# Euclidean Distance Matrix Completions, Sensor Network Localization, and *Graph* Realization

## *EDM, SNL, SDP*

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# Outline

$\boldsymbol{SNL}$ Problem Definition

Problem Formulation - $\boldsymbol{SNL}$ is actually $\boldsymbol{EDMC}$ !!

- (set of anchors equiv. clique in *graph*)

Matrix Reformulation, $\boldsymbol{EDMC}$

- Semidefinite Programming connection

- distance geometry

SDP Relaxation of Hard Constraint $\bar{Y} = PP^T$ (formulations)

Facial Reduction (for regularity/Slater condition)- Reduced Problem Model

- reduction using cliques

Adjoints/Duality/Underdetermined/Stability for $\boldsymbol{EDMC - R}$

Primal-Dual Bilinear Optimality Conditions (overdetermined)

**Robust Interior-Point algorithm**

- Gauss-Newton Direction, crossover, exact p-d feasibility, preconditioning

# Problem

- Ad hoc wireless sensor network
- A few anchors (e.g. with GPS/bulky) have fixed, known locations
- sensors within given (radio) range have known (approx.) distance measurements
- Problem: Determine positions of all sensors
- Params.: Radio range, # anchors, noise level
- Semidefinite Relaxations/Robust Algorithm

# Problem Applications

- health, military, home
- natural habitat monitoring, earthquake detection, weather/current monitoring
- random deployment in inaccessible terrains or disaster relief operations
- Future ?: bicycles (prevent theft); guns; students/teenagers - prevent class absence sensors everywhere - by 2084!

# Problem Example - with Radio Range

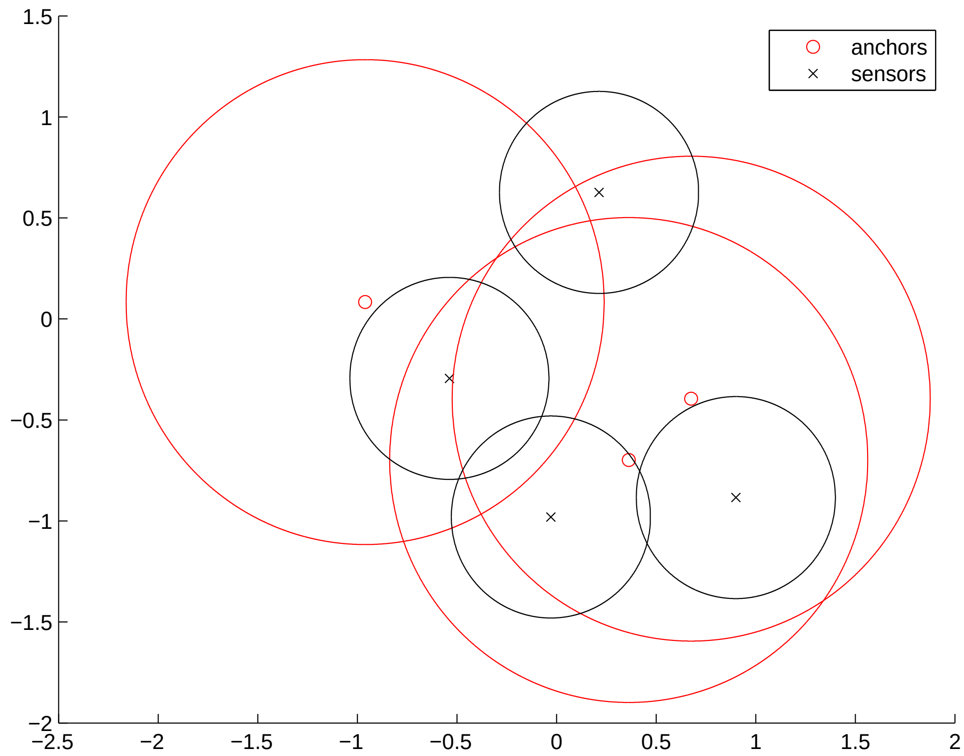
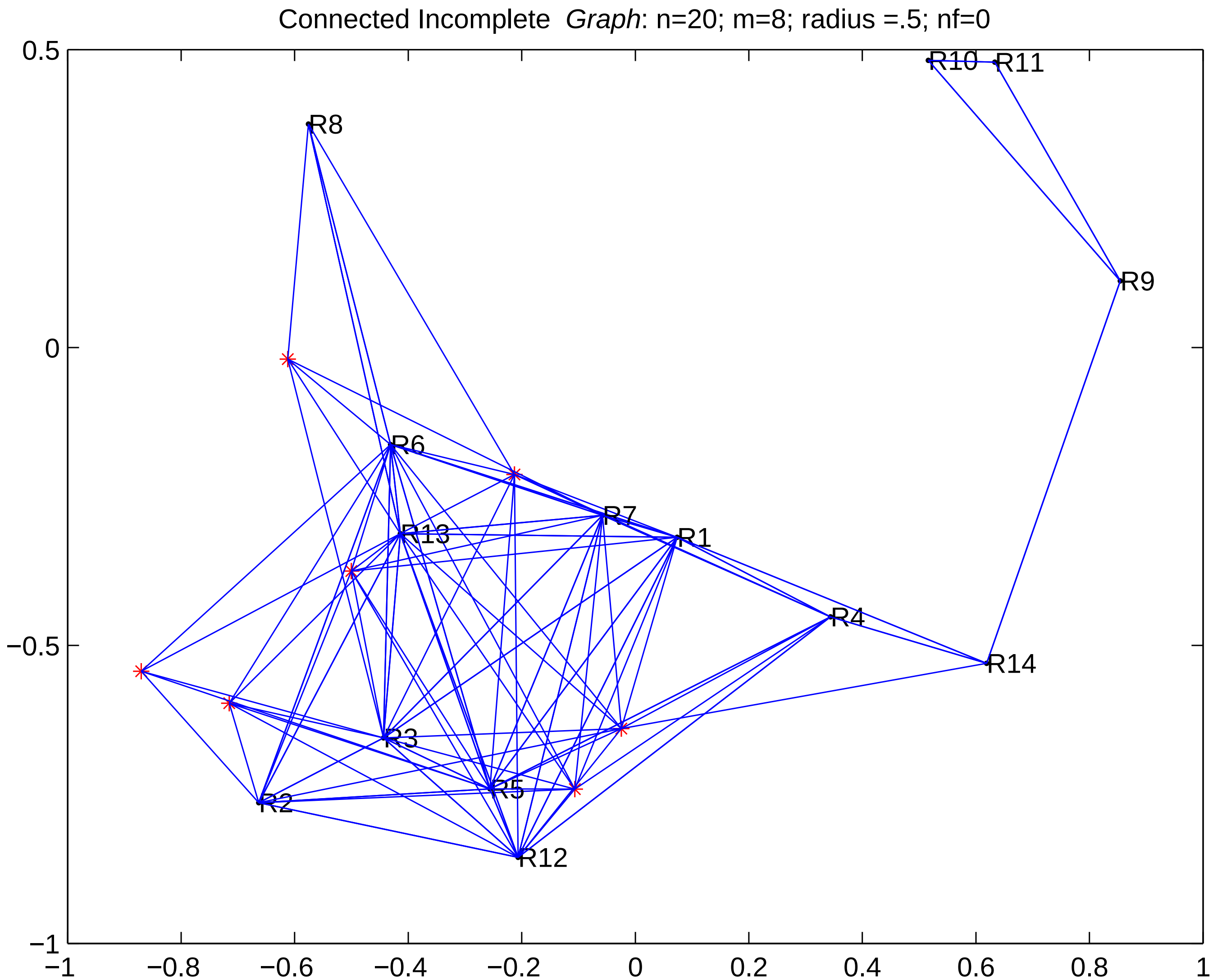



