Anchored Grap
Realization and Se
Localization

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Outline

Problem Formulation, $\boldsymbol{EDMC}$

Matrix Reformulation, $\boldsymbol{EDMC}$

- Semidefinite Programming connection

SDP Relaxation of Hard Constraint $\bar{Y} = PP^T$

Facial Reduction - Reduced Problem Model

Adjoints/Duality for $\boldsymbol{EDMC - R}$

Primal-Dual Bilinear Optimality Conditions (overdetermined)

Robust Interior-Point algorithm

- Gauss-Newton Direction, crossover, exact p-d feasibility, p

MATLAB demonstration

Concluding Remarks

Problem

- Ad hoc wireless sensor network
- A few anchors (e.g. with GPS/bulky
  fixed, known locations
- sensors within a given range have
  known distance measurements (ap
- Problem: Determine positions of al
- Parameters: Radio range, # of anc
  level
- Semidefinite Relaxations/Robust A

Problem Applica

- health, military, home
- natural habitat monitoring, earthqu detection, weather/current monitor
- random deployment in inaccessible disaster relief operations

Problem Exampl
with Radio Rang

Figure 1: Connected Grap

Problem Formula

- $p^1, \ldots, p^n \in \Re^r$ unknown (sensor)
  $a^1, \ldots, a^m \in \Re^r$ known (anchor)
  r embedding dimension (usually
- $A^T := [a^1, a^2, \ldots, a^m]$ $X^T := [p^1, p^2$

$$P^T := \left(p^1, p^2, \ldots, p^n, a^1, a^2, \ldots\right.$$

$$P = \begin{pmatrix} X \\ A \end{pmatrix}$$ rows are sensor/anch

Definitions

- index sets of existing values of dist
  between pairs of sensors, $\{p^i\}_1^n$:
  $\mathcal{N}_e$ distance values
  $\mathcal{N}_u$ upper bounds on distances
  $\mathcal{N}_l$ lower bounds on distances
  Similarly, the index sets $(\mathcal{M}_e, \mathcal{M}_u,$
  pairs from $\{p^i\}_1^n$ (sensors) and $\{a^k$
  (anchors)
- partial EDM matrix E of squared di

$$E_{ij} := \begin{cases} d_{ij}^2 & \text{if} \quad ij \in \mathcal{N}_e \cup \mathcal{M}_e \\ 0 & \text{otherwise.} \end{cases}$$

Definitions

Similarly, we define

- the (partial) matrix of (squared dist
  upper bounds
  U, using $ij \in \mathcal{N}_u \cup \mathcal{M}_u$
- and the (partial) matrix of (squared
  lower bounds
  L, using $ij \in \mathcal{N}_l \cup \mathcal{M}_l$

Weighted Least Squares Error

In the case E_{ij} have errors:
Let W_p, W_a be weight matrices. W
the weighted least squares error. (

$$f_1(P) \;:=\; \sum_{(i,j)\in\mathcal{N}_e} (W_p)_{ij}(\|p^i - p^j\|^2$$
$$+ \sum_{(i,k)\in\mathcal{M}_e} (W_a)_{ik}(\|p^i - a$$

HARD (nonconve
Constrained LS

$EDMC$ Problem:

$$
\begin{array}{ll}
\text{min} & f_1(P) \quad \text{(weighted least square}\\
\text{s.t.} & \|p^i - p^j\|^2 \le U_{ij} \;\; \forall (i,j) \in \mathcal{N}_u \quad \Big(\\
& \|p^i - a^k\|^2 \le U_{ik} \;\; \forall (i,k) \in \mathcal{M}_u\\
& \|p^i - p^j\|^2 \ge L_{ij} \;\; \forall (i,j) \in \mathcal{N}_l \quad \Big(\\
& \|p^i - a^k\|^2 \ge L_{ik} \;\; \forall (i,k) \in \mathcal{M}_l
\end{array}
$$

$\mathcal{K}(\boldsymbol{SDP}) = \boldsymbol{EDM}$

$B = PP^T$ ($\boldsymbol{SDP}$). $B_{ii} = (p^i)^T p^i$; $B_{ij} =$
The squared distance

$$\begin{array}{rcccc} D_{ij} & = & \|p^i - p^j\|^2 & & (\boldsymbol{EDM}) \\ & = & (p^i)^T p^i & + & (p^j)^T p^j \\ & = & \updownarrow & & \updownarrow \\ & = & (\operatorname{diag}(B) e^T & + & e \operatorname{diag}(B)^T \\ & =: & (\mathcal{K}(B))_{ij} & & \end{array}$$

