A *Short Course* on Semidefinite Programming

(in order of appearance)

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About these Notes:

Semidefinite Programming, SDP, refers to optimization problems where the vector variable is a symmetric matrix which is required to be positive semidefinite. Though SDPs (under various names) have been studied as far back as the 1940s, the interest has grown tremendously during the last fifteen years. This is partly due to the many diverse applications in e.g. engineering, combinatorial optimization, and statistics. Part of the interest is due to the great advances in efficient solutions for these types of problems.

These notes summarize the theory, algorithms, and applications for semidefinite programming. They are prepared for a shortcourse given on the first day of the **SIAM Conference on Optimization**, May 15-19, 2005, at City Conference Centre, Stockholm, Sweden, URL: www.siam.org/meetings/op05/index.htm

Preface

The purpose of these notes is three fold: first, they provide a comprehensive treatment of the area of Semidefinite Programming, a new and exciting area in Optimization; second, the notes illustrate the strength of convex analysis in Optimization; third, they emphasize the interaction between theory and algorithms and solutions of practical problems.

1 Introduction and Motivation

1.1 Outline

- Basic Properties and Notation
- Examples/Applications
- Historical Notes

1.2.1 Basic Properties and Notation

Basic <u>linear</u> ***Semidefinite Programming*** looks just like *Linear Programming*

$$
(\mathbf{PSDP}) \qquad \begin{array}{rlll} p^* = & \max & \operatorname{trace} CX & (\langle C, X\rangle) \\ & \text{s.t.} & \mathcal{A}(X) \cong \mathcal{A}X = b & \text{(linear)} \\ & & X \succeq 0, \quad (X \in \mathcal{P}) & \text{(nonneg)} \end{array}
$$

$C, X \in \mathcal{S}^n, \quad b \in \mathbb{R}^m$

$\mathcal{S}^n$:= space of $n \times n$ symmetric matrices

$\mathcal{A}$:= linear operator

Space of Symmetric Matrices

$A = A^T \in \mathcal{S}^n$:= space of $n \times n$ (real) symmetric matrices

A is positive semidefinite (positive definite), $\quad (A \succeq 0$ $(A \succ 0))$
if $x^T A x \geq 0 (> 0), \; \forall x \neq 0$.

$\preceq$ denotes the Löwner partial order, [54]
$A \preceq B$ if $B - A \succeq 0$ (positive semidefinite)

TFAE:

1. $A \succeq 0 \quad (A \succ 0)$
2. the vector of eigenvalues $\lambda(A) \geq 0 \quad (\lambda(A) > 0)$
3. all principal minors ≥ 0 $\quad$ (all leading principal minors > 0)

Some Properties/Equivalences of $\mathcal{S}^n, \mathcal{P}$

1. $A \in \mathcal{S}^n$:
   (a) All eigenvalues are real
   (b) $A = P\Lambda P^T, \; P^T P = I, \; \Lambda = \mathrm{Diag}\,(\lambda)$, i.e. A can be orthogonally diagonalized
2. $A \succeq 0$:
   (a) All eigenvalues are real nonnegative
   (b) $A = S^2, \; S \succeq 0$, i.e. A has a square root in $\mathcal{P}$
3. $A \succ 0$:
   (a) All eigenvalues are real positive
   (b) $A = S^2, \; S \succ 0$, i.e. A has a square root in $\operatorname{int} \mathcal{P}$
   (c) $A = LL^T$, L lower triangular with positive diagonal (Cholesky factorization)

Linear transformation $\mathcal{A}$; Adjoint:

$$\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m, \qquad \mathcal{A}^* : \mathbb{R}^m \to \mathcal{S}^n$$

$$\mathcal{A} \text{ defined by associated set } \{A_i \in \mathcal{S}^n, i = 1, \ldots m\} :$$
$$(\mathcal{A}X)_i = \operatorname{trace}(A_i X); \quad \mathcal{A}^* y = \sum_{i=1}^m y_i A_i,$$

where the adjoint of $\mathcal{A}$ is defined by

$$\langle \mathcal{A}X, y \rangle = \langle X, \mathcal{A}^* y \rangle, \quad \forall X \in \mathcal{S}^n, \forall y \in \mathbb{R}^m$$

$\mathcal{P} = \mathcal{S}^n_+$ - cone of positive semidefinite matrices
replaces
$\mathbb{R}^n_+$ - nonnegative orthant

SDP or LMI

$\mathrm{trace}\, CX = \langle C, X \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} = (\mathrm{vec}\, C)^T (\mathrm{vec}\, C)$

(primal) SDP is equivalent to (Linear Matrix Inequalities):

$$
\begin{array}{llll}
 & p^* = & \max & \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} & \\
(\mathbf{PSDP}) & & \text{s.t.} & \langle A_i, X \rangle = b_i, & i = 1, \ldots m \\
 & & & X \succeq 0, \quad (X \in \mathcal{P}) &
\end{array}
$$

1.2.2 Application - Max-Cut Problem, MC

(MC) is a combinatorial optimization problem on undirected graphs with weights on the edges.

Problem 1.2.2.1 *Find a partition of the set of vertices into two parts that maximizes the sum of the weights on the edges that have one end in each part of the partition.*

One Formulation of MC

vector $v \in \{\pm 1\}^n$, $n = |V|$, represents cut in graph G

$$S = \{i : v_i = +1\} \qquad V \backslash S = \{i : v_i = -1\}.$$

$$(MC) \qquad \begin{array}{rll} \mu^* = & \max & \sum_{1 \le i < j \le n} w_{ij} \left(\frac{1 - v_i v_j}{2} \right) \\ & \text{s.t.} & v \in \{\pm 1\}^n \end{array}$$

Equivalently,

$$(\text{MC1}) \qquad \begin{array}{rll} \mu^* = & \max & v^T Q\, v \\ & \text{s.t.} & v_i^2 = 1, \qquad i = 1, \ldots, n, \end{array}$$

where $Q = \frac{1}{4}(\text{Diag}\,(A\,e) - A)$ is $\frac{1}{4}$ Laplacian matrix, L, $A = (w_{ij})$ is weighted adjacency matrix of G.

Equivalent Formulation of MC

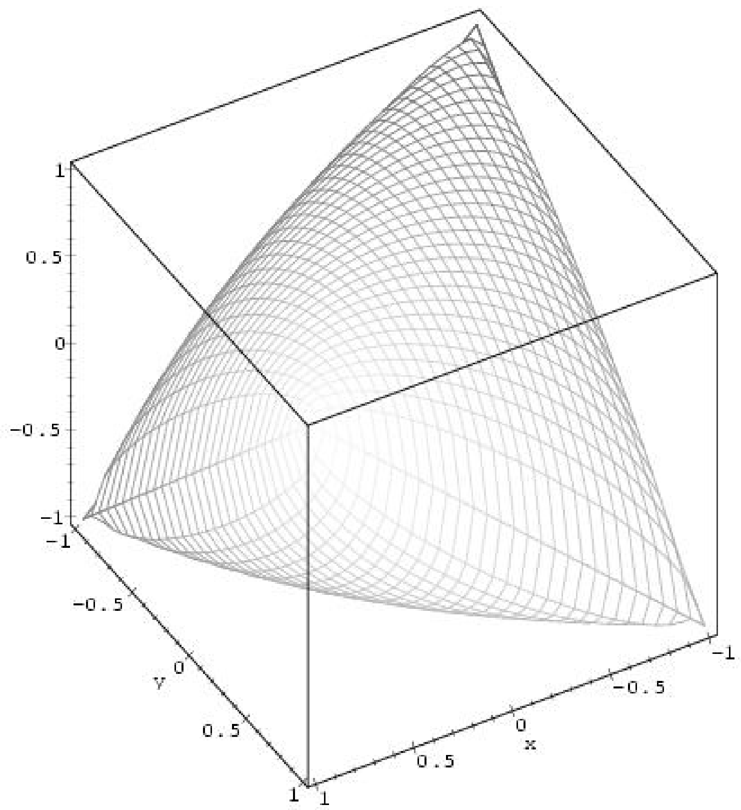
With $Q := \frac{1}{4}L$, $X := vv^T, v \in \{\pm 1\}^n$,
then $v^T Qv = \operatorname{trace} QX$ and equivalent formulation is:

$$
\text{(MC1)} \qquad \begin{array}{rll} \mu^* = & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X) = e \\ & & \operatorname{rank}(X) = 1 \\ & & X \succeq 0, X \in \mathcal{S}^n, \end{array}
$$

relax by deleting the ***hard*** constraint $\operatorname{rank}(X) = 1$ (get *elliptope*)

Note $X \succeq 0$, rank r iff $X = VV^T$ with $V\, n \times r$, full rank

Elliptope for $n = 3$, [50]



GW Approximation Result

Goemans & Williamson proved that
if $w_{ij} \geq 0 \ \forall \, i, j$, then

$$\mu^* \geq \alpha \, \nu_1^*,$$

where $\alpha = \min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos\theta} \approx 0.87856$.

Since $\frac{1}{\alpha} \approx 1.13823 \leq 1.14$, this implies

$$\mu^* \leq \nu_1^* \leq 1.14 \, \mu^*.$$

Other Approximations

Other such convex relaxations have been studied before.

The smallest convex set containing all the rank-one matrices X corresponding to cuts is the *cut polytope*:

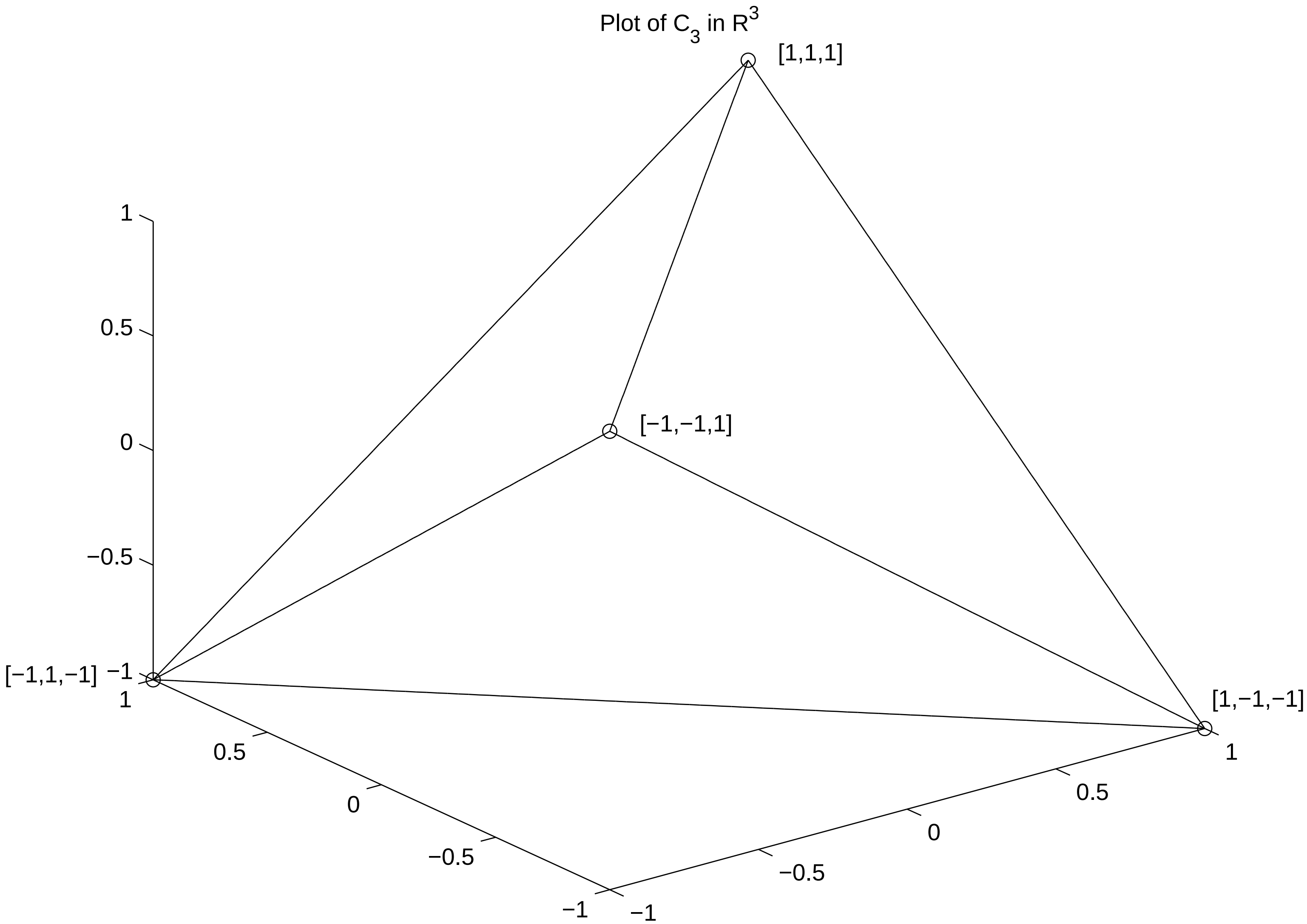
$$C_n := \text{Conv}\{X : X = vv^T, v \in \{\pm 1\}^n\}.$$

In fact,

$$\begin{aligned} \mu^* = \quad & \max \quad \operatorname{trace} QX \\ & \text{s.t.} \quad X \in C_n. \end{aligned}$$

However, it is not known how to optimize in polynomial-time over C_n.

Cut polytope for $n = 3$



Metric Polytope

Another convex relaxation is the *metric polytope* M_n, defined by

$$M_n := \{X \in \mathcal{S}^n : \mathrm{diag}\,(X) = e, \text{ and}$$
$$X_{ij} + X_{ik} + X_{jk} \geq -1, X_{ij} - X_{ik} - X_{jk} \geq -1,$$
$$- X_{ij} + X_{ik} - X_{jk} \geq -1, -X_{ij} - X_{ik} + X_{jk} \geq -1,$$
$$\forall\, 1 \leq i < j < k \leq n\}.$$

These inequalities model the fact that for any three mutually connected vertices in the graph, it is only possible to cut either zero or two of the edges joining them.

It is known that:

$C_n = M_n$ for $n \leq 4$, but $C_n \subsetneq M_n$ for $n \geq 5$.

Primal-Dual Pair

$$\text{(MCSDP)} \quad \begin{array}{rll} \mu^* = & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X) = e \\ & & X \succeq 0, X \in \mathcal{S}^n, \end{array}$$

$$\text{(DMCSDP)} \quad \begin{array}{rll} \mu^* = & \min & e^T y \\ & \text{s.t.} & \operatorname{Diag}(y) \succeq Q \\ & \text{equivalently:} & \operatorname{Diag}(y) - Z = Q \\ & & Z \succeq 0, Z \in \mathcal{S}^n \end{array}$$

See Appendix B, Page 69-19, for information and examples on solving relaxations of Max-Cut problems using NEOS or MATLAB.