Figure 1: Connected Incomplete *Graph*

# Problem Example - with Radio Range



# Graph Realization/ EDM

- $G = (V, E, \omega)$ - incomplete, undirected, edge-weighted simple graph
- node set $V = \{$ sensors,anchors$\}$
- weight $\omega_{ij}$ is the squared distance between two nodes (not all known/possibly inaccurate)
- Realization of $G$ in $\Re^r$: a mapping of nodes $v_i$ into points $p^i$ in $\Re^r$ with squared distances given by the weights
- Solve using EDM (no anchors?)

# $SNL$ with Anchors; Problem Formulation

- $p^1, \ldots, p^n \in \Re^r$ unknown (sensor) points
  $a^1, \ldots, a^m \in \Re^r$ known (anchor) points
  $r$ embedding dimension (usually $2$ or $3$)
- $A^T := [a^1, a^2, \ldots, a^m] \quad X^T := [p^1, p^2, \ldots, p^n]$

$$\begin{aligned} P^T &:= \left(p^1, p^2, \ldots, p^n, a^1, a^2, \ldots, a^m\right) \\ &= \left(p^1, p^2, \ldots, p^n, p^{n+1}, p^{n+2}, \ldots, p^{n+m}\right) \end{aligned}$$

$$P = \begin{pmatrix} X \\ A \end{pmatrix}$$ rows are sensor/anchor points

# Assumption (avoids some trivialities)

- The number of sensors and anchors, and the embedding dimension satisfy

$$n >> m > r, \; A^T e = 0 \text{ and } A \text{ is full rank.}$$

- Translate/center anchors $A$ if needed; translate back at end.

# Definitions

- $(\mathcal{N}_e, \mathcal{N}_u, \mathcal{N}_l)$ are index sets for sensor-sensor existing distance values, upper bounds, and lower bounds, resp.
  Similarly, $(\mathcal{M}_e, \mathcal{M}_u, \mathcal{M}_l)$ for sensor to/from anchor

- partial EDM matrix $E$ of squared distances

$$E_{ij} := \begin{cases} d_{ij}^2 & \text{if} \quad ij \in \mathcal{N}_e \cup \mathcal{M} \\ \|p^i - p^j\|^2 = \|a^{i-n} - a^{j-n}\|^2 & \text{if} \quad i, j > n \\ 0 & \text{otherwise.} \end{cases}$$

# Definitions

Similarly, we define

- the (partial) matrix of (squared distances) upper bounds
  $U^b$, using $ij \in \mathcal{N}_u \cup \mathcal{M}_u$
- and the (partial) matrix of (squared distances) lower bounds
  $L^b$, using $ij \in \mathcal{N}_l \cup \mathcal{M}_l$

