Duality, Complementarity, and Regularization, in Conic Convex Optimization

Henry Wolkowicz
Dept. Comb. & Opt, Univ. Waterloo

Joint work with: Simon Schurr and Levent Tunçel

Tutte Seminar, University of Waterloo, July 4/08

Motivation: Difficulties for Nonpolyhedral Cone Opt.

Strong Duality Failure/Absence of Constraint Qualification, CQ

- Instances: SDP relaxations for hard combinatorial problems (e.g. QAP, GP, strengthened MC, polynomial opt.)

- Fresh look at known Characterizations of Optimality using Subspace Formulation

Regularization, Efficient Solutions

- theme: use minimal representations

Complementarity/Duality/Closure

- Surprising Connections: Complementarity; duality; closure of sums of cones/Numerical implications

Outline

Cone Optimization, (e.g. $K = \mathbb{R}^n_+$, LP; $K = \mathbb{S}^n_+$, SDP)

Primal-Dual Pair of Optimization Problems in Conic Form

(assumed finite) $\quad v_P = \sup_y \{\langle b, y\rangle \ : \ \mathcal{A}^* y \preceq_K c\} \qquad (\mathbb{P})$

(weak dual. $v_P \leq$) $\quad v_D = \inf_x \{\langle c, x\rangle : \mathcal{A}x = b, \ x \succeq_{K^*} 0\} \qquad (\mathbb{DP})$

where

- $\mathcal{A}$ - onto linear transformation; adjoint $\mathcal{A}^*$
- K - a proper convex cone with dual/polar cone $K^* = \{x : \langle s, x\rangle \geq 0, \ \forall s \in K\}$.
- $s' \preceq_K s''(s' \prec_K s'')$ - partial order, $s'' - s' \in K (\in \mathrm{int} K)$

Cone Optimization, (e.g. $K = \mathbb{R}^n_+$, LP; $K = \mathbb{S}^n_+$, SDP)

Primal-Dual Pair of Optimization Problems in Conic Form

(assumed finite) $$v_P = \sup_y \{\langle b, y\rangle \ : \ \mathcal{A}^* y \preceq_K c\} \qquad (\mathbb{P})$$

(weak dual. $v_P \leq$) $$v_D = \inf_x \{\langle c, x\rangle : \mathcal{A}x = b, \ x \succeq_{K^*} 0\} \qquad (\mathbb{DP})$$

where

- $\mathcal{A}$ - onto linear transformation; adjoint $\mathcal{A}^*$
- K - a proper convex cone with dual/polar cone $K^* = \{x : \langle s, x\rangle \geq 0, \ \forall s \in K\}$.
- $s' \preceq_K s''(s' \prec_K s'')$ - partial order, $s'' - s' \in K (\in \mathrm{int} K)$

Faces of Cones

Face

A convex cone F is a face of K, denoted $F \trianglelefteq K$, if

$$(s \in F, 0 \preceq_K u \preceq_K s) \text{ implies } (\mathrm{cone}\{u\} \subset F)$$

If $F \trianglelefteq K$ and $F \neq K$, write $F \triangleleft K$.

Conjugate Face

If $F \trianglelefteq K$, the conjugate face (or complementary face) of F is

$$F^c := F^\perp \cap K^* \trianglelefteq K^*.$$

If $x \in \mathrm{relint}(F)$, then $F^c = \{x\}^\perp \cap K^*$ (face is exposed by $\{x\}^\perp$)

Minimal Face (Minimal Cone)

Feasible sets

$$\mathcal{F}_P^y(c) = \mathcal{F}_P^y := \{y : c - \mathcal{A}^* y \succeq_K 0\}$$
$$\mathcal{F}_P^s(c) = \mathcal{F}_P^s := \{s : s = c - \mathcal{A}^* y \succeq_K 0, \text{ for some } y\} \quad \text{(slacks)}$$
$$\mathcal{F}_D^x(b) = \mathcal{F}_D^x := \{x : \mathcal{A}x = b, x \succeq_K^* 0\}$$

Minimal Faces (smallest face containing set)

$$f_P := \operatorname{face} \mathcal{F}_P^s \unlhd K \qquad f_D := \operatorname{face} \mathcal{F}_D^x \unlhd K^*$$

Minimal Face (Minimal Cone)

Feasible sets

$$\mathcal{F}_P^y(c) = \mathcal{F}_P^y := \{y : c - \mathcal{A}^* y \succeq_K 0\}$$
$$\mathcal{F}_P^s(c) = \mathcal{F}_P^s := \{s : s = c - \mathcal{A}^* y \succeq_K 0, \text{ for some } y\} \quad \text{(slacks)}$$
$$\mathcal{F}_D^x(b) = \mathcal{F}_D^x := \{x : \mathcal{A}x = b, x \succeq_K^* 0\}$$

Minimal Faces (smallest face containing set)

$$f_P := \text{face}\, \mathcal{F}_P^s \trianglelefteq K \qquad f_D := \text{face}\, \mathcal{F}_D^x \trianglelefteq K^*$$

(Modified) SDP Example from Ramana, 1995

Primal SDP

$$0 = v_P = \sup_y \left\{ y_2 : \begin{bmatrix} \underline{0} & 0 & \underline{y_2} \\ 0 & y_2 & 0 \\ \underline{y_2} & 0 & \underline{y_1} \end{bmatrix} \preceq \begin{bmatrix} \underline{0} & 0 & \underline{0} \\ 0 & 1 & 0 \\ \underline{0} & 0 & \underline{0} \end{bmatrix} \right\}$$

$$y^* = \begin{pmatrix} y_1^* & 0 \end{pmatrix}^T, \quad y_1^* \leq 0, \quad s^* = c - \mathcal{A}^* y^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -y_1^* \end{bmatrix}$$

Slater (strict feas.) CQ fails for $(\mathbb{P})$; $v_P = 0$

(SDP Example cont. . .