Rewrite Lagrangian/payoff

payoff function; player Y to player X (Lagrangian)

$$L(X,y) := \operatorname{trace}(CX) + y^T(b - \mathcal{A}X) \;\; (= \langle C, X\rangle + \langle y, b - \mathcal{A}X\rangle)$$

Optimal (worst case) strategy for player X:

$$p^* = \max_{X \succeq 0} \min_{y} L(X,y)$$

For each fixed $X \succeq 0$: y free yields hidden constraint $b - \mathcal{A}X = 0$; recovers primal problem (PSDP).

Lagrangian Duality

$$\begin{aligned} L(X,y) &= \operatorname{trace}(CX) + y^T(b - \mathcal{A}X) \\ &= b^T y + \operatorname{trace}\,(C - \mathcal{A}^* y)\, X \end{aligned}$$

using adjoint operator, $\mathcal{A}^* y = \sum_i y_i A_i$

satisfies $\langle \mathcal{A}^* y, X \rangle = \langle y, \mathcal{A}X \rangle, \quad \forall X, y$

Then, ***WEAK DUALITY (Lagrangian)*** holds:

$$p^* = \max_{X \succeq 0} \min_y L(X,y) \le d^* := \min_y \max_{X \succeq 0} L(X,y)$$

Derive dual: for each <u>fixed</u> y, $X \succeq 0$ yields <u>hidden constraint</u>

$$g(y) := C - \mathcal{A}^* y \preceq 0$$

Hidden Constraint

$$p^* = \max_{X \succeq 0} \min_{y} L(X,y) \leq d^* := \min_{y} \max_{X \succeq 0} L(X,y)$$

dual obtained from optimal strategy of competing player Y;
use *hidden constraint* $g(y) = C - \mathcal{A}^* y \preceq 0$

$$(\textbf{DSDP}) \qquad \begin{array}{rcc} d^* = & \min & b^T y \\ & \text{s.t.} & \mathcal{A}^* y \succeq C \end{array}$$

for the primal

$$(\textbf{PSDP}) \qquad \begin{array}{rcc} p^* = & \max & \operatorname{trace} CX \\ & \text{s.t.} & \mathcal{A} X = b \\ & & X \succeq 0 \end{array}$$

1.2.3 Weak Duality - Optimality

Proposition 1.2.3.1 *(Weak Duality) If X feas. in (PSDP), y feas. in (DSDP), $Z = \mathcal{A}^* y - C \succeq 0$ is slack variable, then*

$$\operatorname{trace} CX - b^T y = -\operatorname{trace} XZ \le 0.$$

Proof. (Direct - using: $\operatorname{trace} ZX = \operatorname{trace} X^{1/2} X^{1/2} Z = \operatorname{trace} X^{1/2} Z^{1/2} Z^{1/2} X^{1/2} = \|X^{1/2} Z^{1/2}\|^2 \ge 0$;
(Note: $XZ = 0$ iff $\operatorname{trace} XZ = 0$)

$$\begin{aligned} \operatorname{trace} CX - b^T y &= \operatorname{trace} (\mathcal{A}^* y - Z) X - b^T y \\ &= \operatorname{trace} y^T \mathcal{A} X - \operatorname{trace} ZX - b^T y \\ &= y^T (\mathcal{A} X - b) - \operatorname{trace} ZX = -\operatorname{trace} ZX, \end{aligned}$$

■

Characterization of optimality

primal-dual pair $X \succeq 0, y$ (and slack $Z \succeq 0$) are optimal iff

$\mathcal{A}^* y - Z = C$	dual feasibility	$(OC1)$
$\mathcal{A}X = b$	primal feasibility	$(OC2)$
$ZX = 0$	complementary slackness	$(OC3)$

And

$ZX = \mu I$ perturbed complementary slackness (POC3)

Forms the basis for:
interior point methods
(central path: X_μ, y_μ, Z_μ solutions of (OC1),(OC2),(OC3));
primal simplex method
dual simplex method

Strong Duality

$$\begin{aligned}
\text{primal value} &= \langle C, X\rangle \\
&= \langle \mathcal{A}^* y - Z, X\rangle && \text{by dual feasibility} \\
&= \langle y, \mathcal{A}X\rangle - \langle Z, X\rangle && \text{by adjoint} \\
&= \langle y, b\rangle - \langle Z, X\rangle && \text{by primal feasibility} \\
&= \langle y, b\rangle && \text{by complementary slackness} \\
&= \text{dual value}
\end{aligned}$$

1.2.4 Preliminary Examples

Example 1.2.4.1 Minimizing the Maximum Eigenvalue
Arises in e.g. stability of differential equations

- *The mathematical problem:*
  - ◇ *given* $A(x) \in \mathcal{S}^n$ *depending linearly on vector* $x \in \mathbb{R}^m$
  - ◇ *Find* x *to minimize the maximum eigenvalue of* $A(x)$
- *SDP Model:*
  - ◇ *the largest eigenvalue* $\lambda_{\max}(A(x)) \leq \alpha$ ***iff*** $\lambda_{\max}(A(x) - \alpha I) \leq 0$ ***iff*** $A(x) - \alpha I \preceq 0$.
  - ◇ *The DSDP (in dual form) is:*

$$\max -\alpha \qquad \textit{s.t.} \quad A(x) - \alpha I \preceq 0.$$

Pseudoconvex Optimization

Example 1.2.4.2 Pseudoconvex (Nonlinear) Optimization

$$(\mathbf{PCP}) \qquad \begin{array}{rll} d^* = & \min & \frac{(c^T x)^2}{d^T x} \\ & \textit{s.t.} & Ax + b \geq 0, \end{array}$$

(given $Ax + b \geq 0 \Rightarrow d^T x > 0$*)*
Then (using 2×2 *determinant), (PCP) is equivalent to*

$$\begin{array}{rll} d^* = & \min & t \\ & \textit{s.t.} & \begin{bmatrix} \mathrm{Diag}\,(Ax+b) & 0 & 0 \\ 0 & t & c^T x \\ 0 & c^T x & d^T x \end{bmatrix} \succeq 0 \end{array}$$

1.2.5 Historical Notes

- See URL: orion.math.uwaterloo.ca/˜hwolkowi/henry/software/sdpbibliog.pdf for a regularly updated annotated bibliograhy.
- Arguably most active area in optimization (see HANDBOOK OF SEMIDEFINITE PROGRAMMING: Theory, Algorithms, and Applications, 2000, [96], for comprehensive results, history, references, ... and e.g. books [16, 98, 99, 7], [93, 69, 13]).
- Lyapunov over 100 years ago on stability analysis of differential equations
- Bohnenblust 1948 on geometry of the cone of SDPs, [11]
- Yakubovitch in the 1960's and Boyd and others on convex optimization in control in the 1980's, e.g. solving Ricatti Equations (called LMIs), e.g. [14],[94, 93]

More: Historical Notes

• matrix completion problems (another name for SDP) started early 1980's, continues to be a very active area of research, [20],[29], and e.g.: [48, 47, 46, 45, 49, 41, 33, 21, 42]. (More recently,it is being used to solve large scale SDPs.)
• combinatorial optimization applications 1980's: Lovász *theta function* [53]; the strong approximation results for the max-cut problem by Goemans-Williamson, e.g. [28], survey papers: [26, 27],[79].
• linear complementarity problems can be extended to problems over the cone of semidefinite matrices, e.g. [23, 40, 44, 43, 60].