$D = \mathcal{K}(B)$ change $\boldsymbol{EDM}$ $D \leftrightarrow \boldsymbol{SD}$

Löwner Partial O

matrix inner-product $\boxed{\langle M, N \rangle = \operatorname{trace} M}$

Frobenius norm $\boxed{\|M\|^2 = \operatorname{trace} M^T M}$

In $\mathcal{S}^n$, $n \times n$ symmetric matrices:

$B \succeq 0$ (is positive semidefin

$$\Longleftrightarrow$$

$\exists P$ with $B = PP^T, \ \operatorname{rank}(B) = \mathrm{ra}$

the positive semidefinite (Löwner) par

$A \succeq B \ (A \succ B)$ if $A - B \succeq 0 \ (A -$

Matrix Reformula
of $EDMC$

Let $\bar{Y} := PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & AA^T \end{pmatrix}$

We get the equivalent $\boldsymbol{EDMC}$

$$\begin{array}{ll} \min & f_2(\bar{Y}) := \frac{1}{2}\|W \circ (\mathcal{K}(\bar{Y}) - \\ \text{subject to} & g_u(\bar{Y}) := H_u \circ (\mathcal{K}(\bar{Y}) \\ & g_l(\bar{Y}) := H_l \circ (\mathcal{K}(\bar{Y}) \\ & \text{hard constraint } \boxed{\bar{Y} - PP^T} \end{array}$$

SDP Relaxation o
Hard Constraint

$$\bar{Y} = PP^T = \begin{pmatrix} XX^T & XA^T \\ AX^T & \boxed{AA^T} \end{pmatrix}$$ hold

$$\iff$$

$\bar{Y}_{11} = XX^T$ and $\bar{Y}_{21} = AX^T$, $\bar{Y}_2$

Relax $\bar{Y} = PP^T$ to (Löwner partial orc

$\boxed{\bar{Y}_{22} = AA^T}$ $\quad PP^T - \bar{Y} \preceq 0$ quadr co

(But .... why this relaxation?)

Convex wrt Löwn
Partial Order

The constraint $g(P,Y) = PP^T - Y \preceq$
$\succeq$-convex, since each function

$\phi_Q(P,Y) = \operatorname{trace} Qg(P,Y)$ is conve

Note

$$\begin{aligned} \operatorname{trace} QPP^T &= \operatorname{trace} QPIP^T \\ &= \operatorname{vec}(P)^T (I \otimes Q) \end{aligned}$$

Hessian is $I \otimes Q \succeq 0$;
and the cone $\boldsymbol{SDP}$ is self-polar.

Linearization of S
Relaxation

$$PP^T - \bar{Y} \preceq 0, \qquad P = \begin{pmatrix} X \\ A \end{pmatrix}, \boxed{\bar{Y}_{22} =}$$

$\iff$ (by Schur complemen

$$Z = \begin{pmatrix} I & P^T \\ P & \bar{Y} \end{pmatrix} \succeq 0, \quad P = \begin{pmatrix} X \\ A \end{pmatrix}, \boxed{\bar{Y}_{2}}$$

$\iff$ (ignore $\bar{\cdot}$)

$$Z = \begin{pmatrix} I & X^T & A^T \\ X & Y & Y_{21}^T \\ A & Y_{21} & \boxed{AA^T} \end{pmatrix} \succeq 0, \qquad \Bigg(\quad \text{NC} \quad \Rightarrow Y$$

Facial Reduction

$$Z_s := \begin{pmatrix} I & X^T \\ X & Y \end{pmatrix}$$

(NE 2×2 blo

(Lin.Tr. but N

THEOREM:

$$Z = \begin{pmatrix} I & X^T & A^T \\ X & Y & Y_{21}^T \\ A & Y_{21} & \boxed{AA^T} \end{pmatrix} \succeq 0$$

$$\Longleftrightarrow$$

$Z_s \succeq 0$ and $Y_{21} = AX^T$

Matrix $\leftrightarrow$ Vector Notation I

vector $\boxed{v = \operatorname{vec} V}$ is matrix V taken co

$\boxed{\operatorname{sblk}_{21}\begin{pmatrix} 0 & X^T \\ X & 0 \end{pmatrix} = \sqrt{2}X}$ pulls out th

the $\sqrt{2}$ is for isometry in Frobenius no

$$x := \operatorname{vec}\left(\operatorname{sblk}_{21}\begin{pmatrix} 0 & X^T \\ X & 0 \end{pmatrix}\right) = \sqrt{}$$