Dual SDP

$$1 = v_D = \inf_{x \succeq 0} \{\langle c, x\rangle = x_{22} \ : \ x_{33} = 0, x_{31} + x_{22} + x_{13} = 1\}$$

$$v_D = 1, x^* = \begin{bmatrix} X_{11} & X_{12} & 0 \\ X_{21} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \succeq 0$$

$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \operatorname{precl} \mathcal{A}((\operatorname{face} s^*)^c)$ primal opt. cond. hold only asymptotically

Slater's CQ fails for primal and dual; $v_D = 1 > v_P = 0$

Minimal Face for Ramana Example

Feasible Set/Minimal Face

$\mathcal{F}_P^y = \{y \in \mathbb{R}^2 : y_1 \leq 0,\ y_2 = 0\}$

$$\begin{aligned} f_P &= \bigcap\{F \trianglelefteq K : \mathcal{F}_P^s = c - \mathcal{A}^*(\mathcal{F}_P^y) \subset F\} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{S}_+^2 \end{bmatrix} \\ &\lhd \ \mathbb{S}_+^3 \end{aligned}$$

Slater CQ and Minimal Face (in general)

If $(\mathbb{P})$ is feasible, then

$$c - \mathcal{A}^* y \not\succ_K 0\ \forall y \text{ (Slater's CQ fails for } (\mathbb{P}) \text{) } \iff f_P \lhd K$$

Minimal Face for Ramana Example

Feasible Set/Minimal Face

$$\mathcal{F}_P^y = \{y \in \mathbb{R}^2 : y_1 \leq 0,\ y_2 = 0\}$$

$$\begin{aligned} f_P &= \bigcap\{F \trianglelefteq K : \mathcal{F}_P^s = c - \mathcal{A}^*(\mathcal{F}_P^y) \subset F\} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{S}^2_+ \end{bmatrix} \\ &\lhd \mathbb{S}^3_+ \end{aligned}$$

Slater CQ and Minimal Face (in general)

If $(\mathbb{P})$ is feasible, then

$$c - \mathcal{A}^* y \not\succ_K 0\ \forall y \text{ (Slater's CQ fails for } (\mathbb{P}) \text{) } \iff f_P \lhd K$$

(SYMMETRIC) Subspace Form for ($\mathbb{P}$) and ($\mathbb{DP}$)

Assume Linear Feasibility for $\tilde{s}, \tilde{y}, \tilde{x}$; with data A, b, c, K

$\mathcal{A}^*\tilde{y} + \tilde{s} = c$ $\qquad$ $\mathcal{A}\tilde{x} = b$

$\mathcal{L}^\perp = \mathcal{R}(\mathcal{A}^*)$ (range) $\qquad$ $\mathcal{L} = \mathcal{N}(\mathcal{A})$ (nullspace)

(innerproducts:) $\langle b, y\rangle = \langle(\mathcal{A}\tilde{x}), y\rangle = \langle \tilde{x}, (\mathcal{A}^* y)\rangle = \langle \tilde{x}, (c - \tilde{s})\rangle$

Equivalent Primal-Dual Pair in Subspace Form, (e.g. N&N '94)

$\underline{\widehat{\text{Particular solution}}} + \underline{\text{solution of homogeneous equation}}$

$$v_P = \langle c, \tilde{x}\rangle - \inf_s \left\{ \langle s, \tilde{x}\rangle : s \in (\tilde{s} + \mathcal{L}^\perp) \cap K \right\}. \qquad (\mathbb{P})$$

$$v_D = \langle \tilde{y}, b\rangle + \inf_x \left\{ \langle \tilde{s}, x\rangle : x \in (\tilde{x} + \mathcal{L}) \cap K^* \right\}. \qquad (\mathbb{D})$$

(SYMMETRIC) Subspace Form for $(\mathbb{P})$ and $(\mathbb{DP})$

Assume Linear Feasibility for $\tilde{s}, \tilde{y}, \tilde{x}$; with data A, b, c, K

$$\mathcal{A}^*\tilde{y} + \tilde{s} = c \qquad\qquad \mathcal{A}\tilde{x} = b$$

$\mathcal{L}^\perp = \mathcal{R}(\mathcal{A}^*)$ (range) $\qquad \mathcal{L} = \mathcal{N}(\mathcal{A})$ (nullspace)

(innerproducts:) $\langle b, y\rangle = \langle(\mathcal{A}\tilde{x}), y\rangle = \langle \tilde{x}, (\mathcal{A}^*y)\rangle = \langle \tilde{x}, (c - \tilde{s})\rangle$

Equivalent Primal-Dual Pair in Subspace Form, (e.g. N&N '94)

$\underline{\text{Particular solution}} + \underline{\text{solution of homogeneous equation}}$

$$v_P = \langle c, \tilde{x}\rangle - \inf_s \left\{ \langle s, \tilde{x}\rangle : s \in (\tilde{s} + \mathcal{L}^\perp) \cap K \right\}. \qquad (\mathbb{P})$$

$$v_D = \langle \tilde{y}, b\rangle + \inf_x \{\langle \tilde{s}, x\rangle : x \in (\tilde{x} + \mathcal{L}) \cap K^*\}. \qquad (\mathbb{D})$$

Numerical/Stability Advantages of Subspace Form

Single Bilinear Equation For Interior Point Methods

Current p-d i-p methods introduce ill-conditioning (ill-posedness)
Instead, solve:

$$0 = F_\mu(y, v) = (c - \mathcal{A}^* y)(\tilde{x} + \mathcal{V}^* v) - \mu I$$

$\mathcal{L}^\perp = \mathcal{R}(\mathcal{A}^*)$ (range) $\quad \mathcal{L} = \mathcal{N}(\mathcal{A}) = \mathcal{R}(\mathcal{V}^*)$ (nullspace)