# Weighted Least Squares Error

If distances $E_{ij}$ have errors/noise; let $W_p, W_a, L_a$ be weight matrices ($L_a$ large). Minimize weighted least squares error. ($\boldsymbol{EDMC}$ - usually underdetermined/ill-conditioned)

$$\begin{aligned} 2f_1(P) &:= \sum_{(i,j)\in\mathcal{N}_e} (W_p)_{ij}(\|p^i - p^j\|^2 - E_{ij})^2 \\ &\quad + \sum_{(i,k)\in\mathcal{M}_e} (W_a)_{ik}(\|p^i - a^k\|^2 - E_{i \end{aligned}$$

(anchors) $\quad \left( + \sum_{i,j>n} (L_a)_{ij}(\|p^i - p^j\|^2 - E_{ij}) \right.$

# HARD (nonconvex) Constrained LS

$EDMC$ Problem:

$$\begin{array}{lll} \text{min} & f_1(P) & \text{(weighted least squares)} \\ \text{s.t.} & \|p^i - p^j\|^2 \le U_{ij} \ \ \forall (i,j) \in \mathcal{N}_u & \left(n_u = \frac{|\mathcal{N}_u|}{2}\right) \\ & \|p^i - a^k\|^2 \le U_{ik} \ \ \forall (i,k) \in \mathcal{M}_u & \left(m_u = \frac{|\mathcal{M}_u|}{2}\right) \\ & \|p^i - p^j\|^2 \ge L_{ij} \ \ \forall (i,j) \in \mathcal{N}_l & \left(n_l = \frac{|\mathcal{N}_l|}{2}\right) \\ & \|p^i - a^k\|^2 \ge L_{ik} \ \ \forall (i,k) \in \mathcal{M}_l & \left(m_l = \frac{|\mathcal{M}_l|}{2}\right) \\ & \left(p^{i+n} = a^i \ \ \forall i = 1, \ldots m \qquad \text{anchors}\right) & \end{array}$$

# Current $SDP$ Relaxation

Let $Y = XX^T$; relaxed to $Z_s = \begin{pmatrix} I_r & X^T \\ X & Y \end{pmatrix} \succeq 0$

$$\operatorname{tr} \begin{pmatrix} 0 \\ e_i - e_j \end{pmatrix} \begin{pmatrix} 0 \\ e_i - e_j \end{pmatrix}^T Z_s = E_{ij} \quad \left(\|x_i - x_j\|^2\right)$$

$$\operatorname{tr} \begin{pmatrix} -a_k \\ e_i \end{pmatrix} \begin{pmatrix} -a_k \\ e_i \end{pmatrix}^T Z_s = E_{i(k+n)} \quad \left(\|x_i - a_k\|^2\right)$$

Connection to EDM?

# $\mathcal{K}(\boldsymbol{SDP}) = \boldsymbol{EDM}$

$\bar{Y} = PP^T$ **($\boldsymbol{SDP}$)**. $\bar{Y}_{ii} = (p^i)^T p^i$; $\bar{Y}_{ij} = (p^i)^T p^j$

The squared distance

$$\begin{array}{rcccccc} D_{ij} & = & \|p^i - p^j\|^2 & & (\boldsymbol{EDM}) & & \\ & = & (p^i)^T p^i & + & (p^j)^T p^j & - & 2(p^i)^T p^j \\ & = & \updownarrow & & \updownarrow & & \updownarrow \\ & = & (\operatorname{diag}(\bar{Y})e^T & + & e\operatorname{diag}(\bar{Y})^T & - & 2\bar{Y})_{ij} \\ & =: & \left(\mathcal{K}(\bar{Y})\right)_{ij} & & & & \end{array}$$

$D = \mathcal{K}(\bar{Y})$ $\quad \boldsymbol{EDM}\ D \leftrightarrow \boldsymbol{SDP}\ \bar{Y}$ $\quad (\bar{Y}e = 0)$

$$\bar{Y} = PP^T$$

$$D_{ij} = \mathcal{K}(\bar{Y})_{ij} = \|p^i - p^j\|^2 \qquad \boldsymbol{EDM}$$

$$\begin{aligned} 2f_1(P) &= \sum_{(i,j)\in\ldots} (W_{\ldots})_{ij}(\|p^i - p^j\|^2 - E_{ij})^2 \text{ quartic} \\ &= \|W \circ (\mathcal{K}(\bar{Y}) - E)\|_F^2 \quad \text{quadratic} \end{aligned}$$

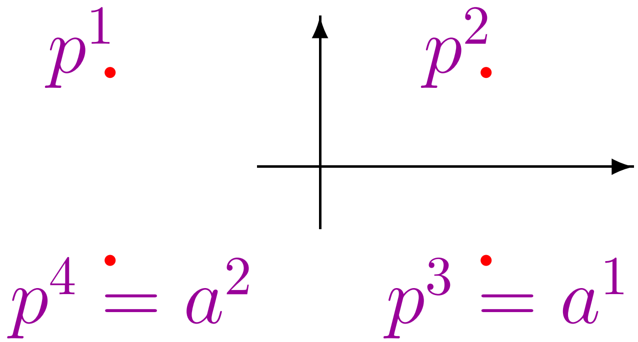
- quartic objective is changed to a quadratic; with additional hard quadratic constraint, $\bar{Y} = PP^T$

# Example of EDM

- $D = \begin{bmatrix} 0 & 4 & 5 & 1 \\ 4 & 0 & 1 & 5 \\ 5 & 1 & \boxed{\begin{matrix} 0 & 4 \\ 4 & 0 \end{matrix}} \\ 1 & 5 & & \end{bmatrix}$ is an EDM.