$\boxed{y := \operatorname{svec}(Y)}$ $(Y = Y^T,$ isom

Matrix ↔ Vector Notation II

(adjoints: $\mathrm{sblk}^*_{21}(X) =$)

$$\mathrm{sBlk}_{21}(X) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & X^T \\ X & 0 \end{pmatrix}$$

$$\mathrm{svec}^{-1}(\cdot) = \mathrm{svec}^*(\cdot) = \mathrm{sMat}(\cdot)$$

$$\mathcal{Z}^x_s(x) := \mathrm{sBlk}_{21}(\mathrm{Mat}(x)), \quad \mathcal{Z}^y_s(y) :=$$

$$\mathcal{Z}_s(x,y) := \mathcal{Z}^x_s(x) + \mathcal{Z}^y_s(y), \quad Z_s := \mathrm{sBl}$$

to build $\quad Z_s = \begin{pmatrix} I & X^T \\ X & Y \end{pmatrix} \in \mathcal{S}^{r+n}$

Matrix $\leftrightarrow$ Vector Notation III

$$\mathcal{Y}^x(x) = \text{sBlk}_{21}(A\text{Mat}(x)^T),\ \mathcal{Y}^y(y) =$$
$$\mathcal{Y}(x,y) = \mathcal{Y}^x(x) + \mathcal{Y}^y(y),\ \boxed{\bar{Y} = \text{sBlk}_2(}$$

$$\bar{E} := W \circ \big[E - \mathcal{K}(\text{sBlk}_2(AA^T$$
$$\bar{U} := H_u \circ \big[\mathcal{K}(\text{sBlk}_2(AA^T)) -$$
$$\bar{L} := H_l \circ \big[L - \mathcal{K}(\text{sBlk}_2(AA^T$$

The unknown matrix $\bar{Y}$ is equal to $\mathcal{Y}($
additional constant $2, 2$ block), i.e. unk
the vectors x, y.

Equivalent Redu
Problem Model

$(\boldsymbol{EDMC-R})$

$$\begin{array}{ll} \min & f_3(x,y) := \frac{1}{2}\|W \circ (\mathcal{K}(\mathcal{Y}(x,y))) - \\ \text{s.t.} & g_u(x,y) := H_u \circ \mathcal{K}(\mathcal{Y}(x,y) \\ & g_l(x,y) := \bar{L} - H_l \circ \mathcal{K}(\mathcal{Y}(\\ & \mathrm{sBlk}_1(I) + \mathcal{Z}_s \end{array}$$

(objective is ℓ_2 rather than ℓ_1 in the lite
H. Jin(05), A. So, Y. Ye(05), P. Biswas
K. Toh, T. Wang, Y. Ye(06).)

Problems with Relaxation

1. $\{(\bar{Y}, P) : \bar{Y} = PP^T\} \subset \{(\bar{Y}, P) : \bar{P}$
   (But, is Lagrangian relaxation stro
2. linearization (using Schur comple
   results in a constraint that is *NOT*
   two relaxations *NOT* numerically e
3. Least squares problem is (usually)
   underdetermined.

Lagrangian of $EDMC - R$

$$L(x, y, \Lambda_u, \Lambda_l, \Lambda) =$$
$$\tfrac{1}{2}\|W \circ \mathcal{K}(\mathcal{Y}(x,y) - \bar{E}\|_F^2$$
$$+ \langle \Lambda_u, H_u \circ \mathcal{K}(\mathcal{Y}(x,y)) - \bar{U}$$
$$+ \langle \Lambda_l, \bar{L} - H_l \circ \mathcal{K}(\mathcal{Y}(x,y$$
$$- \langle \Lambda, \mathrm{sBlk}_1(I) + \mathcal{Z}_s(x$$

where $0 \leq \Lambda_u, 0 \leq \Lambda_l \in \mathcal{S}^{m+n}, \quad 0 \preceq \Lambda$

$$\Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_{21}^T \\ \Lambda_{21} & \Lambda_2 \end{pmatrix}, \qquad \langle A, B \rangle = \mathrm{tra}$$

Matrix ↔ Vector [cut off]
Variable Notation [cut off]

$$\lambda_u := \operatorname{svec}(\Lambda_u), \quad \lambda_l := \operatorname{svec}(\ldots$$
$$h_u := \operatorname{svec}(H_u), \quad h_l := \operatorname{svec}(\ldots$$
$$\lambda := \operatorname{svec}(\Lambda), \quad \lambda_1 := \operatorname{svec}(\Lambda\ldots$$
$$\lambda_2 := \operatorname{svec}(\Lambda_2), \quad \lambda_{21} := \operatorname{vec}\operatorname{sblk}\ldots$$