Some Applications

- Gauss-Newton method for SDP (Kruk et al 2001)
- GN and CG method for SDP relaxation of Max-Cut (W. 2004)
- Large Scale LP (Wei and W. 2004)

Numerical/Stability Advantages of Subspace Form

Single Bilinear Equation For Interior Point Methods

Current p-d i-p methods introduce ill-conditioning (ill-posedness)
Instead, solve:

$$0 = F_\mu(y, v) = (c - \mathcal{A}^* y)(\tilde{x} + \mathcal{V}^* v) - \mu I$$

$\mathcal{L}^\perp = \mathcal{R}(\mathcal{A}^*)$ (range) $\quad \mathcal{L} = \mathcal{N}(\mathcal{A}) = \mathcal{R}(\mathcal{V}^*)$ (nullspace)

Some Applications

- Gauss-Newton method for SDP (Kruk et al 2001)
- GN and CG method for SDP relaxation of Max-Cut (W. 2004)
- Large Scale LP (Wei and W. 2004)

Characterization of Feasibility for $(\mathbb{P})$, $(\mathbb{DP})$

Recall: Equivalent Primal-Dual Pair in Subspace Form

$$\mathcal{A}^*\tilde{y} + \tilde{s} = c \qquad\qquad \mathcal{A}\tilde{x} = b$$

$$\begin{aligned} v_P &= \langle c, \tilde{x}\rangle - \inf_s \left\{ \langle s, \tilde{x}\rangle : s \in (\tilde{s} + \mathcal{L}^\perp) \cap K \right\} \\ v_D &= \langle \tilde{y}, b\rangle + \inf_x \left\{ \langle \tilde{s}, x\rangle : x \in (\tilde{x} + \mathcal{L}) \cap K^* \right\} \end{aligned}$$

Characterizations

$(\mathbb{P})$ is feasible <u>iff</u> $\tilde{s} \in K + \mathcal{L}^\perp$; $(\mathbb{DP})$ is feasible <u>iff</u> $\tilde{x} \in K^* + \mathcal{L}$

Homogeneous Problems from $(\mathbb{P})$ and $(\mathbb{DP})$

Faces of Recession Directions (feasible case)

$$
\begin{array}{lllll}
f_P^0 & := & \text{face}\, \mathcal{F}_P^s(0) & = & \text{face}\, \left(\mathcal{L}^\perp \cap K\right) \qquad (\subset f_P \text{ and } \subset (f_D^0)^c) \\
f_D^0 & := & \text{face}\, \mathcal{F}_D^x(0) & = & \text{face}\, \left(\mathcal{L} \cap K^*\right) \qquad (\subset f_D \text{ and } \subset (f_P^0)^c)
\end{array}
$$

Recall

minimal faces $f_P = \text{face}\, \mathcal{F}_P^s$ $\quad f_D = \text{face}\, \mathcal{F}_D^x$

Minimal Subspaces/Linear Transformations

min. subsp. $\mathcal{L}_{PM}^\perp := \mathcal{L}^\perp \cap (f_P - f_P)$ $\quad \mathcal{L}_{DM} := \mathcal{L} \cap (f_D - f_D)$

min. Lin. Tr. $\mathcal{A}_{PM}^*$ $\quad \mathcal{A}_{DM}$

Homogeneous Problems from ($\mathbb{P}$) and ($\mathbb{DP}$)

Faces of Recession Directions (feasible case)

$$\begin{array}{llllll} f_P^0 & := & \text{face}\,\mathcal{F}_P^s(0) & = & \text{face}\,\left(\mathcal{L}^\perp \cap K\right) & (\subset f_P \text{ and } \subset (f_D^0)^c) \\ f_D^0 & := & \text{face}\,\mathcal{F}_D^x(0) & = & \text{face}\,(\mathcal{L} \cap K^*) & (\subset f_D \text{ and } \subset (f_P^0)^c) \end{array}$$

Recall

minimal faces $f_P = \text{face}\,\mathcal{F}_P^s$ $f_D = \text{face}\,\mathcal{F}_D^x$

Minimal Subspaces/Linear Transformations

$$\begin{array}{lll} \text{min. subsp.} & \mathcal{L}_{PM}^\perp := \mathcal{L}^\perp \cap (f_P - f_P) & \mathcal{L}_{DM} := \mathcal{L} \cap (f_D - f_D) \\ \text{min. Lin. Tr.} & \mathcal{A}_{PM}^* & \mathcal{A}_{DM} \end{array}$$

Example of Nonclosure for $\mathbb{S}^2_+$

$\left(\mathbb{S}^2_+ + \mathrm{span} f^0_P\right)$ not closed

Let $\mathcal{L}^\perp = \mathcal{R}(\mathcal{A}^*) = \mathrm{span}\,\{E_{22}, E_{12}\}$. Then

$$\mathcal{L} = \mathcal{N}(\mathcal{A}) = \mathrm{span}\,\{E_{11}\}\,, \quad f^0_P = \mathrm{cone}\{E_{22}\}, f^0_D = \mathrm{cone}\{E_{11}\}.$$

$f^0_P = (f^0_D)^c, f^0_D = (f^0_P)^c$, i.e.a str. compl. partition. But,

$$\lim_{i\to\infty}\left(\begin{bmatrix} 1/i & 1 \\ 1 & i \end{bmatrix} - iE_{22}\right) = E_{12} \in \left(\mathbb{S}^2_+ + (f^0_D)^\perp\right) \Big\backslash \left(\mathbb{S}^2_+ + \mathrm{span} f^0_P\right)$$

Examples of Nonclosure in $\mathbb{S}^3_+$

$(\mathbb{S}^3_+ + \mathrm{span} f^0_P)$ not closed $(\dim \mathbb{S}^3_+ = 6)$

Let $\mathcal{L}^\perp = \mathcal{R}(\mathcal{A}^*) = \mathrm{span}\left\{E_{33}, E_{22} + E_{12}\right\}$. Then
$\mathcal{L} = \mathcal{N}(\mathcal{A}) = \mathrm{span}\left\{E_{11}, E_{12}, E_{23}, 2E_{22} - E_{13}\right\}$.
$\left(f^0_P = \mathrm{cone}\{E_{33}\}\right) \subsetneq \left(f^0_D = \mathrm{cone}\{E_{11}\}\right)^c$ NOT strict C.P.