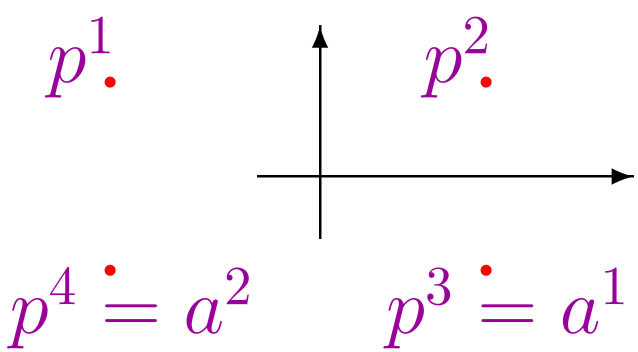
# Example of EDM

- $D = \begin{bmatrix} 0 & 4 & 5 & 1 \\ 4 & 0 & 1 & 5 \\ 5 & 1 & 0 & 4 \\ 1 & 5 & 4 & 0 \end{bmatrix}$ is an EDM.
- points that generate $D$ are:



# Example of EDM

- $D = \begin{bmatrix} \begin{matrix}0 & 4\\ 4 & 0\end{matrix} & \begin{matrix}5 & 1\\ 1 & 5\end{matrix} \\ \begin{matrix}5 & 1\\ 1 & 5\end{matrix} & \boxed{\begin{matrix}0 & 4\\ 4 & 0\end{matrix}} \end{bmatrix}$ is an EDM.
- points that generate $D$ are:




- (Since $\sum_j p^j = 0$:) Embedding dim of $D$ is 2.

# The EDM Completion Problem

- Given a *partial* matrix $A$ where only some of its elements are specified; the EDMCP is the problem of determining whether or not $A$ can be completed into an EDM.

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# The EDM Completion Problem

- Given a *partial* matrix $A$ where only some of its elements are specified; the EDMCP is the problem of determining whether or not $A$ can be completed into an EDM.
- $A = \begin{bmatrix} 0 & 4 & 9 & 1 \\ 4 & 0 & 1 & 1 \\ 9 & 1 & 0 & 4 \\ 1 & 1 & 4 & 0 \end{bmatrix}$ Embedding dim = 1.

# EDM Applications

- molecular conformation problems in chemistry
- multidimensional scaling and multivariate analysis problems in statistics genetics, geography, and others,
- Many of these applications require a low embedding dimension, e.g. $r = 2, 3$.

# Distance Geometry; Operator $\mathcal{K}$

- Define centered and hollow subspaces:

$$\begin{array}{rcll} \mathcal{S}_C & := & \{B \in \mathcal{S}^n : Be = 0\}, & \text{centered} \\ \mathcal{S}_H & := & \{D \in \mathcal{S}^n : \operatorname{diag}(D) = 0\}, & \text{hollow} \end{array}$$

- $J = I - \frac{1}{n}$ONES; $\operatorname{offDiag} :=$ zero-out diagonal

$$\begin{array}{rcl} \mathcal{K}(B) & := & \textsf{diag}(B)\, e^T + e\, \textsf{diag}(B)^T - 2B \\ & := & \mathcal{D}_e(B) - 2(B); \\ \mathcal{K}^\dagger(D) & = & -\frac{1}{2} J \operatorname{offDiag}(D) J \quad \text{M.P. Gen. Inverse} \\ \mathcal{K}^*(D) & = & \mathcal{D}_e^*(D) - 2D = 2\operatorname{Diag}(De) - 2D \end{array}$$

# Properties of $\mathcal{K}, \mathcal{T} := \mathcal{K}^\dagger$

COROLLARY The linear operators $\mathcal{T}, \mathcal{K}$ are one-one and onto mappings between the $\boldsymbol{EDM}$ cone in $S_H$ and the face of the semidefinite cone

$$\mathcal{F}_{\mathcal{E}_n} := \boldsymbol{SDP} \cap \mathcal{S}_C \quad \text{(centered } \boldsymbol{SDP}\text{)}$$

i.e.

$$\mathcal{T}(\boldsymbol{EDM}) = \mathcal{F}_{\mathcal{E}_n}, \qquad \mathcal{K}(\mathcal{F}_{\mathcal{E}_n}) = \boldsymbol{EDM}\,.$$

■

(Note $\mathcal{F}_{\mathcal{E}_n}$ has empty interior - a problem for interior-point methods!)

# Schoenberg Theorem

- COROLLARY
  An $n \times n$ symmetric $D$ with $\text{diag}(D) = 0$ is $\boldsymbol{EDM}$ iff $Y := -\frac{1}{2}JDJ \succeq 0$, where $J = I - ee^T/n$, $e = (1, \ldots, 1)^T$, and embedding dim of $D = \operatorname{rank} Y$.

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- (i.e. if and only if $Y = \mathcal{T}(D) \succeq 0$).
- The points $p^1, \ldots, p^n$ that generate $D$ are given by the rows of $P$ where $Y = PP^T$.

