# Generating and Measuring Instances of Hard Semidefinite Programs

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# Outline

# SDP Models

## Primal SDP

$$(PSDP) \qquad \begin{array}{rll} p^* := & \min & \operatorname{trace} CX \\ & \text{s.t.} & \mathcal{A}(X) = b \\ & & X \succeq 0 \end{array}$$

## Dual SDP

$$(DSDP) \qquad \begin{array}{rll} d^* := & \max & b^T y \\ & \text{s.t.} & \mathcal{A}^*(y) + Z = C \\ & & Z \succeq 0, \end{array}$$

where $C, X, Z \in \mathcal{S}^n$, $n \times n$ real symmetric matrices, $y, b \in \Re^m$,
$(\succeq) \succ$ denotes positive (semi)definiteness.
$\mathcal{P}^*$ and $\mathcal{D}^*$ - are sets of optimal primal and dual solutions

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# (Near) Loss of Slater Condition/Strict Feasibility

## Primal/Dual Slater

- $\mathcal{A}(X) = b, X \succ 0$
- $\mathcal{A}^*(y) \prec C$

## Theoretical/Numerical Difficulties

- Primal Slater condition implies strong duality, i.e. zero duality gap AND dual attainment.
- (Near) loss of strict feasibility is used as a measure in complexity theory. (e.g. Renegar/95, Freund/01, Lara and Tuncel/02)
- (Near) loss of strict feasibility correlates with number of iterations and loss of accuracy in interior-point methods (e.g. Freund/Ordonez/Toh 2006)

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# Loss of Strict Complementarity, (SC)

## Strict Complementary Optimal Primal-Dual Pair

- There exists an optimal primal-dual pair $X, Z$ such that

$$X + Z \succ 0$$

## Theoretical Difficulties

- Convergence proofs for asymptotic quadratic superlinear convergence require SC.
- Proofs of convergence to the analytic center require SC

## Numerical Difficulties ???

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# Other Theoretical/Numerical Difficulties

- The Primal-Dual: Feasibility , Complementary Slackness Optimality Conditions are overdetermined; symmetrizations to use Newton's method result in ill-posed problems.
- High accuracy solutions are hard/impossible to obtain; sparsity is hard to exploit; large scale is still an open area of research.

# Hard SDP Instances/Definitions

## Maximal Complementary Solution Pair:

- A primal-dual pair of optimal solutions $(\bar{X}, \bar{Z}) \in \mathcal{P}^* \times \mathcal{D}^*$ is called a *maximal complementary solution pair* to the problems (PSDP) and (DSDP), if the pair maximizes the sum $\operatorname{rank}(X) + \operatorname{rank}(Z)$ over all primal-dual optimal solution pairs $(X, Z)$.

## Strict Complementarity Nullity:

- $g = n - \operatorname{rank}(\bar{X}) - \operatorname{rank}(\bar{Z})$, where $(\bar{X}, \bar{Z})$ is a maximal complementary solution pair

## Hard SDP Instances:

- problems where strict complementarity fails (nullity is nonzero)

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# Algorithm for Given Nullity $g$

## Algorithm

- Given: positive integers $r > 0$ and $m > 1$: the rank of an optimum $X$ and the number of constraints, resp.
- Let $Q = [Q_P | Q_N | Q_D]$ be an orthogonal matrix, where the dimensions of $Q_P$, $Q_N$, $Q_D$ are $n \times r$, $n \times g$, $n \times (n - r - g)$, respectively, and $r > 0$. Construct positive semidefinite matrices $X$ and $Z$ as follows:

$$X := Q_P D_X Q_P^T, \qquad Z := Q_D D_Z Q_D^T,$$

  where $D_X$ and $D_Z$ are diagonal positive definite.
- cont...