More: Historical Notes

• Complexity, Distance to Ill-Posedness, and Condition Numbers SDP is a convex program and it can be solved to any desired accuracy in polynomial time, see seminal work of Nesterov and Nemirovski e.g. [64, 65, 69, 67, 63, 66, 68]. Another measure of complexity is the distance to ill-posedness: e.g. work by Renegar [81, 85, 84, 83, 82].

• Cone Programming this is a generalization of SDP, also called *generalized linear programming*, in paper by Bellman and Fan 1963, [9]. Other books deal with problems over cones date back to 60s, e.g. [71],[35, 55, 39, 38, 77]. More recently, generalization of SDP to more general cones, e.g. Güler and Tuncel, [31, 92] and also Hauser [32]

More: Historical Notes

- Other Related Areas e.g.: Eigenvalue Functions, e.g. [15], [73, 74]; Financial Applications; Generalized Convexity, e.g. [86, 57]; Statistics; Nonlinear Programming; ...

Further historical notes thanks Didier Henrion

LMI Terminology coined by Jan Willems in 1971

Historically, first LMIs 1890; Lyapunov - differential equation

$\frac{d}{dt}x(t) = \dot{x}(t) = Ax(t)$ is stable (all trajectories converge to zero)

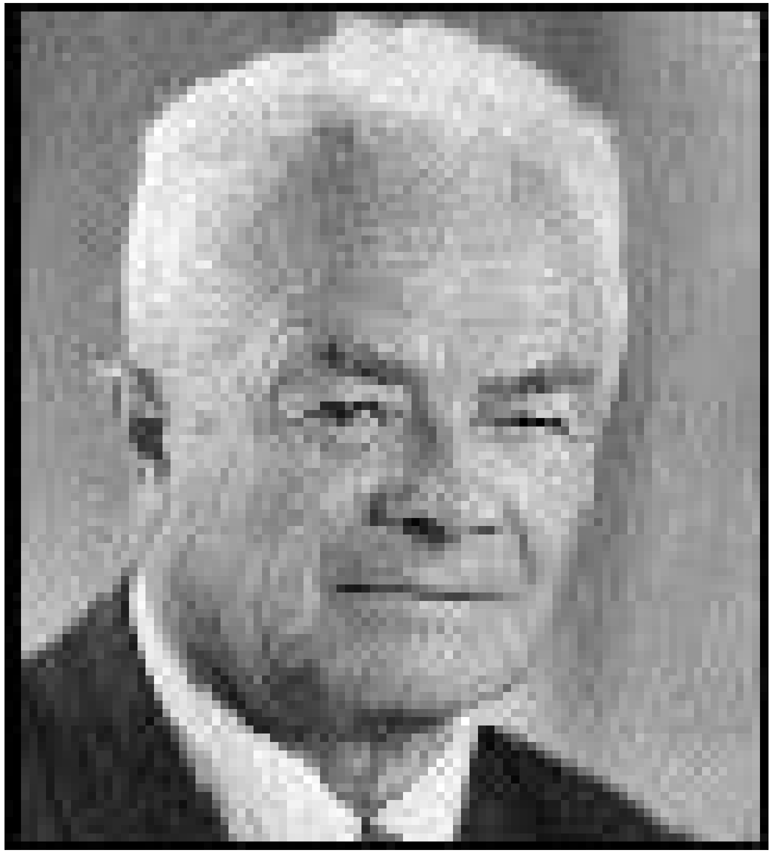
iff there exists a solution to the matrix inequalities

$$A^T P + PA \prec 0 \quad P = P^T \succ 0$$

Aleksandr Mikhailovich Lyapunov (1857 - 1918)

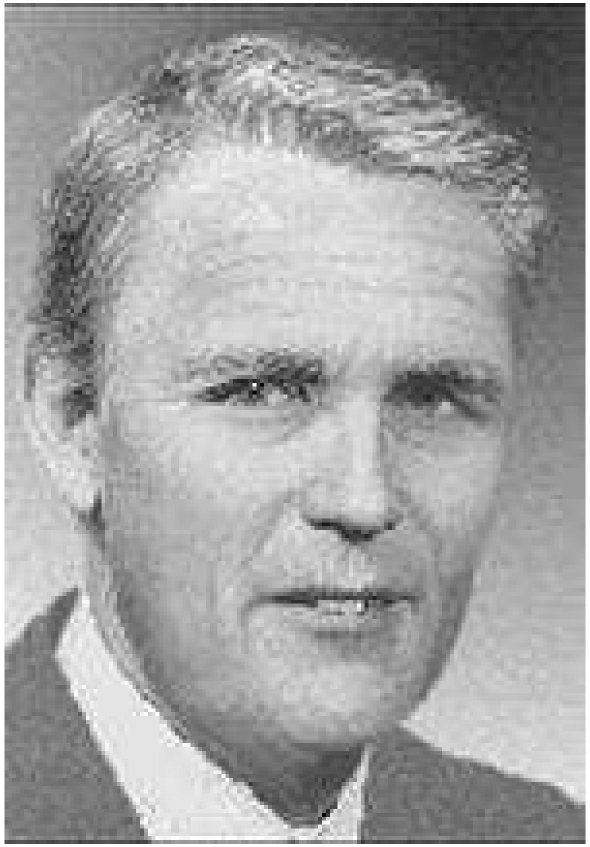
Some history (2)

1960s: Yakubovich, Popov, Kalman, Anderson et al obtained the positive real lemma
Reduces solution of LMI to a simple graphical criterion (Popov, circle and Tsypkin criteria)



Vladimir Andreevich Yakubovich

(1926 Novosibirsk)



Rudolf Emil Kalman

(1930 Budapest)

1970s: Willems focused on solving algebraic equations such as Lyapunov’s or Riccati’s
equations (AREs), rather than LMIs
Then most of the work dedicated to numerical algebra, development of Matlab (1984), focus on solving control AREs, until..

Mathematical programming

1979: ellipsoid algorithm of Khachiyan:
polynomial bound on worst case interation count for LP
1984: Karmarkar introduces interior-point (IP) methods for LP:
improved complexity bound and efficiency
1988: Nesterov, Nemirovski, Alizadeh extend IP methods for SDP
1994: Goemans and Williamson prove than LMI relaxations of MAXCUT (a problem of combinatorial optimization) provide solutions at least 88% the optimal value
1994: Research effort in control culminated in the LMI book by Boyd, El Ghaoui, Feron, Balakrishnan - Contributions also by Apkarian, Bernussou, Gahinet, Geromel, Peres and many others..

1.3 Motivation/Examples/Applications

- Quadratic constrained quadratic programs
- Lovász theta function
- Statistics
- minimizing the L_2-operator norm of a matrix
- linear programming
- robust mathematical programming
- engineering, e.g. control theory
- Combinatorial Problems

1.3.1 Quadratic Constrained Quadratic Programs

What is SEMIDEFINITE PROGRAMMING?
Why use it?

- Quadratic approximations are better than linear approximations.

(For example, model $x \in \{0, 1\}$ using $x^2 - x = 0$.)