Nonclosed/closed Odd Behaviour for $\mathbb{S}^n_+$

$$\mathbb{S}^3_+ + \mathrm{span} f^0_P \subsetneq \overline{\mathbb{S}^3_+ + \mathrm{span} f^0_P} = \mathbb{S}^3_+ + ((f^0_P)^c)^\perp \subsetneq \mathbb{S}^3_+ + (f^0_D)^\perp.$$

And BOTH $\mathbb{S}^3_+ + \mathcal{L}^\perp$ and $\mathbb{S}^3_+ + \mathcal{L}$ are not closed.

Examples of Nonclosure in $\mathbb{S}^3_+$ cont...

$\left(\mathbb{S}^3_+ + \operatorname{span} f_P^0\right)$ not closed

Moreover, if we choose $\bar{s} = E_{22} \in (f_P^0)^c \cap (f_D^0)^c$, then
$x = E_{12} \in \operatorname{precl}\left(\mathcal{L} + (\operatorname{face}\bar{s})^c\right)$, and
$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \operatorname{precl}\mathcal{A}\left((\operatorname{face}\bar{s})^c\right)$.
(Asymptotic optimality conditions hold at $\bar{s}$.)

Examples of Nonclosure for Nonpolyhedral Cones

K is Necessarily Nonpolyhedral

$F_1 \trianglelefteq K$, $F_2 \trianglelefteq K^*$, are both exposed faces, and $F_2 \subset F_1^c$. If

$$0 \neq \tilde{x} \in F_1^\perp \cap (F_1^c)^\perp \cap F_2^\perp \cap (F_2^c)^\perp,$$

then

$$\tilde{x} \in \mathrm{precl}\,(\overline{K} + \mathrm{span} F_1) \cap \mathrm{precl}\,(K^* + \mathrm{span} F_1^c)$$

and

$$\tilde{x} \in \mathrm{precl}\,(\overline{K} + \mathrm{span} F_2^c) \cap \mathrm{precl}\,(K^* + \mathrm{span} F_2)$$

Regularization of ($\mathbb{P}$) Using Minimal Face

Borwein-W (1981), $f_P = \mathrm{face}\, \mathcal{F}_P^S$

($\mathbb{P}$) is equivalent to regularized ($\mathbb{P}$)

$$v_{RP} := \sup_y \{\langle b, y\rangle \ : \ \mathcal{A}^* y \preceq_{f_P} c\}. \qquad (\mathbb{RP})$$

Lagrangian Dual DRP Satisfies Strong Duality:

$$v_P = v_{RP} = v_{DRP} := \inf_x \{\langle c, x\rangle \ : \ \mathcal{A} x = b, \ x \succeq_{f_P^*} 0\} \qquad (\mathbb{DRP})$$

and v_{DRP} is attained

Regularization of ($\mathbb{P}$) Using Minimal Face

Borwein-W (1981), $f_P = \mathrm{face}\, \mathcal{F}_P^S$

($\mathbb{P}$) is equivalent to regularized ($\mathbb{P}$)

$$v_{RP} := \sup_y \{\langle b, y\rangle \ : \ \mathcal{A}^* y \preceq_{f_P} c\}. \qquad (\mathbb{RP})$$

Lagrangian Dual DRP Satisfies Strong Duality:

$$v_P = v_{RP} = v_{DRP} := \inf_x \{\langle c, x\rangle \ : \ \mathcal{A}x = b,\ x \succeq_{f_P^*} 0\} \qquad (\mathbb{DRP})$$

and v_{DRP} is attained

Regularization of ($\mathbb{P}$) Using Minimal Subspace

Assume K Facially Dual Complete, FDC (Pataki/07, 'nice')

i.e. $F \trianglelefteq K \implies K^* + F^\perp$ is closed. (e.g. $\mathbb{S}^n_+, \mathbb{R}^n_+$,SOC).

$\mathcal{L}_{PM}^\perp = \mathcal{L}^\perp \cap (f_P - f_P)$

$$v_P = v_{RP} = c\tilde{x} - \inf_s \left\{ s\tilde{x} : s \in (\tilde{s} + \mathcal{L}_{PM}^\perp) \cap K \right\} \qquad (\mathbb{RP})$$

Lagrangian Dual DRP Satisfies Strong Duality:

$$v_P = v_{RP} = v_{DRP} = \tilde{y}b + \inf_x \{\tilde{s}x : x \in (\tilde{x} + \mathcal{L}_{PM}) \cap K^*\} \quad (\mathbb{DRP})$$

and v_{DRP} is attained

Regularization of ($\mathbb{P}$) Using Minimal Subspace

Assume K Facially Dual Complete, FDC (Pataki/07, 'nice')

i.e. $F \trianglelefteq K \implies K^* + F^\perp$ is closed. (e.g. $\mathbb{S}^n_+, \mathbb{R}^n_+$, SOC).

