Strong Duality and Stability in Conic Convex Optimization

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Motivation

Strong Duality Failure/Absence of Constraint Qualification, CQ

- Fresh look at known Characterizations of Optimality using Subspace Formulation
- Instances: SDP relaxations for hard combinatorial problems (e.g. QAP, GP, strengthened MC)

Regularization, EFficient Solutions

- We can still solve many problems if they are properly regularized.

Connections to Complementarity (for nonpolyhedral problems)

- Surprising Connections to Complementarity of Homogeneous Problem

Outline

Cone Optimization, (e.g. $K = \mathbb{S}^n_+$, SDP)

Primal-Dual Pair of Optimization Problems in Conic Form

$$\text{(finite)} \quad v_P = \sup_y \{\langle b, y\rangle \; : \; \mathcal{A}^* y \preceq_K c\}, \qquad (\mathbb{P})$$

$$(v_P \le) \quad v_D = \inf_x \{\langle c, x\rangle \; : \; \mathcal{A}x = b, \; x \succeq_{K^*} 0\}. \qquad (\mathbb{D})$$

where

- $\mathcal{A}$ - an onto linear transformation; adjoint is $\mathcal{A}^*$
- K - a proper convex cone with dual/polar cone $K^* = \{x : \langle s, x\rangle \ge 0, \; \forall s \in K\}$.
- $s' \preceq_K s'' (s' \prec_K s'')$ - partial order, $s'' - s' \in K (\in \operatorname{int} K)$

Cone Optimization, (e.g. $K = \mathbb{S}^n_+$, SDP)

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Faces of Cones

Face

A convex cone F is a face of K, denoted $F \trianglelefteq K$, if

$$x, y \in K \text{ and } x + y \in F \implies x, y \in F.$$

If $F \trianglelefteq K$ and $F \neq K$, write $F \triangleleft K$.

Conjugate Face

If $F \trianglelefteq K$, the conjugate face (or complementary face) of F is

$$F^c := F^\perp \cap K^* \trianglelefteq K^*.$$

If $x \in \mathrm{ri}(F)$, then $F^c = \{x\}^\perp \cap K^*$.

Minimal Face (Minimal Cone)

Feasible sets

$$\mathcal{F}_P^y := \{y : c - \mathcal{A}^* y \succeq_K 0\}$$
$$\mathcal{F}_P^s := \{s : s = c - \mathcal{A}^* y \succeq_K 0, \text{ for some } y\}$$
$$\mathcal{F}_D^x := \{x : \mathcal{A}x = b, x \succeq_K^* 0\}$$

Minimal Faces

$$f_P := \text{face} \mathcal{F}_P^s \trianglelefteq K \qquad f_D := \text{face} \mathcal{F}_D^x \trianglelefteq K^*$$

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SDP Example from Ramana, 1995

Primal SDP

$$0 = v_P = \sup_y \left\{ y_2 \; : \; \begin{pmatrix} y_2 & 0 & 0 \\ 0 & y_1 & y_2 \\ 0 & y_2 & 0 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$y^* = \begin{pmatrix} y_1^* & 0 \end{pmatrix}^T, \quad y_1^* \leq 0, \quad s^* = c - \mathcal{A}^* y^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -y_1^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Slater's CQ fails for primal and dual; $v_D = 1 > v_P = 0$

Minimal Face for Ramana Example

Feasible Set/Minimal Face

$\mathcal{F}_P^y = \{y \in \mathbb{R}^2 : y_1 \leq 0,\ y_2 = 0\}$

$$\begin{array}{rcl} f_P & = & \bigcap\{F \trianglelefteq K : c - \mathcal{A}^*(\mathcal{F}_P^y) \subset F\} \\ & = & \begin{pmatrix} \mathbb{S}^2_+ & 0 \\ 0 & 0 \end{pmatrix} \\ & \lhd & \mathbb{S}^3_+ \end{array}$$