And, we can solve relaxations of quadratic approximations efficiently using semidefinite programming.

HOW DOES SDP arise from quadratic approximations?

Let $\quad q_i(y) = \frac{1}{2} y^T Q_i y + y^T b_i + c_i, \; y \in \mathbb{R}^n, \; i = 0, 1, \ldots, m$

$$
\textbf{(QQP)} \quad \begin{cases} q^* = & \min & q_0(y) \\ & \text{s.t.} & q_i(y) = 0 \\ & & i = 1, \ldots m \end{cases}
$$

- Lagrangian: $\quad L(y, x) = \quad$ (with x as Lagrange multipliers)

quadratic in y | linear in y | constant in y

$$\frac{1}{2} y^T (Q_0 - \textstyle\sum_{i=1}^m x_i Q_i) y \;+\; y^T (b_0 - \textstyle\sum_{i=1}^m x_i b_i) \;+\; (c_0 - \textstyle\sum_{i=1}^m x_i c_i)$$

- Primal-Dual pair: $\quad q^* = \min\limits_y \max\limits_x L(y, x) \geq d^* = \max\limits_x \min\limits_y L(y, x)$

Homogenization

- homogenize (add y_0): $\quad y_0 y^T(b_0 - \sum_{i=1}^m x_i b_i), \;\; y_0^2 = 1.$

$$
\begin{aligned}
d^* &= \max_x \min_y && L(y,x) \\
&= \max_x \min_{y_0^2=1} && \tfrac{1}{2} y^T (Q_0 - \textstyle\sum_{i=1}^m x_i Q_i) y \quad (+t y_0^2) \\
& && + y_0 y^T (b_0 - \textstyle\sum_{i=1}^m x_i b_i) \\
& && + (c_0 - \textstyle\sum_{i=1}^m x_i c_i) \quad (-t)
\end{aligned}
$$

Hidden Constraint

with t as Lagrange mulitplier for $y_0^2 = 1$ constraint;

use *hidden semidefinite constraint* to yield SDP constraint

$$(\mathcal{A} : \mathbb{R}^{m+1} \rightarrow \mathcal{S}_{n+1}) \qquad B - \mathcal{A}\begin{pmatrix} t \\ x \end{pmatrix} \succeq 0.$$

$$B = \begin{pmatrix} 0 & b_0^T \\ b_0 & Q_0 \end{pmatrix}, \quad \mathcal{A}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{bmatrix} -t & \sum_{i=1}^m x_i b_i^T \\ \sum_{i=1}^m x_i b_i & \sum_{i=1}^m x_i Q_i \end{bmatrix}$$

Dual Program

The dual program is equivalent to the SDP (with $c_0 = 0$)

$$(\mathbf{D}) \qquad \begin{aligned} d^* = \quad & \sup \quad -t - \textstyle\sum_{i=1}^m x_i c_i \\ & \text{s.t.} \quad \mathcal{A}\begin{pmatrix} t \\ x \end{pmatrix} \preceq B \\ & \qquad x \in \mathbb{R}^m, t \in \mathbb{R} \end{aligned}$$

As in LP: dual obtained from
optimal strategy of competing player:

$$(\mathbf{DD}) \qquad \begin{aligned} p^* = \quad & \inf \quad \operatorname{trace} BU \\ & \text{s.t.} \quad \mathcal{A}^* U = \begin{pmatrix} -1 \\ -c \end{pmatrix} \\ & \qquad U \succeq 0. \end{aligned}$$

1.3.2 Generalized Eigenvalue Problems for $X = X^T$

1. **The Generalized Eigenvalue Problem**
   Let M, A be two $n \times n$ symmetric matrices, $M \succ 0$.
   The set of eigenvalues of the *matrix pencil*, denoted $[M, A]$, is $\{\lambda \in \mathbb{R} : \lambda M - A \text{ is singular}\}$.

$$\lambda_{\max}([M, X]) \le t \quad \text{is equivalent to} \quad tM - X \succeq 0$$

2. **Spectral Norm of Symmetric X**

$$\{|\lambda_i(X)| \le t,\ \forall i\} \text{ iff } \{tI - X \succeq 0,\ tI + X \succeq 0\}$$

3. **Sum of k largest eigenvalues of Symmetric X**
   Let $S_k(X)$ denote the sum of the largest k eigenvalues of X.

$$S_k(X) \le t \ \underline{\text{iff}}\ \ t - ks - \operatorname{trace} Z \succeq 0, Z \succeq 0, Z - X + sI \succeq 0.$$

1.3.3 SDP Application in Statistics

If $m_0, m_1, m_2, \ldots, m_{2n}$ are moments of some distribution, then

$$H(m_0, m_1, \ldots, m_{2n}) = \begin{bmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & \cdots & \cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{bmatrix} \succeq 0$$

Find distribution with maximal variance and $l_i \leq m_i \leq u_i$:

$$\begin{array}{ll} \max & y \\ \text{s.t.} & \begin{bmatrix} m_2 - y & m_1 \\ m_1 & 1 \end{bmatrix} \succeq 0 \\ & l_i \leq m_i \leq u_i \; (i = 1, \ldots 2n) \end{array}$$

2 Theory Of Cone Programming

2.1 Outline

- Convex cones and Partial Orders
- Convex Cone Programs
- Strong Duality
- Log Barrier and the Central Path

2.2 Convex Cones; Löwner Partial Order

Definition 2.2.0.1 *Let* $\alpha \in \mathbb{R}$ *and* $S, T \subset \mathbb{R}^n$. *Then* $\alpha S = \{y : y = \alpha s,$ *for some* $s \in S\}$ *and* $S + T = \{y : y = s + t,$ *for some* $s \in S, t \in T\}$

Definition 2.2.0.2 $\mathcal{K} \subset \mathbb{R}^n$ *is a* *cone* *if* $\alpha\mathcal{K} \subset K, \quad \forall \alpha > 0$.

Definition 2.2.0.3 *the cone* $\mathcal{K}$ *is a* *convex cone* *if* $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$.

Definition 2.2.0.4 *A cone* $\mathcal{K}$ *is a* *pointed cone* *if* $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$.

Definition 2.2.0.5 *A cone* $\mathcal{K} \subset \mathbb{R}^n$ *is a* *proper cone* *if it is closed, pointed, and convex and has nonempty interior.*

Examples of Cones

Example 2.2.0.1 *open half line:* $\{x \in \mathbb{R} : x > 0\}$.

Example 2.2.0.2 *closed half line:* $\{x \in \mathbb{R} : x \geq 0\}$.

Example 2.2.0.3 *psd matrices, $\mathcal{P}$:* $\{X \in \mathcal{S}^n : X \succeq 0\}$.

Example 2.2.0.4 *Lorentz cone*
(ice-cream cone, second-order cone):
$L^m = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_m \geq \sqrt{x_1^2 + \ldots + x_{m-1}^2}$.

Polar Cone

And, an important closed convex cone is:

Definition 2.2.0.6 *Polar (dual, conjugate) cone of set* $\mathcal{S}$*:*

$\mathcal{S}^+ := \{z : \langle x, z \rangle \geq 0, \forall x \in \mathcal{S}\}$.

Example 2.2.0.5 *nonnegative orthant, psd cone* $\mathcal{P}$*, and Lorentz cone* L^m*, are all self-polar, i.e.* $\mathcal{K} = \mathcal{K}^+$.