$\mathcal{L}^\perp_{PM} = \mathcal{L}^\perp \cap (f_P - f_P)$

$$v_P = v_{RP} = c\tilde{x} - \inf_s \left\{ s\tilde{x} : s \in (\tilde{s} + \mathcal{L}^\perp_{PM}) \cap K \right\} \qquad (\mathbb{RP})$$

Lagrangian Dual DRP Satisfies Strong Duality:

$$v_P = v_{RP} = v_{DRP} = \tilde{y}b + \inf_x \{\tilde{s}x : x \in (\tilde{x} + \mathcal{L}_{PM}) \cap K^*\} \quad (\mathbb{DRP})$$

and v_{DRP} is attained

Regularization of ($\mathbb{P}$) Using Minimal Subspace

Assume K Facially Dual Complete, FDC (Pataki/07, 'nice')

i.e. $F \trianglelefteq K \implies K^* + F^\perp$ is closed. (e.g. $\mathbb{S}^n_+, \mathbb{R}^n_+$,SOC).

$\mathcal{L}_{PM}^\perp = \mathcal{L}^\perp \cap (f_P - f_P)$

$$v_P = v_{RP} = c\tilde{x} - \inf_s \left\{ s\tilde{x} : s \in (\tilde{s} + \mathcal{L}_{PM}^\perp) \cap K \right\} \qquad (\mathbb{RP})$$

Lagrangian Dual DRP Satisfies Strong Duality:

$$v_P = v_{RP} = v_{DRP} = \tilde{y}b + \inf_x \{\tilde{s}x : x \in (\tilde{x} + \mathcal{L}_{PM}) \cap K^*\} \quad (\mathbb{DRP})$$

and v_{DRP} is attained

Nice and Devious Cones

Alternate Form of BW Characterization

$v_P \geq \sup_{y \in g^{-1}(f_P - f_P)} \langle b, y \rangle + \langle (c - \mathcal{A}^* y), \bar{x} \rangle$, for some $\bar{x} \in f_P^* = K^* + f_P^\perp$; WLOG $\bar{x} \in K^*$

Lemma for SDP Case (Ramana,Tuncel,W./97)

Let $0 \neq F \lhd \mathbb{S}_+^n$. Then: $\mathbb{S}_+^n + F^\perp$ is closed (nice)
But: $\mathbb{S}_+^n + \mathrm{span} F^c$ is not closed (devious)

$$\mathbb{S}_+^n + F^\perp = \overline{\mathbb{S}_+^n + \mathrm{span} F^c}$$

Infinite Duality Gap Using $0 \neq F \lhd \mathbb{S}_+^n$.

Let $\mathcal{L} = \mathrm{span} F^c$; $c = \tilde{s} = 0$; $\tilde{x} \in (\mathbb{S}_+^n + F^\perp) \backslash (\mathbb{S}_+^n + \mathrm{span} F^c)$;
then: $0 = v_P < v_D = \infty$.

Nice and Devious Cones

Alternate Form of BW Characterization

$v_P \geq \sup_{y \in g^{-1}(f_P - f_P)} \langle b, y \rangle + \langle (c - \mathcal{A}^* y), \bar{x} \rangle$, for some
$\bar{x} \in f_P^* = K^* + f_P^\perp$; WLOG $\bar{x} \in K^*$

Lemma for SDP Case (Ramana,Tuncel,W./97)

Let $0 \neq F \lhd \mathbb{S}_+^n$. Then: $\mathbb{S}_+^n + F^\perp$ is closed (nice)
But: $\mathbb{S}_+^n + \mathrm{span} F^c$ is <u>not</u> closed (devious)

$\mathbb{S}_+^n + F^\perp = \overline{\mathbb{S}_+^n + \mathrm{span} F^c}$

Infinite Duality Gap Using $0 \neq F \lhd \mathbb{S}_+^n$.

Let $\mathcal{L} = \mathrm{span} F^c$; $c = \tilde{s} = 0$; $\tilde{x} \in (\mathbb{S}_+^n + F^\perp) \backslash (\mathbb{S}_+^n + \mathrm{span} F^c)$;
then: $0 = v_P < v_D = \infty$.

Strong Duality for (P) ($v_P = v_D$ and v_D is attained)

Minimal Face and Minimal Subspace CQs for (P)

1. $f_P = K$ is a CQ
   (from BW: $f_P^* = K^*$)
2. $\mathcal{L}^\perp \cap (f_P - f_P) = \mathcal{L}_{PM}^\perp = \mathcal{L}^\perp$ is a CQ (if K is FDC (nice))
   ($\tilde{s} \in f_P - f_P : x^* = x_K^* + x_f^* \in f_P^* = K^* + f_P^\perp \implies$
   $x^*(\tilde{s} + \mathcal{L}^\perp) = x_K^*(\tilde{s} + \mathcal{L}^\perp)$)

Universal CQ, UCQ for (P) (indep. of feasible data c, b)

$\mathcal{L}^\perp \subset f_P^0 - f_P^0$ is a UCQ (if K is FDC)
(wlog choose $\tilde{s} \in K$, $\tilde{x} \in K^*$; shows that $f_P^0 \subset f_P, f_D^0 \subset f_D$)

Strong Duality for (P) ($v_P = v_D$ and v_D is attained)

Minimal Face and Minimal Subspace CQs for (P)

1. $f_P = K$ is a CQ
   (from BW: $f_P^* = K^*$)
2. $\mathcal{L}^\perp \cap (f_P - f_P) = \mathcal{L}_{PM}^\perp = \mathcal{L}^\perp$ is a CQ (if K is FDC (nice))
   ($\tilde{s} \in f_P - f_P : x^* = x_K^* + x_f^* \in f_P^* = K^* + f_P^\perp \implies$
   $x^*(\tilde{s} + \mathcal{L}^\perp) = x_K^*(\tilde{s} + \mathcal{L}^\perp)$)

Universal CQ, UCQ for (P) (indep. of feasible data c, b)

$\mathcal{L}^\perp \subset f_P^0 - f_P^0$ is a UCQ (if K is FDC)
(wlog choose $\tilde{s} \in K$, $\tilde{x} \in K^*$; shows that $f_P^0 \subset f_P, f_D^0 \subset f_D$)