# Matrix Reformulation of $EDMC$

$$P = \begin{pmatrix} X \\ A \end{pmatrix}, \bar{Y} := PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & AA^T \end{pmatrix}$$

$$\begin{array}{lrcl}
\min & \frac{1}{2}\|W \circ (\mathcal{K}(\bar{Y}) - E)\|_F^2 & & \\
\text{subject to} & H_u \circ (\mathcal{K}(\bar{Y}) - U^b) & \leq & 0 \\
 & H_l \circ (\mathcal{K}(\bar{Y}) - L^b) & \geq & 0 \\
\text{hard constraint} & \boxed{\bar{Y} - PP^T = 0} & & \\
 & \bar{Y} \succeq 0, \quad \operatorname{sblk}_2(\bar{Y}) = AA^T & &
\end{array}$$

NOTE: Discard hard constraint. Relaxation to $\bar{Y} \succeq PP^T$ is redundant/NOT needed

# Convex wrt Löwner Partial Order

The constraint $g(P,Y) = PP^T - Y \preceq 0$ is $\succeq$-convex, since each function

$$\phi_Q(P,Y) = \operatorname{trace} Qg(P,Y) \quad \text{is convex } \forall Q \succeq 0.$$

Note

$$\begin{aligned} \operatorname{trace} QPP^T &= \operatorname{trace} QPIP^T \\ &= \operatorname{vec}(P)^T (I \otimes Q) \operatorname{vec}(P) \end{aligned}$$

Hessian is $I \otimes Q \succeq 0$;
and the cone $\boldsymbol{SDP}$ is self-polar.

# Useful Lemma

$$\bar{Y} = \begin{pmatrix} Y_{11} & Y_{21}^T \\ Y_{21} & AA^T \end{pmatrix} \succeq 0 \;\Rightarrow\; Y_{21}^T = XA^T,$$ for some $X$

■

Slater's CQ fails if we use EDM with constraint
$\operatorname{sblk}_2(\bar{Y}) = AA^T$ or equiv: $\bar{Y}_{22} = AA^T$
or equivalently,
$\operatorname{sblk}_2(\mathcal{K}(\bar{Y})) = \mathcal{K}(AA^T)$ (fixed clique of distances)
Take advantage/PROJECT onto Minimal Cone

# Projection onto Minimal Cone

- Suppose $\mathcal{F}$ is a face of $\boldsymbol{SDP}^{n+m}$.
- $Z \in \operatorname{relint} \mathcal{F}$ iff $\operatorname{rank} Z = t \leq n+m$ is max rank in $\mathcal{F}$
- If $Z = UDU^T$, $\mathcal{R}(Z) = \mathcal{R}(U)$ (same range)
- Then

$$\mathcal{F} = U(\boldsymbol{SDP}^{t})U^T, \quad U \text{ is } (n+m) \times t$$

- Here we have

$$0 \preceq Z = \bar{Y} = \begin{pmatrix} I_n & 0 \\ 0 & AA^T \end{pmatrix}, \quad \operatorname{rank}(Z) = t \leq n+r.$$

# Projection onto Minimal Cone

$A = U\Sigma_r V^T$ $m \times r$, rank $r$, compact SVD

$$\bar{Y} = \begin{pmatrix} Y_{11} & Y_{21}^T \\ Y_{21} & Y_{22} \end{pmatrix} \succeq 0, \bar{Y}_{22} = AA^T, \bar{Y} \in \mathcal{S}^{n+m}$$

IFF

$$\bar{Y} = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix} Z \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}^T \succeq 0, Z_{22} = I_r, Z \in \mathcal{S}^{n+r}$$

IFF

$$\bar{Y} = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} Z \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}^T \succeq 0, Z_{22} = \Sigma_r, Z \in \mathcal{S}^{n+r}$$

# Summary of Projection

$$\bar{Y} = \begin{pmatrix} Y_{11} & XA^T \\ AX^T & Y_{22} \end{pmatrix} \succeq 0, \bar{Y}_{22} = AA^T$$

$$\bar{Y} = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{21}^T \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}^T \succeq 0, Z_{22} = I_r$$

- $Z_{21}$ plays role of $X^T$.
- $E = \mathcal{K}(\bar{Y})$
- $E_{22} = \mathcal{K}(AA^T)$ (clique)

# Equivalent Reduced Problem Model

$(\boldsymbol{EDMC-R})$

$$
\begin{array}{ll}
\min & \frac{1}{2}\|W \circ \mathcal{K}(\mathcal{Y}(Z)) - \bar{E}\|_F^2 \\
\text{s.t.} & H_u \circ \mathcal{K}(\mathcal{Y}(Z)) - \bar{U}^b \leq 0 \\
& \bar{L}^b - H_l \circ \mathcal{K}(\mathcal{Y}(Z)) \leq 0 \\
& \mathcal{Y}(Z) = \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix} Z \begin{pmatrix} I_n & 0 \\ 0 & A \end{pmatrix}^T \\
& Z_{22} = I_r, Z \succeq 0
\end{array}
$$

# Equivalent Reduced Problem Model

objective is $\ell_2$ rather than $\ell_1$ in the literature, e.g. H. Jin(05), A. So, Y. Ye(05), P. Biswas, T. Liang, K. Toh, T. Wang, Y. Ye(06).

# Problems/Questions with Relaxation

1. $\{(\bar{Y}, P) : \bar{Y} = PP^T\} \subset \{(\bar{Y}, P) : \bar{P}P^T - \bar{Y} \preceq 0\}$
   (But, is Lagrangian relaxation stronger?)
2. Least squares problem is (usually) underdetermined/ill-conditioned.
3. linearization (using Schur complement) results in constraint *NOT* onto, i.e. two relaxations *NOT* numerically equivalent quadratic constraint $XX^T - Y \preceq 0$ better!
4. Exploit $\boldsymbol{EDM}$ model?