Further Examples

Definition 2.2.0.7 *Direct sum of two cones*

$\mathcal{K} \oplus \mathcal{L} := \{(k\ l) : k \in \mathcal{K}, l \in \mathcal{L}\}$.

Example 2.2.0.6 *Let L denote the half line in $\mathbb{R}$, then $\underbrace{L \oplus L \cdots \oplus L}_{n}$ is the n-dimension nonnegative orthant.*

Properties of Cones

Suppose $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ are proper cones, then:

- $\mathcal{K}^+$ is proper.
- $(\mathcal{K}^+)^+ = \mathcal{K}$.
- $(\mathcal{K}_1 \cap \mathcal{K}_2)^+ = \overline{\mathcal{K}_1^+ + \mathcal{K}_2^+}$.
- $(\mathcal{K}_1 + \mathcal{K}_2)^+ = \mathcal{K}_1^+ \cap \mathcal{K}_2^+$.
- $\mathcal{K}_1 \oplus \mathcal{K}_2$ is proper.
- $(\mathcal{K}_1 \oplus \mathcal{K}_2)^+ = \mathcal{K}_1^+ \oplus \mathcal{K}_2^+$.

Useful Lemma, e.g. to Prove Farkas' Lemma

Lemma 2.2.0.1

$$\mathcal{K} \textit{ is a closed convex cone} \iff \mathcal{K} = \mathcal{K}^{++}$$

Partial Orders

Definition 2.2.0.8 $x \succeq_{\mathcal{K}} y$ *(respectively* $x \succ_{\mathcal{K}} y$*) if* $x - y \in \mathcal{K}$ *(respectively* $x - y \in \operatorname{int} \mathcal{K}$*).*

Remark 2.2.0.1 *If* $\mathcal{K}$ *is a pointed, convex cone, then "*$\succeq_{\mathcal{K}}$*" is a* *(linear) partial order* *(reflexive, transitive, and antisymmetric):*

- $0 \in \mathcal{K} \Rightarrow x \succeq_{\mathcal{K}} x$ *(reflexive);*
- $\mathcal{K}$ *is convex* $\Rightarrow$ *if* $x \succeq_{\mathcal{K}} y$ *and* $y \succeq_{\mathcal{K}} z$*, then* $x \succeq_{\mathcal{K}} z$ *(transitive);*
- $\mathcal{K}$ *is pointed* $\Rightarrow$ *if* $x \succeq_{\mathcal{K}} y$ *and* $y \succeq_{\mathcal{K}} x$*, then* $x = y$ *(antisymmetric);*
- $\mathcal{K}$ *is convex cone* $\Rightarrow$
  if $a, b \geq 0, u \succeq_{\mathcal{K}} x$ *and* $v \succeq_{\mathcal{K}} y$*, then* $au + bv \succeq_{\mathcal{K}} ax + by$ *(linear - homogeneous, additive);*

2.3 Convex Cone Program

Let: $\mathcal{K}, \mathcal{L}$ be convex cones;
f real valued convex function, $\mathcal{G}$ is $\mathcal{K}$-convex, i.e.
$\mathcal{G}(\alpha u + (1-\alpha)v) \preceq_{\mathcal{K}} \alpha\mathcal{G}(u) + (1-\alpha)\mathcal{G}(v), \quad \forall 0 \le \alpha \le 1, \forall u, v$

$$(CP) \qquad \begin{array}{rl} \mu^* = \min & f(x) \\ \text{s.t.} & \mathcal{G}(x) \preceq_{\mathcal{K}} 0 \\ & x \succeq_{\mathcal{L}} 0 \end{array}$$

$$\mu^* = \min_{x \in \mathcal{L}} \max_{y \in \mathcal{K}^+} f(x) + \langle y, \mathcal{G}(x) \rangle$$

-

with Lagrangian dual (weak duality)

$$(DCP) \qquad \mu^* \ge \nu^* = \max_{y \in \mathcal{K}^+} \min_{x \in \mathcal{L}} \; f(x) + \langle y, \mathcal{G}(x) \rangle$$

Linear Cone Optimization

Example 2.3.0.7 *If* $f(x) = \langle c, x \rangle$, $g(x) = b - \mathcal{A}x$, *then we have a linear cone programming problem, (LCP). The dual*

$$(DLCP) \qquad \begin{aligned} \mu^* \geq \nu^* &= \max_{y \in \mathcal{K}^+} \min_{x \in \mathcal{L}} \langle c, x \rangle + \langle y, b - \mathcal{A}x \rangle \\ &= \max_{y \in \mathcal{K}^+} \min_{x \in \mathcal{L}} \langle c - \mathcal{A}^* y, x \rangle + \langle y, b \rangle \\ &= \max_{\substack{y \in \mathcal{K}^+ \\ c - \mathcal{A}^* y \in \mathcal{L}^+}} \min_{x \in \mathcal{L}} \langle c - \mathcal{A}^* y, x \rangle + \langle y, b \rangle \end{aligned}$$

reduces to elegant (LP or SDP type)

$$(DLCP) \qquad \begin{array}{rc} \mu^* \geq \nu^* = \max & \langle y, b \rangle \\ \textit{s.t.} & \mathcal{A}^* y \preceq_{\mathcal{L}^+} c \\ & y \succeq_{\mathcal{K}^+} 0 \end{array}$$

Linear Cone Optimization cont...

elegant (LP or SDP type) symmetric dual pair

$$(PLCP) \qquad \begin{array}{rcl} \mu^* = \min & & \langle c, x\rangle \\ \text{s.t.} & & \mathcal{A}x \succeq_{\mathcal{K}} b \\ & & x \succeq_{\mathcal{L}} 0 \end{array}$$

$$(DLCP) \qquad \begin{array}{rcl} \mu^* \geq \nu^* = \max & & \langle y, b\rangle \\ \text{s.t.} & & \mathcal{A}^* y \preceq_{\mathcal{L}^+} c \\ & & y \succeq_{\mathcal{K}^+} 0 \end{array}$$

Abstract Convex Program

general abstract convex program is

$$(CP) \qquad \begin{array}{rll} \mu^* := & \inf & f(x) \\ & \text{subject to} & \mathcal{G}(x) \preceq_L 0 \\ & & x \succeq_K 0, \end{array}$$

$f : X \rightarrow$ real valued convex function on K;
X, Y Banach spaces;
$x \succeq_K y$ partial order induced convex cone K, i.e. $y - x \in K$;
$\mathcal{G} : X \rightarrow Y$ is L-convex function on K, i.e.

$$\mathcal{G}(\lambda x + (1 - \lambda y) \preceq_L \lambda \mathcal{G}(x) + (1 - \lambda)\mathcal{G}(y), \quad \forall x, y \in K, \forall 0 < \lambda < 1$$

Definitions

1. $K \subset \mathbb{R}^n$ is a <u>cone</u> if $\lambda K \subset K, \forall \lambda > 0$.
2. The cone $K \subset \mathbb{R}^n$ is a <u>convex cone</u> if $K + K \subset K$.
3. The <u>polar cone</u> of the set Ω is
   $\Omega^+ := \{\phi \in X^* : \langle \phi, x \rangle \geq 0, \ \forall x \in \Omega\}$, where X^* is the (topological) dual space of X.