Our NUMERICAL Goals:

Goals: Derive an Algorithm that Satisfies

1. recognizes if Slater's CQ holds and if ($\mathbb{P}$)–($\mathbb{DP}$) has a zero duality gap (improves on stability/efficiency of B-W algorithm)
2. size of any intermediate cone program solved does not exceed that of ($\mathbb{P}$) or ($\mathbb{DP}$) (improves on size/efficiency of Ramana's dual)
3. intermediate cone programs to be solved are well behaved (in the Slater CQ sense)

Theorem of the Alternative for Slater's CQ

THEOREM

Suppose that $(\mathbb{P})$ is feasible. Then exactly one of the following two systems is consistent:

(1) $\mathcal{A}x = 0$, $\langle c, x\rangle = 0$, and $0 \neq x \succeq_{K^*} 0$

(2) $\mathcal{A}^* y \prec_K c$ (Slater's CQ holds for $(\mathbb{P})$)

Difficult?

In theory, we can solve

$(*)$ $\min\{0 : x \text{ satisfies } (1)\}$

to determine if Slater's CQ fails for $(\mathbb{P})$.

But this problem (*) need not satisfy the generalized Slater CQ!

Theorem of the Alternative for Slater's CQ

THEOREM

Suppose that ($\mathbb{P}$) is feasible. Then exactly one of the following two systems is consistent:

(1) $\mathcal{A}x = 0, \ \langle c, x \rangle = 0, \ \text{and } 0 \neq x \succeq_{K^*} 0$
(2) $\mathcal{A}^* y \prec_K c$ (Slater's CQ holds for ($\mathbb{P}$))

Difficult?

In theory, we can solve

$(*) \quad \min\{0 : x \text{ satisfies } (1)\}$

to determine if Slater's CQ fails for ($\mathbb{P}$).
But this problem (*) need not satisfy the generalized Slater CQ!

Stable Theorem of the Alternative

Stable Auxiliary Problem

Let $e \in \operatorname{int}(K) \cap \operatorname{int}(K^*)$; define $\mathcal{A}_c x := \begin{pmatrix} \mathcal{A}x \\ \langle c, x \rangle \end{pmatrix}$

$$\alpha^* := \left\{ \inf_{x,\alpha} \alpha : \mathcal{A}_c x = 0, x + \alpha e \succeq_{K^*} 0, \langle e, x \rangle \leq 1 \right\} \qquad (\mathcal{A})$$

Properties/Advantages

- size of $(\mathcal{A})$ essentially that of $(\mathbb{DP})$
- A strictly feasible primal-dual point is easily found.
- Apply primal-dual IPM; assume a barrier for K^* such that the central path defined by it converges to a point in the relative interior of the optimal face; follow central path closely at end of algorithm.

Slater's Condition and the Auxiliary problem

Solution to ($\mathcal{A}$) yields info on ($\mathbb{P}$)–($\mathbb{DP}$)

Theorem: The x component of the central path for ($\mathcal{A}$) converges to a point in $\operatorname{relint}(\operatorname{face}(G_P))$, where

$$G_P := \{x \ : \ Ax = 0, \ \langle c, x\rangle = 0, \ x \succeq_{K^*} 0\}.$$

Moreover, since $f_P \subset \{x^*\}^\perp \cap K = [\operatorname{face}(G_P)]^c \trianglelefteq K$, one of the following holds:

1. $\underline{\alpha^* = 0 \text{ and } x^* = 0}$, so Slater's CQ holds for ($\mathbb{P}$), or
2. $\underline{\alpha^* = 0 \text{ and } 0 \neq x^* \succeq_{K^*} 0}$, so $f_P \subset \{x^*\}^\perp \cap K \triangleleft K$, or
3. $\underline{\alpha^* < 0 \text{ and } x^* \succ_{K^*} 0}$, so the generalized Slater CQ holds for ($\mathbb{DP}$).

Algorithm Alternates to Obtain Minimal Representations

Step 1: For Minimal Face

From auxiliary problem, find:
$0 \neq x \in K^*, \{x\}^\perp = H, \{x\}^\perp \cap K \supset f_P$

Step 2: For Minimal Subspace

Find $\mathcal{A}_H$ so that $\mathcal{R}(A_H^*) = \mathcal{R}(A^*) \cap H$
to get reduced problem in H

Repeat Steps 1 and 2.

Dimension of problem is reduced after step 2.
Stops in finite number of iterations.

Previous SDP with $K = \mathbb{S}^3_+$ and a Duality Gap of 1

SeDuMi 1.1 Results

$y^* = \begin{bmatrix} -0.321 \times 10^6 & 0.372 \end{bmatrix}^T$

$s^* = \begin{bmatrix} 0 & 0 & -0.372 \\ - & 0.628 \times 10^5 & 0 \\ - & - & -0.321 \times 10^6 \end{bmatrix};$

desired accuracy (10^{-6}) achieved but!!

$\langle c, x^* \rangle - \langle b, y^* \rangle \approx -0.12$! and s^* is not pos. semidef.

After One Step of the Reduction

Our code yields correct primal solution:

$$y^* = \begin{pmatrix} -1.50 \\ 0 \end{pmatrix}, \quad s^* = \begin{bmatrix} 0 & 0 & 0 \\ - & 1.00 & 0 \\ - & - & 1.50 \end{bmatrix}$$

Higher Dimensional Numerical Experiments

SDP with $m = n \geq 3$, $b = e_2$, $c = 0$

$$\mathcal{A}^* y = \begin{bmatrix} y_1 & y_2 & y_3 & \cdots & y_{n-1} & y_n \\ y_2 & y_3 & & & & \\ \vdots & & & \ddots & & \\ y_{n-1} & & & & y_n & \\ y_n & & & & & 0 \end{bmatrix}$$

SeDuMi/Our Algorithm

SeDuMi gives <u>incorrect</u> primal/dual solution; duality gap of -1; our algorithm gives correct solution

$F_P = \{y \in \mathbb{R}^n : y_1 \leq 0,\ y_2 = \cdots = y_n = 0\}$

min. face $f_P = \{s \in \mathbb{S}^n_+ : s_{11} \geq 0,\ s_{ij} = 0\ \forall (i,j) \neq (1,1)\}$,

and ($\mathbb{DP}$) is infeasible.