# Algorithm cont...

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- Define

$$A_1 = [Q_P|Q_N|Q_D] \begin{bmatrix} 0 & 0 & Y_2^T \\ 0 & Y_1 & Y_3^T \\ Y_2 & Y_3 & Y_4 \end{bmatrix} [Q_P|Q_N|Q_D]^T,$$

  where $Y_1$, $Y_2$, $Y_3$, and $Y_4$ are block matrices of appropriate dimensions, $Y_1 \succ 0$, $Y_4$ symmetric, and $Q_D Y_2 \neq 0$.
- Choose $A_i \in \mathcal{S}^n, i = 2, \ldots, m$, such that $\{A_1 Q_P, A_2 Q_P, \ldots, A_m Q_P\}$ is a linearly independent set. (Note that $A_1 Q_P = Q_D Y_2 \neq 0$.)
- Set $\quad b := \mathcal{A}(X), \qquad C := \mathcal{A}^*(y) + Z,$ with $y \in \Re^m$ randomly generated.

# Theorem for Generating Hard Instances

## Theorem

The data $(\mathcal{A}, b, C)$ constructed in the above algorithm gives a *hard* SDP instance with a strict complementarity nullity $g$.

## Proof Outline

- Step 2 guarantees $X, Z \succeq 0, XZ = 0$ but strict complementarity fails.
- step 5 guarantees primal-dual feasibility, i.e. XZ are an optimal pair.
- Steps 3,4 guarantee $X, Z$ are a maximal complementary solution pair.

# Generating Hard Instances with Slater Condition

## Corollary

With data $(\mathcal{A}, b, C)$ constructed using above algorithm:

1. If the following additional condition on $A_2$ is satisfied

$$[Q_P|Q_N]^T A_2 [Q_P|Q_N] \succ 0,$$

then Slater's condition holds for the dual program (DSDP).

2. If the following additional conditions on $A_i, i = 1, \ldots, m$, are satisfied,

$$\begin{array}{c} \operatorname{trace} Y_4 = -\operatorname{trace} Y_1 \\ \alpha > 0, \quad \operatorname{trace} A_i X = \alpha \operatorname{trace} A_i, \quad i = 2, \ldots, m, \end{array}$$

then $\hat{X} = \alpha I \succ 0$ is feasible for (PSDP).

# Measures $g_t, g_s, \kappa$

## $g_t, g_s, \kappa$

- $g_t := |\{w_i^d : T_l < w_i^d < T_u\}|$,
  where $w^d = \frac{1}{2}\lambda\left(X^{-1}Z + ZX^{-1}\right)$; $T_l$, $T_u$ are two tolerances.
  ($O(1)$ indicates nullity; $0, \infty$ otherwise)
- $g_s := |\{w_j^s : w_j^s \leq T\}|$, where $w^s := \frac{1}{2\sqrt{\mu}}\lambda(X+Z)$; $T$ is a tolerance.
- (proposed by Freund, Ordóñez, and Toh.) given tolerance $T$, $T^s := \{j : w_j^s \leq T\}$.,

$$\kappa := -\sum_{j \in T^s} \ln(w_j^s) / |\{j : w_j^s \leq T\}|. \qquad (1)$$

[small (large) for str. compl holds (fails) ]

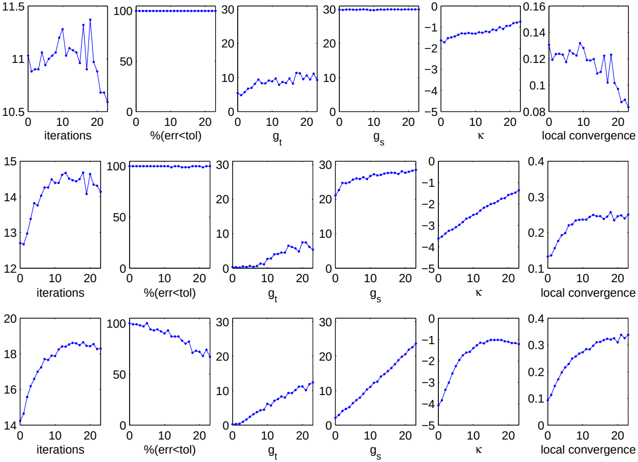
# Randomly Generated Instances

## Random Instances

- We generate 100 instances for each nullity value 0:23.
- The Slater conditions hold for both primal and dual.
- The x-axis of each figure represents the complementarity nullity ranging from 0 to 23.
- On y-axes from left to right:

  *iteration numbers; number of instances that satisfy the desired accuracy; measure* $g_t$*; measure* $g_s$*; measure* $\kappa$*; local convergence rate.*

# Plots




Slater conditions hold; stop tolerances: $10^{-8}, 10^{-10}, 10^{-12}$
(using average of instances that attained desired accuracy)

# Observations

## Observations

- There is a *strong correlation* between the iteration number to achieve the desired stopping tolerance and the size of the complementarity nullity, when the accuracy requirement is high.
- Large nullity instances cause problems for SDPT3 solver.
- The measures $g_t$, $g_s$ and $\kappa$ all improve as the accuracy increases. Measure $\kappa$ correlates well with the iteration number.
- Local asymptotic convergence rate is slower when nullity is larger.