# More Proj./Facial Reduction?

sensors $S_c := \{p^{t+1}, \ldots, p^n\}$; all distances $\|p^i - p^j\|$ known $\forall t+1 \leq i,j \leq n$ (clique); $\mathcal{N}(K) = \mathcal{R}(\mathcal{D})$

$$E = \begin{pmatrix} E_1 & \cdot & \cdot \\ \cdot & \underline{E_2} & \cdot \\ \cdot & \cdot & \underline{E_3} \end{pmatrix} \quad \begin{pmatrix} \cdot \\ \underline{\text{clique for sensors}} \\ \underline{\text{clique for anchors}} \end{pmatrix}$$

$$\bar{Y} = \begin{pmatrix} Y_{11} & \cdot & \cdot \\ \cdot & Y_{22} \in \mathcal{K}^\dagger(E_2) + \mathcal{D}_e(y) \succeq 0 & \cdot \\ \cdot & \cdot & Y_{33} \end{pmatrix}$$

# Again: Exploit Loss of Slater CQ

- $B = \mathcal{K}^\dagger(E_{22}), B \in \mathcal{S}_+^{n-t}, Be = 0$
- $\exists y$: $\bar{Y}_{22} = P_2 P_2^T = B + ye^T + eY^T$ is rank $\leq r$
- $\hat{B} := B + ee^T + ee^T$ is rank $r_2 \leq r + 1$

  If $r_2 < n - t$!, Slater's CQ fails! Advantage!
- $\hat{B} = U_2 D_2 U_2^T \in \operatorname{relint} \mathcal{F}_2$, $\mathcal{R}(\hat{B}) = \mathcal{R}(U_2)$

$$\bar{Y} = \begin{pmatrix} I_t & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & A \end{pmatrix} Z \begin{pmatrix} I_t & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & A \end{pmatrix}^T, Z \in \mathcal{S}^{(t+r_2)+r}.$$

# Best Feasible Solution, I

- Once optimal $\bar{Y}$ found; can find sensor positions $X$ using relaxation $\bar{Y} \succeq PP^T$.

Equivalently: $Y \succeq XX^T$ or $\begin{pmatrix} Y & X^T \\ X & I \end{pmatrix} \succeq 0$.

- But $\boldsymbol{EDM}$ : factor $\bar{Y}$ (Schoenberg Thm), e.g. use spectral decomposition; largest $r$ eigenvalues/best $\operatorname{rank} r$ approximation

$$\mathcal{K}(\bar{Y}) \approx \mathcal{K}(\bar{Y}_r) = \mathcal{K}(\sum_{i=1}^{r} \lambda_i(\bar{Y}) v_i v_i^T)$$

# Best Feasible Solution, II

- $QQ^T = I$, $V = \left[\sqrt(\lambda_1)v_1 \dots \sqrt(\lambda_r)v_r\right] = \begin{pmatrix} \bar{X} \\ \bar{A} \end{pmatrix}$

$$\bar{Y}_r = (VQ)(VQ)^T \approx \begin{pmatrix} X \\ A \end{pmatrix} \begin{pmatrix} X \\ A \end{pmatrix}^T$$

- need to: $\min_{Q^TQ=I} \|\bar{A}Q - A\|_F^2$; analytically:

$Q^* = V_Q U_Q^T$; $U_Q \Sigma_Q V_Q$ is SVD of $A^T \bar{A}$

# GN Path-Following Method

- primal-dual strictly feasible starting point is available
- Use Wolfe Dual and eliminate primal-dual feasibility equations
- exact primal-dual feasibility at each iteration
- NLF (crossover - asymptotic quadratic convergence - high accuracy)

# Lagrangian of $EDMC-R$

$$
\begin{aligned}
L(x,y,\Lambda_u,\Lambda_l,\Lambda) = \\
&\frac{1}{2}\|W \circ \mathcal{K}(\mathcal{Y}(x,y) - \bar{E}\|_F^2 \\
&\quad + \langle \Lambda_u, H_u \circ \mathcal{K}(\mathcal{Y}(x,y)) - \bar{U} \rangle \\
&\quad\quad + \langle \Lambda_l, \bar{L} - H_l \circ \mathcal{K}(\mathcal{Y}(x,y)) \rangle \\
&\quad\quad\quad - \langle \Lambda, \mathrm{sBlk}_1(I) + \mathcal{Z}_s(x,y) \rangle ,
\end{aligned}
$$

where $0 \leq \Lambda_u, 0 \leq \Lambda_l \in \mathcal{S}^{m+n}, \quad 0 \preceq \Lambda \in \mathcal{S}^{m+n}$

$$
\Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_{21}^T \\ \Lambda_{21} & \Lambda_2 \end{pmatrix}, \qquad \langle A, B \rangle = \mathrm{trace}\, A^T B.
$$

# Primal-Dual Optimal. Conditions 1

**THEOREM:** The primal-dual variables $x, y, \Lambda, \lambda_u, \lambda_l$ are optimal for $\boldsymbol{EDMC-R}$ if and only if:

1. Primal Feasibility:

   The slack variables satisfy

$$S_u = \bar{U} - H_u \circ (\mathcal{K}(\mathcal{Y}(x,y))), \quad s_u = \operatorname{svec} S_u \geq 0$$

$$S_l = H_l \circ (\mathcal{K}(\mathcal{Y}(x,y))) - \bar{L}, \quad s_l = \operatorname{svec} S_l \geq 0$$

   and

$$\begin{array}{rcl} Z_s & = & \mathrm{sBlk}_1(I) + \mathrm{sBlk}_2\mathrm{sMat}(y) + \mathrm{sBlk}_{21}\mathrm{Mat} \\ & \succeq & 0 \end{array}$$

# Primal-Dual Optimal. Conditions 2a

**2a.** Dual Feasibility:

The stationarity equations ($\Rightarrow$ exact p-d feas.)

$$
\begin{aligned}
(\mathcal{Z}_s^x)^*(\Lambda) &= \lambda_{21} \quad \text{(eliminated)} \\
&= \left[W \circ (\mathcal{K}\mathcal{Y}^x)\right]^* \big(W \circ \mathcal{K}(\mathcal{Y}(x,y)) - \\
&\qquad + \left[H_u \circ (\mathcal{K}\mathcal{Y}^x)\right]^* (\Lambda_u) \\
&\qquad\quad - \left[H_l \circ (\mathcal{K}\mathcal{Y}^x)\right]^* (\Lambda_l) \\
(\mathcal{Z}_s^y)^*(\Lambda) &= \lambda_{2} \quad \text{(eliminated)} \\
&= \left[W \circ (\mathcal{K}\mathcal{Y}^y)\right]^* \big(W \circ \mathcal{K}(\mathcal{Y}(x,y)) - \\
&\qquad + \left[H_u \circ (\mathcal{K}\mathcal{Y}^y)\right]^* (\Lambda_u) \\
&\qquad\quad - \left[H_l \circ (\mathcal{K}\mathcal{Y}^y)\right]^* (\Lambda_l)
\end{aligned}
$$

# Primal-Dual Optimal. Conditions 2b

**2b.** Dual Feasibility:

Nonnegativity

$$\begin{aligned}\Lambda \;&=\; \mathrm{sBlk}\,_1\mathrm{sMat}\,(\lambda_1) + \mathrm{sBlk}\,_2\mathrm{sMat}\,(\lambda_2)\\ &\quad +\mathrm{sBlk}\,_{21}\mathrm{Mat}\,(\lambda_{21}) \succeq 0;\end{aligned}$$

$$\lambda_u \geq 0; \lambda_l \geq 0$$

$$\Lambda = \Lambda(\lambda_1, x, y, \lambda_u, \lambda_l)$$ (from stationarity)

# Primal-Dual Optimal. Conditions 3 (C.S.)

3. Complementary Slackness:

$$\begin{aligned} \lambda_u \circ s_u &= 0 \\ \lambda_l \circ s_l &= 0 \\ \Lambda Z_s &= 0 \end{aligned}$$

(equivalently $\operatorname{trace} \Lambda Z_s = 0$)

# Perturbed Compl. Slack. Conditions

$$F_\mu(x, y, \lambda_u, \lambda_l, \lambda_1) := \begin{pmatrix} \lambda_u \circ s_u - \mu_u e \\ \lambda_l \circ s_l - \mu_l e \\ \boxed{\Lambda Z_s - \mu_c I} \end{pmatrix} = 0,$$

where $s_u = s_u(x, y)$, $s_l = s_l(x, y)$,
$\Lambda = \Lambda(\lambda_1, x, y, \lambda_u, \lambda_l)$, $Z_s = Z_s(x, y)$
an **overdetermined** bilinear system with
$(m_u + n_u) + (m_l + n_l) + (n + r)^2$ equations
$nr + t(n) + (m_u + n_u) + (m_l + n_l) + t(r)$ variables.

# Gauss-Newton Search Direction

$$\Delta s := \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta \lambda_u \\ \Delta \lambda_l \\ \Delta \lambda_1 \end{pmatrix}$$

overdetermined linearized system is:

$$F'_\mu(\Delta s) \cong F'_\mu(x, y, \lambda_u, \lambda_l, \lambda_1)(\Delta s) = -F_\mu(x, y, \lambda_u, \lambda_l,$$

# GN: Three blocks of Equations

1. $\lambda_u \circ \operatorname{svec} \mathcal{K}_{H_u}(\Delta x, \Delta y) + s_u \circ \Delta\lambda_u = \mu_u e - \lambda_u \circ s_u$
2. $\lambda_l \circ \operatorname{svec} \mathcal{K}_{H_l}(\Delta x, \Delta y) + s_l \circ \Delta\lambda_l = \mu_l e - \lambda_l \circ s_l$
3. $$\begin{aligned} &\Lambda \mathcal{Z}_s(\Delta x, \Delta y) + [\operatorname{sBlk}_1(\operatorname{sMat}(\Delta\lambda_1)) \\ &\quad + \operatorname{sBlk}_2\left(\operatorname{sMat}\left\{(\mathcal{K}^y_W)^* \mathcal{K}_W(\Delta x, \Delta y) + \right.\right. \\ &\qquad \left.\left.(\mathcal{K}^y_{H_u})^*(\operatorname{sMat}(\Delta\lambda_u)) - (\mathcal{K}^y_{H_l})^*(\operatorname{sMat}(\Delta\lambda_l))\right\}\right) \\ &\qquad\quad + \operatorname{sBlk}_{21}\left(\operatorname{Mat}\left\{(\mathcal{K}^x_W)^* \mathcal{K}_W(\Delta x, \Delta y)\right.\right. \\ &\qquad\qquad + (\mathcal{K}^x_{H_u})^*(\operatorname{sMat}(\Delta\lambda_u)) - (\mathcal{K}^x_{H_l})^*(\operatorname{sMat}(\Delta\lambda_l)) \\ &= \mu_c I - \Lambda Z_s \end{aligned}$$

# Initial Str. Feas. Start Heuristic

If the graph is connected, we can use the stationarity equations and get a strictly feasible primal-dual starting point and *maintain exact numerical primal-dual feasibility* throughout the iterations.