Adjoints

To differentiate the Lagrangian, we ne
adjoints of the various linear transform
part of $\mathcal{K}$:

$$\begin{aligned}
\mathcal{D}_e(B) &= \operatorname{diag}(B)\, e^T + e \operatorname{diag}(\\
\mathcal{D}_e^*(D) &= 2\operatorname{Diag}(De) \\
\langle \mathcal{D}_e(B), D \rangle &= \operatorname{trace}(\operatorname{diag}(B) e^T D + \\
&= \operatorname{trace}(De(\operatorname{diag} B)^T + \\
&= 2\operatorname{trace}\ (\operatorname{diag} B)^T (De) \\
&= \langle B, \mathcal{D}_e^*(D) \rangle, \forall D, B
\end{aligned}$$

Primal-Dual Opti
Conditions 1

THEOREM: The primal-dual variables x
are optimal for $\boldsymbol{EDMC-R}$ if and o

1. Primal Feasibility:

The slack variables satisfy

$S_u = \bar{U} - H_u \circ (\mathcal{K}(\mathcal{Y}(x,y)))\,,\ s_u =$

$S_l = H_l \circ (\mathcal{K}(\mathcal{Y}(x,y))) - \bar{L}, \quad s_l =$

and

$$\begin{aligned} Z_s &= \text{sBlk}_1(I) + \text{sBlk}_2\text{sMat}(y) - \\ &\succeq 0 \end{aligned}$$

Primal-Dual Opti
Conditions 2a

2a. Dual Feasibility:

The stationarity equations (⇒ exac

$$
\begin{aligned}
(\mathcal{Z}_s^x)^*(\Lambda) &= \lambda_{21} \quad \text{(eliminated)} \\
&= [W \circ (\mathcal{K}\mathcal{Y}^x)]^* \big(W \circ \mathcal{K} \\
&\quad + [H_u \circ (\mathcal{K}\mathcal{Y}^x)]^* (\Lambda \\
&\quad - [H_l \circ (\mathcal{K}\mathcal{Y}^x)]^* \\
(\mathcal{Z}_s^y)^*(\Lambda) &= \lambda_2 \quad \text{(eliminated)} \\
&= [W \circ (\mathcal{K}\mathcal{Y}^y)]^* \big(W \circ \mathcal{K} \\
&\quad + [H_u \circ (\mathcal{K}\mathcal{Y}^y)]^* (\Lambda \\
&\quad - [H_l \circ (\mathcal{K}\mathcal{Y}^y)]^*
\end{aligned}
$$

Primal-Dual Opti
Conditions 2b

2b. Dual Feasibility:
Nonnegativity

$$\Lambda = \mathrm{sBlk}_1\mathrm{sMat}(\lambda_1) + \mathrm{sBlk}_2\mathrm{sMa}$$
$$+\mathrm{sBlk}_{21}\mathrm{Mat}(\lambda_{21}) \succeq 0;$$

$$\lambda_u \geq 0; \lambda_l \geq 0$$

$$\Lambda = \Lambda(\lambda_1, x, y, \lambda_u, \lambda_l)$$ (from st

Primal-Dual Opti
Conditions 3 (C.S

3. Complementary Slackness:

$$\begin{aligned} \lambda_u \circ s_u &= 0 \\ \lambda_l \circ s_l &= 0 \\ \Lambda Z_s &= 0 \end{aligned}$$

(equivalently tra

Perturbed Compl
Slack. Condition

$$F_\mu(x, y, \lambda_u, \lambda_l, \lambda_1) := \begin{pmatrix} \lambda_u \circ s_u - \mu_u \\ \lambda_l \circ s_l - \mu_l \\ \boxed{\Lambda Z_s - \mu_c I} \end{pmatrix}$$

where $s_u = s_u(x, y)$, $s_l = s_l(x, y)$,
$\Lambda = \Lambda(\lambda_1, x, y, \lambda_u, \lambda_l)$, $Z_s = Z_s(x, y)$

an overdetermined bilinear system w
$(m_u + n_u) + (m_l + n_l) + (n + r)^2$ equa
$nr + t(n) + (m_u + n_u) + (m_l + n_l) + t($

Gauss-Newton Search Direction

$$\Delta s := \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta \lambda_u \\ \Delta \lambda_l \\ \Delta \lambda_1 \end{pmatrix}$$

overdetermined linearized system is:

$$F'_\mu(\Delta s) \cong F'_\mu(x, y, \lambda_u, \lambda_l, \lambda_1)(\Delta s) = -F$$

Notation - Compo
of Lin. Tr.

