orthogonal Procrustes problem:

$$\boxed{\begin{array}{ll} \min & \|A - P_2 Q\| \\ \text{s.t.} & Q^T Q = I \end{array}}$$

  using SVD (Golub/Van Loan, Algorithm 12.4.1)
- Set $X := P_1 Q$

Algorithm: Four Cases

	Clique Union	Node Absorption
Rigid		
Non-rigid		

ALGOR: clique union; facial reduct.; delay compl.

Initialize: Find initial set of cliques.

$C_i := \{j : (D_p)_{ij} < (R/2)^2\}$, for $i = 1, \ldots, n$

Iterate

- For $|C_i \cap C_j| \geq r+1$, do Rigid Clique Union
- For $|C_i \cap \mathcal{N}(j)| \geq r+1$, do Rigid Node Absorption
- For $|C_i \cap C_j| = r$, do Non-Rigid Clique Union (lower bnds)
- For $|C_i \cap \mathcal{N}(j)| = r$, do Non-Rigid Node Absorp. (lower bnds)

Finalize

When $\exists$ a clique containing all anchors, use computed facial representation and positions of anchors to solve for X

ALGOR: clique union; facial reduct.; delay compl.

Initialize: Find initial set of cliques.

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10 random noiseless probs; $r = 2, m = 9$

Rigid Clique Union

n / R	0.7	0.6	0.5	0.4
2000	1	7	91	362
4000	1	1	1	16
6000	1	1	1	1
8000	1	1	1	1
10000	1	1	1	1

Remaining Cliques

n / R	0.7	0.6	0.5	0.4
2000	4.8	4.6	4.2	4.1
4000	9.2	9.4	9.1	9.2
6000	16.0	14.7	15.3	14.9
8000	22.9	22.5	20.9	21.0
10000	38.3	32.7	29.1	30.7

CPU Seconds

n / R	0.7	0.6	0.5	0.4
2000	-10.1	-10.8	$-$	$-$
4000	-10.9	-11.0	-10.5	-9.6
6000	-11.6	-10.7	-10.6	-10.0
8000	-11.1	-11.0	-10.7	-9.2
10000	-11.0	-11.0	-10.2	-10.4

Max log(Error)

Rigid Clique Union and Node Absorption

n / R	0.7	0.6	0.5	0.4
2000	1	1	2	78
4000	1	1	1	1
6000	1	1	1	1
8000	1	1	1	1
10000	1	1	1	1

Remaining Cliques

n / R	0.7	0.6	0.5	0.4
2000	4.9	4.9	6.1	13.2
4000	9.2	9.5	9.1	9.8
6000	16.1	15.1	15.1	14.8
8000	22.7	22.4	21.0	21.3
10000	32.5	32.4	28.8	30.6

CPU Seconds

n / R	0.7	0.6	0.5	0.4
2000	-10.1	-10.8	-9.8	-8.8
4000	-10.9	-11.0	-10.5	-9.6
6000	-11.6	-10.7	-10.6	-10.0
8000	-11.1	-11.0	-10.7	-9.2
10000	-11.0	-11.0	-10.2	-10.4

Max log(Error)

Locally Recover Exact EDMs

Nearest EDM

- Given clique α; corresp. EDM $D_\epsilon = D + N_\epsilon$, N_ϵ noise
- we need to find the smallest face containing $\mathcal{E}^n(\alpha, D)$.
- $$\left\{ \begin{array}{ll} \min & \|\mathcal{K}(X) - D_\epsilon\| \\ \text{s.t.} & \operatorname{rank}(X) = r, Xe = 0, X \succeq 0 \\ & X \succeq 0. \end{array} \right.$$
- Eliminate the constraints: $Ve = 0, V^T V = I$, $\mathcal{K}_V(X) := \mathcal{K}(VXV^T)$:

$$\begin{array}{rcll} U_r^* & \in & \operatorname{argmin} & \frac{1}{2}\left\|\mathcal{K}_V(UU^T) - D_\epsilon\right\|_F^2 \\ & & \text{s.t.} & U \in M^{(n-1)r}. \end{array}$$

The nearest EDM is $D^* = \mathcal{K}_V(U_r^*(U_r^*)^T)$.

Solve Overdetermined Nonlin. Least Squares Prob.

Newton (expensive) or Gauss-Newton (less accurate)

$$F(U) := \text{us2vec}\left(\mathcal{K}_V(UU^T) - D_\epsilon\right), \quad \min_U f(U) := \frac{1}{2}\|F(U)\|^2$$

Derivatives: gradient and Hessian

$$\nabla f(U)(\Delta U) = \langle 2\left(\mathcal{K}_V^*\left[\mathcal{K}_V(UU^T) - D_\epsilon\right]\right)U, \Delta U\rangle$$

$$\nabla^2 f(U) = 2\text{vec}\left(\mathcal{L}_U^*\mathcal{K}_V^*\mathcal{K}_V\mathcal{S}_\Sigma\mathcal{L}_U + \mathcal{K}_V^*\left(\mathcal{K}_V(UU^T) - D_\epsilon\right)\right)\text{Mat}$$

where $\mathcal{L}_U(\cdot) = \cdot U^T$; $\mathcal{S}_\Sigma(U) = \frac{1}{2}(U + U^T)$

random noisy probs; $r = 2, m = 9, nf = 1e - 6$

- Using only Rigid Clique Union, preliminary results:

remaining cliques

n / R	1.0	0.9	0.8	0.7	0.6
1000	1.00	5.00	11.00	40.00	124.00
2000	1.00	1.00	1.00	1.00	7.00
3000	1.00	1.00	1.00	1.00	1.00
4000	1.00	1.00	1.00	1.00	1.00
5000	1.00	1.00	1.00	1.00	1.00

cpu seconds

n / R	1.0	0.9	0.8	0.7	0.6
1000	9.43	6.98	5.57	5.04	4.05
2000	12.46	12.18	12.43	11.18	9.89
3000	18.08	18.50	19.07	18.33	16.33
4000	25.18	24.01	24.02	23.80	22.12
5000	38.13	31.66	30.26	30.32	29.88

max-log-error

n / R	1.0	0.9	0.8	0.7	0.6
1000	-3.28	-4.19	-2.92	*Inf*	*Inf*
2000	-3.63	-3.81	-3.82	-2.39	-3.73
3000	-3.51	-3.98	-3.25	-3.90	-3.28
4000	-4.15	-4.05	-3.52	-3.04	-3.33
5000	-4.80	-4.38	-3.89	-4.13	-3.40

Summary

- SDP relaxation of SNL is highly (implicitly) degenerate: The feasible set of this SDP is restricted to a low dim. face of the SDP cone, causing the Slater constraint qualification (strict feasibility) to fail
- We take advantage of this degeneracy by finding explicit representations of intersections of faces of the SDP cone corresponding to unions of intersecting cliques
- Without using an SDP-solver (eg. SeDuMi or SDPT3), we quickly compute the exact solution to the SDP relaxation (except for round-off error from computing eigenvectors, etc.)

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Thanks for your attention!

Explicit Sensor Network Localization using Semidefinite Programming and Clique Reductions

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