Slater CQ and Minimal Face

If $(\mathbb{P})$ is feasible, then

$$c - \mathcal{A}^* y \not\succ_K 0\ \forall y \text{ (Slater's CQ fails for } (\mathbb{P}) \text{) } \iff f_P \lhd K$$

(SYMMETRIC) Subspace Form for $(\mathbb{P})$ and $(\mathbb{D})$

Assume Linear Feasibility for $\tilde{s}, \tilde{y}, \tilde{x}$; with data A, b, c, K

$$\mathcal{A}^*\tilde{y} + \tilde{s} = c \qquad \mathcal{A}\tilde{x} = b$$

$$\mathcal{L}^\perp = \mathcal{R}(\mathcal{A}^*) \text{ (range)} \qquad \mathcal{L} = \mathcal{N}(\mathcal{A}) \text{ (nullspace)}$$

Equivalent Primal-Dual Pair in Subspace Form

Particular solution + solution of homogeneous equation

$$v_P = c\tilde{x} - \inf_s \left\{ s\tilde{x} : s \in (\tilde{s} + \mathcal{L}^\perp) \cap K \right\}. \qquad (\mathbb{P})$$

$$v_D = \tilde{y}b + \inf_x \{ \tilde{s}x : x \in (\tilde{x} + \mathcal{L}) \cap K^* \}. \qquad (\mathbb{D})$$

(SYMMETRIC) Subspace Form for ($\mathbb{P}$) and ($\mathbb{D}$)

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For $(\mathbb{P})$ and $(\mathbb{D})$

Faces of Recession Directions

$$f_P^0 := \text{face}\left(\mathcal{L}^\perp \cap K\right), \qquad f_D^0 := \text{face}\left(\mathcal{L} \cap K^*\right)$$

Recall

minimal faces $\quad f_P = \text{face}\mathcal{F}_P^s \quad f_D = \text{face}\mathcal{F}_D^x$

Minimal Subspaces/Linear Transformations

min. subsp. $\quad \mathcal{L}_{PM}^\perp := \mathcal{L}^\perp \cap (f_P - f_P) \quad \mathcal{L}_{DM} := \mathcal{L} \cap (f_D - f_D)$

min. Lin. Tr. $\quad \mathcal{A}_{PM}^* \quad \mathcal{A}_{DM}$

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Regularization of $(\mathbb{P})$ Using Minimal Face

Borwein-W (1981), $f_P = \mathrm{face}\mathcal{F}_P^s$

$(\mathbb{P})$ is equivalent to regularized $(\mathbb{P})$

$$v_{RP} := \sup_y \{\langle b, y\rangle \ : \ \mathcal{A}^* y \preceq_{f_P} c\}. \qquad (\mathbb{RP})$$

Lagrangian Dual DRP Satisfies Strong Duality:

$$v_P = v_{RP} = v_{DRP} := \inf_x \{\langle c, x\rangle \ : \ \mathcal{A}x = b, \ x \succeq_{f_P^*} 0\} \qquad (\mathbb{DRP})$$

and v_{DRP} is attained

Regularization of $(\mathbb{P})$ Using Minimal Subspace

Assume K Facially Dual Complete, FDC (Pataki/07, 'nice')

$0 \neq F \lhd K \implies K^* + F^\perp$ is closed. (e.g. $\mathbb{S}^n_+, \mathbb{R}^n_+$,SOC).