Weak/Strong Duality

Lagrangian: $\mathcal{L}(x,\phi) := f(x) + \langle \phi, \mathcal{G}(x) \rangle$

$$\mu^* = \min_{x \in K} \max_{\phi \in L^+} \mathcal{L}(x,\phi) \geq \nu^* := \max_{\phi \in L^+} \min_{x \in K} \mathcal{L}(x,\phi) \qquad \text{Weak Duality}$$

$$\mu^* = \nu^* = \min_{x \in K} \mathcal{L}(x,\phi^*), \ \text{for some } \phi^* \in L^* \qquad \text{Strong Duality}$$

i.e. Strong Duality means equality and dual attainment for some Lagrange multiplier vector ϕ^*.

CQ and CS

$$\mu^* = \nu^* = \min_{x \in K} \mathcal{L}(x, \phi^*), \text{ for some } \phi^* \in L^* \qquad \text{Strong Duality}$$

Strong duality holds under a constraint qualification, e.g.

$$\exists \hat{x} \in K \text{ with } \mathcal{G}(\hat{x}) \prec 0 \qquad \text{Slater's Condition - strict feasibility}$$

primal attainment, $f(x^*) = \mu^*$, implies

$$\langle \phi^*, \mathcal{G}(x^*) \rangle = 0 \qquad \text{Complementary Slackness}$$

Optimality Conditions

x^* optimal implies

$$\nabla \mathcal{L}(x^*, \lambda^*) \in (K - x^*)^+$$

equivalently,

$$\begin{array}{rclllr} \nabla \mathcal{L}(x^*, \phi^*) - Z & = & 0, & Z \succeq_{K^+} 0 & \phi^* \succeq_{L^+} 0 & \text{D.F} \\ \mathcal{G}(x^*) + W & = & 0, & X^* \succeq_K 0 & W \succeq_L 0 & \text{D.F} \end{array}$$

$$\langle Z, X^* \rangle = 0, \quad \langle \phi^*, W \rangle = 0 \qquad \text{C.S.}$$

LP Example

$$LP \qquad \begin{array}{rrcl} \min & c^T x & & \\ \text{s.t.} & a - Ax & = & 0 \\ & b - Bx & \leq & 0 \\ & x & \geq & 0 \end{array}$$

$$\nabla \mathcal{L}(x^*, \psi^*, \phi^*) = c - A^T \psi^* - B^T \phi^* = w \in (\Re^n_+ - x^*)^+$$

$w^T(0 - x^*) \geq 0$ and $w^T(\alpha x^* - x^*) \geq 0$ implies $w^T x^* = 0$.
$w^T(\alpha y^* - x^*) \geq 0, y \geq 0$ implies $w \geq 0$.
(and strong duality implies $(\psi^*)^T(b - Bx) = 0$)

Failures

$$(CP) \qquad \begin{array}{rll} \mu^* := & \inf & f(x) \\ & \text{subject to} & b - \mathcal{A}(x) = 0 \\ & & x \succeq_K 0, \end{array}$$

$\mathcal{F} = K \cap \{\hat{x} + \mathcal{N}(A)\}$ Feasible Set

$$\begin{array}{rcll} \nabla f(x^*) \in (\mathcal{F} - x^*)^+ & = & \mathrm{cl}\,\{(K - x^*)^+ + \mathbb{R}(\mathcal{A}^*)\} & \text{Opt. Cond.} \\ & \stackrel{?}{=} & (K - x^*)^+ + \mathbb{R}(\mathcal{A}^*) & \text{sum of 2 ccc} \end{array}$$

equivalently

$$\nabla f(x^*) + \mathcal{A}^* \phi \stackrel{?}{\in} (K - x^*)^+ \quad \text{Lagr. Mult. } \phi$$

Specific Failure

$$\begin{pmatrix} \frac{1}{i} & 1 \\ 1 & i \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -i \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin \mathbb{S}^n_+ + \operatorname{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

Let $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$0 = \mu^* := \min \quad \operatorname{trace} CX \text{ s.t. } \operatorname{trace} A_1 X = 0, \qquad X \succeq 0.$$

But

$$C + \phi A_1 \notin (\mathbb{S}^n_+ - 0)^+ = \mathbb{S}^n_+, \qquad \forall \phi \in \Re$$

Fix: Replace $\mathbb{S}^n_+$ with the Minimal Cone

$K = \mathbb{S}^n_+ \cap \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^\perp = \operatorname{cone}\{A_1\}$; so $K^+ = \{A_1\}^+$

2.4 Strong Duality

2.4.1 Outline

- Faces and minimal cones
- Optimality Conditions without constraint qualifcations

2.4.2 Facial Structure

The cone $K \subset T$ is a *face* of the cone T, denoted $K \lhd T$, if

$$x, y \in T, \; x + y \in K \Rightarrow x, y \in K.$$

The faces of $\mathcal{P}$ have a very special structure. Each face, $K \lhd \mathcal{P}$, is characterized by a unique subspace, $S \subset \Re^n$:

$$K = \{X \in \mathcal{P} : \mathcal{N}(X) \supset S\}.$$

Moreover,

$$\operatorname{relint}(K) = \{X \in \mathcal{P} : \mathcal{N}(X) = S\}.$$

(Or use $\mathcal{R}(X)$ and $S^{\perp}$.)

Complementary (or conjugate) Faces

The *complementary* (or conjugate) face of K is $K^c = K^\perp \cap \mathcal{P}$ and

$$K^c = \{X \in \mathcal{P} : \mathcal{N}(X) \supset S^\perp\}.$$

Moreover,

$$\operatorname{relint}(K^c) = \{X \in \mathcal{P} : \mathcal{N}(X) = S^\perp\}.$$

Two additional facts:

1. Each face $\mathcal{K}$ (resp., $\mathcal{K}^c$) is exposed, i.e.

$$\mathcal{K} = \mathcal{P} \cap \phi^{\perp}, \qquad \text{for some } \phi \in \mathcal{P}^{+}$$

   i.e. it equals the intersection of $\mathcal{P}$ with a supporting hyperplane; the supporting hyperplane corresponds to any $X \in \operatorname{relint}(K^c)$ (resp., $\operatorname{relint}(K)$). Also, complementary faces are orthogonal and satisfy $XY = 0, \forall X \in K, Y \in K^c$.

2. The cone $\mathcal{P}$ is projectionally exposed, i.e. every face of $\mathcal{P}$ is the image of $\mathcal{P}$ under some projection. In fact, if $Q \in \mathcal{S}_n$ is the projection onto the subspace S, the null space of matrices in $\operatorname{relint}(K)$, then the face K satisfies

$$K = (I - Q)\mathcal{P}(I - Q).$$

2.4.3 Primal-Dual Pair

Consider the semidefinite linear programming problem

$$\textbf{(P)} \qquad \begin{array}{rcl} p^* = & \sup & c^t x \\ & \text{subject to} & Ax \preceq b \\ & & x \in \Re^m, \end{array}$$

If Slater's CQ holds

$$\exists \hat{x} : A\hat{x} \prec b$$

then strong duality holds

$$\textbf{(D)} \qquad \begin{array}{rcl} p^* = d^* := & \boxed{\min} & \operatorname{trace} bU \\ & \text{subject to} & A^*U = c \\ & & U \succeq 0 \end{array}$$

2.4.4 Minimal Cone

The minimal cone of $(\mathbf{P})$ is defined as

$$\mathcal{P}^f = \cap\{K \lhd \mathcal{P} : K \supset (b - A(F))\},$$

i.e. the minimal cone is the intersection of all faces of $\mathcal{P}$ containing the feasible slacks.