# Diagonal Preconditioning

Given $A \in \mathcal{M}^{m\times n}$, $m \geq n$ full rank matrix; and using condition number of $K \succ 0$:
$\omega(K) = \frac{\operatorname{trace}(K)/n}{\det(K)^{1/n}}$, the optimal diagonal scaling

$$\min_{D \succ 0} \omega\left((AD)^T(AD)\right), \quad D^* = \operatorname{Diag}\left(1/\|A_{:,i}\|\right)$$

(cite Dennis-W.) Therefore, need to evaluate columns of $F'_\mu(\cdot)$ (can be done explicitly/efficiently)

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(Partial block Cholesky precondioning)

# dens: W .75,L .8; n 15, m 5, r 2

| nf | optvalue | relaxation | cond.number | sv($\mathcal{Z}_s$) | sv($F'_\mu$) |
|---|---|---|---|---|---|
| 0.0000e+000 | 3.9909e-009 | 1.1248e-005 | 3.8547e+006 | 15 | 19 |
| 5.0000e-002 | 7.5156e-004 | 4.4637e-002 | 1.0244e+011 | 6 | 27 |
| 1.0000e-001 | 3.7103e-003 | 1.1286e-001 | 1.9989e+010 | 5 | 25 |
| 1.5000e-001 | 6.2623e-003 | 1.3125e-001 | 1.0065e+010 | 6 | 14 |
| 2.0000e-001 | 1.3735e-002 | 1.3073e-001 | 6.8833e+009 | 7 | 12 |
| 2.5000e-001 | 2.3426e-002 | 2.4828e-001 | 2.4823e+010 | 8 | 6 |
| 3.0000e-001 | 6.0509e-002 | 2.3677e-001 | 3.4795e+010 | 7 | 7 |
| 3.5000e-001 | 5.5367e-002 | 3.7260e-001 | 2.3340e+008 | 6 | 4 |
| 4.0000e-001 | 7.6703e-002 | 3.6343e-001 | 8.9745e+010 | 8 | 3 |
| 4.5000e-001 | 1.2493e-001 | 6.9625e-001 | 3.2590e+010 | 6 | 9 |
| 5.0000e-001 | 1.3913e-001 | 3.9052e-001 | 2.2870e+005 | 8 | 0 |
| 5.5000e-001 | 8.8552e-002 | 3.8742e-001 | 5.8879e+007 | 8 | 2 |
| 6.0000e-001 | 4.2425e-001 | 4.1399e-001 | 4.9251e+012 | 8 | 4 |

# dens: W .75,L .8; n 15, m 5, r 2

| nf | optvalue | relaxation | cond.number | sv($\mathcal{Z}_s$) | sv($F'_\mu$) |
|---|---|---|---|---|---|
| 0.0000e+000 | 3.9909e-009 | 1.1248e-005 | 3.8547e+006 | 15 | 19 |
| 5.0000e-002 | 7.5156e-004 | 4.4637e-002 | 1.0244e+011 | 6 | 27 |
| 5.5000e-001 | 8.8552e-002 | 3.8742e-001 | 5.8879e+007 | 8 | 2 |
| 6.0000e-001 | 4.2425e-001 | 4.1399e-001 | 4.9251e+012 | 8 | 4 |
| 6.5000e-001 | 2.0414e-001 | 6.6054e-001 | 2.4221e+010 | 7 | 4 |
| 7.0000e-001 | 1.2028e-001 | 3.4328e-001 | 1.9402e+010 | 7 | 6 |
| 7.5000e-001 | 2.6590e-001 | 7.9316e-001 | 1.3643e+011 | 7 | 4 |
| 8.0000e-001 | 4.7155e-001 | 3.7822e-001 | 6.6910e+009 | 8 | 2 |
| 8.5000e-001 | 1.8951e-001 | 5.8652e-001 | 1.4185e+011 | 6 | 7 |
| 9.0000e-001 | 2.1741e-001 | 9.8757e-001 | 2.9077e+005 | 8 | 0 |
| 9.5000e-001 | 4.4698e-001 | 4.6648e-001 | 2.7013e+006 | 9 | 2 |

Table 0: Robust Algorithm for Ill-posed Problem

# Conclusion

- Sensor localization is a special case of $EDM$ where a clique of distances are known.
- different formulations/models for SDP relaxation influence stability of algorithms.
- It can be disadvantageous to linearize the convex constraint $XX^T - Y \preceq 0$ to the LMI $\begin{pmatrix} I & X^T \\ X & Y \end{pmatrix}$.
- Using Wolfe dual, get a stable GN algorithm; with exact primal-dual fesibility and high accuracy.