$\mathcal{L}_{PM}^\perp = \mathcal{L}^\perp \cap (f_P - f_P)$

$$v_{RP} = c\tilde{x} - \inf_s \left\{ s\tilde{x} : s \in (\tilde{s} + \mathcal{L}_{MP}^\perp) \cap K \right\} \qquad (\mathbb{RP})$$

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and v_{DRP} is attained

Strong Duality for (P) ($v_P = v_D$ and v_D is attained)

Minimal Face and Minimal Subspace CQs for (P)

1. $f_P = K$ is a CQ
2. $\mathcal{L}^\perp \cap (f_P - f_P) = \mathcal{L}_{PM}^\perp = \mathcal{L}^\perp$ is a CQ (if K is FDC)

Universal CQ, UCQ (i.e. independent of data c, b) for (P)

$\mathcal{L}^\perp \subset f_P^0 - f_P^0$ is a UCQ (if K is FDC)

Strong Duality for (P) ($v_P = v_D$ and v_D is attained)

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Our Goals:

Goals: Derive an Algorithm that Satisfies

1. recognizes if Slater's CQ holds and if $(\mathbb{P})$–$(\mathbb{D})$ has a zero duality gap (improves on stability/efficiency of B-W algorithm)
2. size of any intermediate cone program solved does not exceed that of $(\mathbb{P})$ or $(\mathbb{D})$ (improves on size/efficiency of Ramana's dual)
3. intermediate cone programs to be solved are well behaved

Theorem of the Alternative for Slater's CQ

THEOREM

Suppose that ($\mathbb{P}$) is feasible. Then exactly one of the following two systems is consistent:

(1) $\mathcal{A}x = 0,\ \langle c, x\rangle = 0,$ and $0 \neq x \succeq_{K^*} 0$
(2) $\mathcal{A}^* y \prec_K c$ (Slater's CQ holds for ($\mathbb{P}$))

Difficult?

In theory, we can solve $\min\{0 : x \text{ satisfies } (1)\}$ to determine if Slater's CQ fails for ($\mathbb{P}$).
But this problem need not satisfy the generalized Slater CQ.
So how can we solve (1)?

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Stable Theorem of the Alternative

Stable Auxiliary Problem

Let $e \in \operatorname{int}(K) \cap \operatorname{int}(K^*)$; define $\mathcal{A}_c x := \begin{pmatrix} \mathcal{A}x \\ \langle c, x \rangle \end{pmatrix}$

$$\alpha^* := \left\{ \inf_{x,\alpha} \alpha : \mathcal{A}_c x = 0, x + \alpha e \succeq_{K^*} 0, \langle e, x \rangle \leq 1 \right\} \qquad (\mathcal{A})$$

Properties/Advantages

- size of $(\mathcal{A})$ essentially that of $(\mathbb{D})$
- A strictly feasible primal-dual point is easily found.
- Apply primal-dual IPM; assume a barrier for K^* such that the central path defined by it converges to a point in the relative interior of the optimal face; follow central path closely at end of algorithm.

Slater's Condition and the Auxiliary problem

Solution to ($\mathcal{A}$) yields info on ($\mathbb{P}$)–($\mathbb{D}$)

Theorem: The x component of the central path for ($\mathcal{A}$) converges to a point in $\mathrm{ri}(\mathrm{face}(G_P))$, where

$$G_P := \{x \ : \ Ax = 0, \ \langle c, x\rangle = 0, \ x \succeq_{K^*} 0\}.$$

Moreover, since $f_P \subset \{x^*\}^\perp \cap K = [\mathrm{face}(G_P)]^c \trianglelefteq K$, one of the following holds:

1. $\alpha^* = 0 \text{ and } x^* = 0$, so Slater's CQ holds for ($\mathbb{P}$), or
2. $\alpha^* = 0 \text{ and } 0 \neq x^* \succeq_{K^*} 0$, so $f_P \subset \{x^*\}^\perp \cap K \triangleleft K$, or
3. $\alpha^* < 0 \text{ and } x^* \succ_{K^*} 0$, so the generalized Slater CQ holds for ($\mathbb{D}$).