2.4.5 Strong Duality Without CQ

$$(\mathbf{P}) \qquad \begin{array}{rcc} p^* = & \sup & c^t x \\ & \text{subject to} & Ax \preceq_{\mathcal{P}^f} b \\ & & x \in \Re^m, \end{array}$$

$$(\mathbf{D}) \qquad \begin{array}{rcc} p^* = d^* := & \boxed{\min} & \operatorname{trace} bU \\ & \text{subject to} & A^*U = c \\ & & U \succeq_{(\mathcal{P}^f)^+} 0 \end{array}$$

Applications in e.g. QAP,GP,MC ...: [102, 97, 3, 91]

A Related Papers

A.1 Outline

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B Solving SDPs with NEOS and MATLAB

B.1 SDP at NEOS

Solvers for SDP are available at e.g. NEOS: (URL www-neos.mcs.anl.gov/neos/server-solvers.html#SDP) The list of solvers and type of input is:
CSDP [Matlab Binary Input] [Sparse SDPA Input]
DSDP [Sparse SDPA Input]
PENBMI [Matlab Binary Input] [Matlab input]
PENSDP [Matlab Binary Input] [Sparse SDPA Input]
SDPA [Matlab Binary Input] [Sparse SDPA Input]
SDPLR [Matlab Binary Input] [Sparse SDPA Input] [SDPLR input]
SDPT3 [Matlab Binary Input] [Sparse SDPA Input]

SeDuMi [Matlab Binary Input] [Sparse SDPA Input]
You can solve SDP problems with various NEOS interfaces including email.

The Sparse SDPA Input format is described at ftp://plato.asu. and included below. (The notation differs from that we use above. In particular, the primal and dual problems are interchanged.)

We work with a semidefinite programming problem that has been written in the following standard form:

```
(P)     min c1*x1+c2*x2+...+cm*xm
        st  F1*x1+F2*x2+...+Fm*xn-F0=X
                                   X >= 0

The dual of the problem is:

(D)     max tr(F0*Y)
        st  tr(Fi*Y)=ci    i=1,2,...,m
                  Y >= 0
```

Here all of the matrices F0, F1, ..., Fm, X, and Y are assumed to be symmetric of size n by n. The constraints X GE 0 and Y GE 0 mean that X and Y must be positive semidefinite.
Note that several other standard forms for SDP have been

used by a number of authors- these can generally be translated into the SDPA standard form with little effort.
File Format:
The file consists of six sections:
1. Comments. The file can begin with arbitrarily many lines of comments. Each line of comments must begin with '"' or ''.*
2. The first line after the comments contains m, the number of constraint matrices. Additional text on this line after m is ignored.
3. The second line after the comments contains nblocks, the number of blocks in the block diagonal structure of the matrices. Additional text on this line after nblocks is ignored.
4. The third line after the comments contains a vector of numbers that give the sizes of the individual blocks. Negative numbers may be used to indicate that a block is actually a diagonal submatrix. Thus a block size of "-5" indicates a 5 by 5 block in which only the diagonal elements are nonzero.
5. The fourth line after the comments contains the objective function vector c.
6. The remaining lines of the file contain entries in the constraint matrices, with one entry per line. The format for each line is

```
<matno> <blkno> <i> <j> <entry>
```

Here matno *is the number of the matrix to which this entry belongs,* blkno *specifies the block within this matrix, i and j specify a location within the block, and* entry *gives the value of the entry in the matrix. Note that since all matrices are assumed to be symmetric, only entries in the upper triangle of a matrix are given.*
Example:

```
          min 10x1+20x2

          st  X=F1x1+F2x2-F0

              X >= 0

where


  F0=[1 0 0 0
      0 2 0 0
      0 0 3 0
      0 0 0 4]

  F1=[1 0 0 0
```

```
      0 1 0 0
      0 0 0 0
      0 0 0 0]

  F2=[0 0 0 0
      0 1 0 0
      0 0 5 2
      0 0 2 6]
```

In SDPA sparse format,
this problem can be written as:

```
"A sample problem.
2 =mdim
2 =nblocks
2 2
10.0 20.0
0 1 1 1 1.0
0 1 2 2 2.0
0 2 1 1 3.0
0 2 2 2 4.0
1 1 1 1 1.0
1 1 2 2 1.0
2 1 2 2 1.0
```

```
2 2 1 1 5.0
2 2 1 2 2.0
2 2 2 2 6.0
```

B.2 SDP with MATLAB

Following is a MATLAB program that solves the relaxation of the MAXCUT problem taken from [34]. The input is the Laplacian matrix L; the optimal primal matrix X satisfies $\mathrm{diag}\,(X) = \frac{1}{4}e$.

```
function [phi, X, y] = maxcut( L);
% solves: max trace(LX) s.t. X psd, diag(X) = b;  b = ones(n,1)/4
%         min b'y       s.t. Diag(y) - L psd, y unconstrained,
% input:  L ... symmetric matrix
% output: phi ... optimal value of primal, phi =trace(LX)
%         X   ... optimal primal matrix
%         y   ... optimal dual vector
% call:   [phi, X, y] = psd_ip( L);

digits = 6;                        % 6 significant digits of phi
[n, n1] = size( L);                % problem size
b = ones( n,1 ) / 4;               % any b>0 works just as well
X = diag( b);                      % initial primal matrix is pos. def.
y = sum( abs( L))' * 1.1;          % initial y is chosen so that
Z = diag( y) - L;                  % initial dual slack Z is pos. def.
phi = b'*y;                        % initial dual
psi = L(:)' * X( :);               % and primal costs
mu = Z( :)' * X( :)/( 2*n);        % initial complementarity
iter=0;                            % iteration count

disp(['      iter    alphap    alphad      gap     lower      upper']);

while phi-psi > max([1,abs(phi)]) * 10^(-digits)

      iter = iter + 1;             % start a new iteration
      Zi = inv( Z);                % inv(Z) is needed explicitly
      Zi = (Zi + Zi')/2;
      dy =  (Zi.*X) \ (mu * diag(Zi) - b);      % solve for dy
      dX = - Zi * diag( dy) * X + mu * Zi - X;  % back substitute for dX
      dX = ( dX + dX')/2;          % symmetrize
```

```
% line search on primal
      alphap = 2;               % initial steplength changed to 2
%     alphap = 1;               % initial steplength
      [dummy,posdef] = chol( X + alphap * dX );  % test if pos.def
      while posdef > 0,
              alphap = alphap * .8;
              [dummy,posdef] = chol( X + alphap * dX );
              end;
      alphap = alphap * .95;  % stay away from boundary
% line search on dual; dZ is handled implicitly: dZ = diag( dy);
%     alphad = 1;
      alphad = 2;               % initial steplength changed to 2
      [dummy,posdef] = chol( Z + alphad * diag(dy) );
      while posdef > 0;
              alphad = alphad * .8;
              [dummy,posdef] = chol( Z + alphad * diag(dy) );
              end;
      alphad = alphad * .95;
% update
      X = X + alphap * dX;
      y = y + alphad * dy;
      Z = Z + alphad * diag(dy);
      mu = X( :)’ * Z( :) / (2*n);
      if alphap + alphad > 1.8, mu = mu/2; end; % speed up for long steps
      phi = b’ * y; psi = L( :)’ * X( :);
% display current iteration
        disp([ iter alphap alphad (phi-psi) psi  phi ]);

        end;            % end of main loop
```