# Correlation Matrix

The local convergence rate is for stop tolerance is $10^{-12}$.

| | Nullity | Iterations | $g_t$ | $g_s$ | $\kappa$ | Cvg. Rate |
|---|---|---|---|---|---|---|
| Nullity | 1.0000 | 0.8341 | 0.9921 | 0.9993 | 0.8430 | 0.9263 |
| Iterations | 0.8341 | 1.0000 | 0.8417 | 0.8191 | 0.9958 | 0.9763 |
| $g_t$ | 0.9921 | 0.8417 | 1.0000 | 0.9897 | 0.8567 | 0.9274 |
| $g_s$ | 0.9993 | 0.8191 | 0.9897 | 1.0000 | 0.8280 | 0.9164 |
| $\kappa$ | 0.8430 | 0.9958 | 0.8567 | 0.8280 | 1.0000 | 0.9785 |
| Cvg. Rate | 0.9263 | 0.9763 | 0.9274 | 0.9164 | 0.9785 | 1.0000 |

- nullity correlates well with iterations as well as local convergence rate.
- The measure $\kappa$ has highest correlation with iterations.

# SDPT3 Iteration Numbers on SDPLIB Instances

- Measures $g_t$, $g_s$ and $\kappa$ are applied on the SDPLIB instances.
- We only consider those SDP instances (47 such instances), where the error obtained was less than $10^{-7}$. The correlation between the measures and the iteration numbers are:

$$\operatorname{corr}(g_t, \text{its}) = 0.1472, \ \operatorname{corr}(g_s, \text{its}) = 0.4509,$$

$$\text{and } \operatorname{corr}(\kappa, \text{its}) = 0.4371.$$

- Low accuracy in solutions yield low correlations.

# SDPLIB Instances

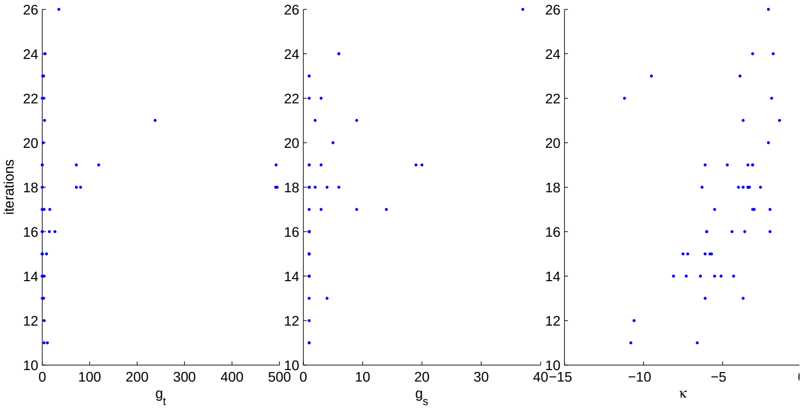



Figure: Scatter plots of $g_t, g_s, \kappa$ versus # iterations for SDPLIB instances with attained tolerance $< 10^{-7}$.

# Conclusion

## Generating Hard Instances

We have presented an algorithm for generating hard SDP instances, i.e. problem instances where we can control the complementarity nullity, $g$.

## Two Empirical Measures for $g$

We introduced two empirical measures $g_t$ and $g_s$ for $g$. These two measures provide accurate measurement of $g$ if high accuracy solutions are used.

## Measures: Positive Correlation with Numerical Difficulties

The numerical tests show: number of iterations and the numerical accuracy are positively correlated to $g$.

# Thanks!! Enjoy ICCOPT/MOPTA

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