$$\mathcal{K}_H^x(x) := H \circ (\mathcal{K}(\mathcal{Y}^x(x))$$
$$\mathcal{K}_H^y(y) := H \circ (\mathcal{K}(\mathcal{Y}^y(y))$$
$$\mathcal{K}_H(x,y) := H \circ (\mathcal{K}(\mathcal{Y}(x,y)$$

GN: Three blocks
Equations

1. $\lambda_u \circ \operatorname{svec} \mathcal{K}_{H_u}(\Delta x, \Delta y) + s_u \circ \Delta \lambda_u =$
2. $\lambda_l \circ \operatorname{svec} \mathcal{K}_{H_l}(\Delta x, \Delta y) + s_l \circ \Delta \lambda_l = \mu$
3.

$$\Lambda \mathcal{Z}_s(\Delta x, \Delta y) + [\text{sBlk}_1 (\text{sMat}(\Delta \lambda_1))$$
$$+ \text{ sBlk}_2 (\text{sMat } \{(\mathcal{K}_W^y)^* \mathcal{K}_W(\Delta x, \Delta y)$$
$$(\mathcal{K}_{H_u}^y)^* (\text{sMat}(\Delta \lambda_u)) - (\mathcal{K}_{H_l}^y)^* (\text{sM}$$
$$+ \text{sBlk}_{21} (\text{Mat } \{(\mathcal{K}_W^x)^* \mathcal{K}_W(\Delta x, \Delta y$$
$$+ (\mathcal{K}_{H_u}^x)^* (\text{sMat}(\Delta \lambda_u)) - (\mathcal{K}_{H_l}^x)^*$$
$$= \mu_c I - \Lambda Z_s$$

Initial Str. Feas. S
Heuristic

If the graph is connected, we can use
stationarity equations and get a strictl
primal-dual starting point and *maintai*
numerical primal-dual feasibility throu
iterations.

Diagonal Preconditioning

Given $A \in \mathcal{M}^{m\times n}$, $m \geq n$ full rank mat
using condition number of $K \succ 0$:
$\omega(K) = \frac{\mathrm{trace}\,(K)/n}{\det(K)^{1/n}}$, the optimal diagona

$$\min_{D \succ 0} \omega\left((AD)^T(AD)\right), \quad D^* = \mathrm{Diag}\,($$

(cite Dennis-W.) Therefore, need to evaluate
$F'_\mu(\cdot)$ (can be done explicitly/efficiently)

(Partial block Cholesky precondioning)

dens: W .75,L .8; n 15, m 5, r 2

nf	optvalue	relaxation	cond.number
0.0000e+000	3.9909e-009	1.1248e-005	3.8547e+006
5.0000e-002	7.5156e-004	4.4637e-002	1.0244e+011
1.0000e-001	3.7103e-003	1.1286e-001	1.9989e+010
1.5000e-001	6.2623e-003	1.3125e-001	1.0065e+010
2.0000e-001	1.3735e-002	1.3073e-001	6.8833e+009
2.5000e-001	2.3426e-002	2.4828e-001	2.4823e+010
3.0000e-001	6.0509e-002	2.3677e-001	3.4795e+010
3.5000e-001	5.5367e-002	3.7260e-001	2.3340e+008
4.0000e-001	7.6703e-002	3.6343e-001	8.9745e+010
4.5000e-001	1.2493e-001	6.9625e-001	3.2590e+010
5.0000e-001	1.3913e-001	3.9052e-001	2.2870e+005
5.5000e-001	8.8552e-002	3.8742e-001	5.8879e+007
6.0000e-001	4.2425e-001	4.1399e-001	4.9251e+012

Anchored Graph Re

dens: W .75,L .8;
n 15, m 5, r 2

nf	optvalue	relaxation	cond.number
0.0000e+000	3.9909e-009	1.1248e-005	3.8547e+006
5.0000e-002	7.5156e-004	4.4637e-002	1.0244e+011
5.5000e-001	8.8552e-002	3.8742e-001	5.8879e+007
6.0000e-001	4.2425e-001	4.1399e-001	4.9251e+012
6.5000e-001	2.0414e-001	6.6054e-001	2.4221e+010
7.0000e-001	1.2028e-001	3.4328e-001	1.9402e+010
7.5000e-001	2.6590e-001	7.9316e-001	1.3643e+011
8.0000e-001	4.7155e-001	3.7822e-001	6.6910e+009
8.5000e-001	1.8951e-001	5.8652e-001	1.4185e+011
9.0000e-001	2.1741e-001	9.8757e-001	2.9077e+005
9.5000e-001	4.4698e-001	4.6648e-001	2.7013e+006

Table 1: Robust Algorithm for Ill-pose

Anchored Graph Re