(Near) Loss of Slater Condition/Strict Feasibility

Theoretical/Numerical Difficulties

- Primal Slater condition implies strong duality, i.e. zero duality gap AND dual attainment.
- (Near) loss of strict feasibility is used as a measure in complexity theory. (e.g. Renegar/95, Freund/01, Lara and Tuncel/02)
- (Near) loss of strict feasibility correlates with number of iterations and loss of accuracy in interior-point methods (e.g. Freund/Ordonez/Toh 2006)

(Near) Loss of Slater Condition/Strict Feasibility

Theoretical/Numerical Difficulties

- Primal Slater condition implies strong duality, i.e. zero duality gap AND dual attainment.
- (Near) loss of strict feasibility is used as a measure in complexity theory. (e.g. Renegar/95, Freund/01, Lara and Tuncel/02)
- (Near) loss of strict feasibility correlates with number of iterations and loss of accuracy in interior-point methods (e.g. Freund/Ordonez/Toh 2006)

Loss of Strict Complementarity, (SC)

Strict Complementary Optimal Primal-Dual Pair

- There exists an optimal primal-dual pair x, s such that
  $x + s \succ 0 \qquad (\in \operatorname{int}(K + K^*))$

Theoretical Difficulties/Convergence

- Convergence proofs for asymptotic quadratic superlinear convergence require SC.
- Proofs of convergence to the analytic center require SC

Numerical Difficulties/Relation to Duality Gaps???

increased number of iterations? loss of accuracy?

Loss of Strict Complementarity, (SC)

Strict Complementary Optimal Primal-Dual Pair

- There exists an optimal primal-dual pair x, s such that
  $x + s \succ 0 \qquad (\in \operatorname{int}(K + K^*))$

Theoretical Difficulties/Convergence

- Convergence proofs for asymptotic quadratic superlinear convergence require SC.
- Proofs of convergence to the analytic center require SC

Numerical Difficulties/Relation to Duality Gaps???

increased number of iterations? loss of accuracy?

Loss of Strict Complementarity, (SC)

Strict Complementary Optimal Primal-Dual Pair

- There exists an optimal primal-dual pair x, s such that
  $x + s \succ 0 \qquad (\in \operatorname{int}(K + K^*))$

Theoretical Difficulties/Convergence

- Convergence proofs for asymptotic quadratic superlinear convergence require SC.
- Proofs of convergence to the analytic center require SC

Numerical Difficulties/Relation to Duality Gaps???

increased number of iterations? loss of accuracy?

Hard SDP Instances (Wei and W. 2006)

Maximal Complementary Solution Pair:

- A p-d pair of optimal solutions $(\bar{s}, \bar{x})$ is a *maximal complementary solution pair* if the pair maximizes the sum $\mathrm{rank}(s) + \mathrm{rank}(x)$ over all p-d optimal (s, x).

Strict Complementarity Nullity, g:

- $g = n - \mathrm{rank}(\bar{s}) - \mathrm{rank}(\bar{x})$, where $(\bar{s}, \bar{x})$ is a maximal complementary solution pair

Hard SDP Instances:

- problems where nullity is nonzero

Hard SDP Instances (Wei and W. 2006)

Maximal Complementary Solution Pair:

- A p-d pair of optimal solutions $(\bar{s}, \bar{x})$ is a *maximal complementary solution pair* if the pair maximizes the sum $\mathrm{rank}(s) + \mathrm{rank}(x)$ over all p-d optimal (s, x).

Strict Complementarity Nullity, g:

- $g = n - \mathrm{rank}(\bar{s}) - \mathrm{rank}(\bar{x})$, where $(\bar{s}, \bar{x})$ is a maximal complementary solution pair

Hard SDP Instances:

- problems where nullity is nonzero

Hard SDP Instances (Wei and W. 2006)

Maximal Complementary Solution Pair:

- A p-d pair of optimal solutions $(\bar{s}, \bar{x})$ is a *maximal complementary solution pair* if the pair maximizes the sum $\mathrm{rank}(s) + \mathrm{rank}(x)$ over all p-d optimal (s, x).

Strict Complementarity Nullity, g:

- $g = n - \mathrm{rank}(\bar{s}) - \mathrm{rank}(\bar{x})$, where $(\bar{s}, \bar{x})$ is a maximal complementary solution pair

Hard SDP Instances:

- problems where nullity is nonzero

Algorithm for Given Nullity g

Algorithm

- Given: rank of optimum x is $r > 0$; number constraints is $m > 1$
- Let $Q = [Q_P|Q_N|Q_D]$ be orthogonal matrix; dimensions of Q_P, Q_N, Q_D are $n \times r$, $n \times g$, $n \times (n - r - g)$, resp.
  Construct: $x := Q_P D_x Q_P^T \quad s := Q_D D_s Q_D^T$,
  where $D_x \succ 0$ and $D_s \succ 0$.
- cont...

Algorithm cont...

Algorithm cont...

- Define
$$A_1 = [Q_P|Q_N|Q_D] \begin{bmatrix} 0 & 0 & Y_2^T \\ 0 & Y_1 & Y_3^T \\ Y_2 & Y_3 & Y_4 \end{bmatrix} [Q_P|Q_N|Q_D]^T,$$
where $Y_1 \succ 0$, Y_4 symmetric, and $Q_D Y_2 \neq 0$.
- Choose $A_i \in \mathbb{S}^n$, with $\{A_1 Q_P, A_2 Q_P, \dots, A_m Q_P\}$ lin. indep. (Note $A_1 Q_P = Q_D Y_2 \neq 0$.)
- Set $b := \mathcal{A}(x)$, $\quad c := \mathcal{A}^*(y) + s$, with $y \in \Re^m$ random.