Algorithm Alternates to Obtain Minimal Representations

For Minimal Face

From auxiliary problem, find:
$0 \neq x \in K^*, \{x\}^\perp = H, \{x\}^\perp \cap K \supset f_P$

For Minimal Subspace

Find $\mathcal{A}_H$ so that $\mathcal{R}(A_H^*) = \mathcal{R}(A^*) \cap H$
to get reduced problem in H

Previous SDP with $K = \mathbb{S}^3_+$ and a Duality Gap of 1

SeDuMi 1.1 Results

$$y^* = \begin{pmatrix} -0.321 \times 10^6 & 0.372 \end{pmatrix}^T$$
$$s^* = \begin{pmatrix} 0.628 \times 10^5 & 0 & 0 \\ 0 & -0.321 \times 10^6 & -0.372 \\ 0 & -0.372 & 0 \end{pmatrix};$$

desired accuracy (10^{-6}) achieved but!!

$\langle c, x^* \rangle - \langle b, y^* \rangle \approx -0.12$! and s^* is not pos. semidef.

After One Step of the Reduction

Our code yields correct primal solution:

$$y^* = \begin{pmatrix} -1.50 \\ 0 \end{pmatrix}, \quad s^* = \begin{pmatrix} 1.00 & 0 & 0 \\ 0 & 1.50 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Higher Dimensional Numerical Experiments

SDP with $m = n \geq 3$, $b = e_2$, $c = 0$

$$\mathcal{A}^* y = \begin{pmatrix} y_1 & y_2 & y_3 & & \cdots & y_{n-1} & y_n \\ y_2 & y_3 & & & & & \\ \vdots & & & & \ddots & & \\ y_{n-1} & & & & & y_n & \\ y_n & & & & & & 0 \end{pmatrix}$$

SeDuMi/Our Algorithm

SeDuMi gives incorrect primal/dual solution; duality gap of -1; our algorithm gives correct solution
$F_P = \{y \in \mathbb{R}^n : y_1 \leq 0,\ y_2 = \cdots = y_n = 0\}$
min. face $f_P = \{Z \in \mathbb{S}^n_+ : Z_{11} \geq 0,\ Z_{ij} = 0\ \forall (i,j) \neq (1,1)\}$,
and $(\mathbb{D})$ is infeasible.

Complementarity Partition

Recall Faces of Recession Directions

$$f_P^0 := \text{face}\left(\mathcal{L}^\perp \cap K\right), \qquad f_D^0 := \text{face}\left(\mathcal{L} \cap K^*\right)$$

The pair f_P^0, f_D^0 define a Complementarity Partition

$\text{face}(f_P^0) \subset \text{face}(f_D^0)^c$ and $\text{face}(f_D^0) \subset \text{face}(f_P^0)^c$.
it is a strict complementarity partition if also
$[\text{face}(f_P^0)]^c = \text{face}(f_D^0)$ (equiv. $[\text{face}(f_P^0)]^c \cap [\text{face}(f_D^0)]^c = \{0\}$);
it is proper if f_P^0 and f_D^0 are both nonempty.

Strict Complementarity and Nonzero Gaps

Theorem: K is a proper cone

(1) If f_P^0, f_D^0 define a proper complementarity partition but not a strict complementarity partition, then there exists $\bar{s}$ and $\bar{x}$ such that $(\mathbb{P})$–$(\mathbb{D})$ with data $(\mathcal{L}, K, \bar{s}, \bar{x})$ has a finite nonzero duality gap.

(Partial Converse)

(2) If (a) $(\mathbb{P})$–$(\mathbb{D})$ with data $(\mathcal{L}, K, \bar{s}, \bar{x})$ has a finite nonzero duality gap with both optimal values attained, and (b) the objective functions are constant along all recession directions of $(\mathbb{P})$ and $(\mathbb{D})$, then f_P^0, f_D^0 has a proper complementarity partition but not a strict complementarity partition.

Conclusion

- Minimal Representations of the data regularize (P) min. face f_P and/or the min. L.T. $\mathcal{A}_{PM}$ or $\mathcal{L}^*_{PM}$
- presented a stable algorithm to solve (feasible) conic problems for which Slater's CQ fails
- Failure of strict complementarity for the associated recession problems is closely related to the existence of instances having a finite nonzero duality gap; provides a means of generating instances for testing.