Theorem for Generating Hard Instances

Theorem

The data $(\mathcal{A}, b, c)$ constructed in the above algorithm gives a *hard* SDP instance with a strict complementarity nullity g.

Proof Outline

- Step 2 guarantees $s, x \succeq 0, sx = 0$ but strict complementarity fails.
- step 5 guarantees primal-dual feasibility, i.e. s,x are an optimal pair.
- Steps 3,4 guarantee s, x are a maximal complementary solution pair.

Generating Hard Instances with Slater Condition

Corollary

With data $(\mathcal{A}, b, c)$ constructed using above algorithm:

1. If the following additional condition on A_2 is satisfied $[Q_P|Q_N]^T A_2 [Q_P|Q_N] \succ 0$, then Slater's CQ holds for (P).
2. If the following additional conditions on $A_i, i = 1, \ldots, m$, are satisfied,

$$\begin{array}{c} \mathrm{trace} Y_4 = -\mathrm{trace} Y_1 \\ \alpha > 0, \quad \mathrm{trace} A_i x = \alpha \mathrm{trace} A_i, \quad i = 2, \ldots, m, \end{array}$$

then $\hat{x} = \alpha I \succ 0$ is feasible for (D)

Empirical Observations

Numerical Difficulties Correlate with Large Nullity

- There is a strong correlation between the iteration number to achieve the desired stopping tolerance and the size of the complementarity nullity, when the accuracy requirement is high.
- Large nullity instances cause problems for SDPT3 solver.
- Local asymptotic convergence rate is slower when nullity is larger.

Theoretical Connections Complementarity/Duality?

Numerical Difficulties

(Both) loss of Slater CQ (strict feasibility) and loss of strict complementarity independently result in numerical difficulties for interior-point methods.

Theoretical Connection?

Is there a theoretical connection between loss of duality (from loss of a CQ) and loss of strict complementarity?

Complementarity Partition

Recall Faces of Recession Directions

$$f_P^0 := \mathrm{face}\left(\mathcal{L}^\perp \cap K\right), \qquad f_D^0 := \mathrm{face}\left(\mathcal{L} \cap K^*\right)$$

The pair f_P^0, f_D^0 define a Complementarity Partition

$\mathrm{face}\,(f_P^0) \subset \mathrm{face}\,(f_D^0)^c$ and $\mathrm{face}\,(f_D^0) \subset \mathrm{face}\,(f_P^0)^c$.
it is a strict complementarity partition if both
$\left(\mathrm{face}\,(f_P^0)\right)^c = \mathrm{face}\,(f_D^0)$ and $\left(\mathrm{face}\,(f_D^0)\right)^c = \mathrm{face}\,(f_P^0)$;
it is proper if f_P^0 and f_D^0 are both nonempty.

Homogeneous Problem in LP Case

Proper compl. part. is strict for LP! (Goldman-Tucker)

SDP Picture

For SDP (after a rotation)

$$\begin{bmatrix} f_D^0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & f_P^0 \end{bmatrix}$$

Form Primal-Dual Pair

$$\tilde{x} = \tilde{s} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & v \succ 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \langle s, x \rangle \geq \|v\|_F^2,$$

for all feasible pairs s, x.

Strict Complementarity and Nonzero Gaps

Theorem: K is a proper cone

(1) If f_P^0, f_D^0 define a proper complementarity partition but not a strict complementarity partition, then there exists $\bar{s}$ and $\bar{x}$ such that $(\mathbb{P})$–$(\mathbb{DP})$ with data $(\mathcal{L}, K, \bar{s}, \bar{x})$ has a finite nonzero duality gap.

(Partial Converse)

(2) If (a) $(\mathbb{P})$–$(\mathbb{DP})$ with data $(\mathcal{L}, K, \bar{s}, \bar{x})$ has a finite nonzero duality gap with both optimal values attained, and (b) the objective functions are constant along all recession directions of $(\mathbb{P})$ and $(\mathbb{DP})$, then f_P^0, f_D^0 has a proper complementarity partition but not a strict complementarity partition.

Characterization of Zero Duality Gap

Cones of Feasible Directions (Tangent Cones)

$\mathcal{D}_P^{\leq}(\bar{y}) = \operatorname{cone}(\mathcal{F}_P^y - \bar{y})$ $\mathcal{D}_D^{\leq}(\bar{x}) = \operatorname{cone}(\mathcal{F}_D^x - \bar{x})$

Assume both (P),(DP) attained

THEOREM Suppose that K is closed and $\mathcal{A}^*\bar{y} + \bar{s} = c$. Then

$$\bar{s} \in \mathcal{O}_P^s, \bar{x} \in \mathcal{O}_D^x, \quad \bar{s}\bar{x} > 0,$$

if and only if

$$b \in \operatorname{precl} \mathcal{A}((K - \bar{s})^*), \quad c \in \operatorname{precl} \left(\mathcal{L}^\perp + (K^* - \bar{x})^*\right).$$

$((K - \bar{s})^* = \operatorname{face} \bar{s})^c)$

Conclusion

- Minimal Representations of the data regularize (P) min. face f_P and/or the min. L.T. $\mathcal{A}_{PM}$ or $\mathcal{L}^*_{PM}$
- presented a stable algorithm to solve (feasible) conic problems for which Slater's CQ fails
- Failure of strict complementarity for the associated recession problems is closely related to the existence of instances having a finite nonzero duality gap; and, both related to lack of closure; provides a means of generating instances for testing.

HAPPY JULY FOURTH/ Questions?