Cone Preserving Maps

Invariant Ellipsoidal and SDP ***cones***

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ICCOPT$_{\mathrm{I}}$ 2004

First Mathematical Programming Society

International Conference on Continuous Optimization

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OUTLINE

- Background on Ellipsoidal and SDP Cones;
  Notation and Motivation
- Invariant Ellipsoidal Cones
- Invariant examples on EDM and SDP cones
- Extensions to Invariant Cones in $\mathcal{S}^n$

Notation and Motivation

Vandergraft68-Elsner70 Extension of classical
Perron-Frobenius Theory of Nonnegative Matrices:
(Google uses Perron vector in its page rank algorithm cf Moler
- current interest in cones of matrices)

> spectral conditions on A, $n \times n$, to characterize existence of a proper cone $K \subset \mathbb{R}^n$ such that A is K-nonnegative, denoted $A \succeq^K 0$,
>
> $$A(K) \subset K.$$

K proper cone - closed,
convex ($\lambda K \subset K, \forall \lambda \geq 0, K + K \subset K$),
pointed ($K \cap -K = \{0\}$),
int $K \neq \emptyset$

Preliminaries

Also, A is: K-positive, K-irreducible, K-strongly nonnegative

Here we add the restriction: K is ellipsoidal
(ellipsoidal cross sections, translations of Lorentz (ice-cream, 2-nd order) cone; subset of rotund cones, i.e. all proper faces are 1-dimensional exposed rays)

Preview of Results:

Stern-W91: Characterizations unchanged for stronger than K-nonnegativity. But, if degree of spectral radius of A is > 3, then no invariant ellipsoidal cone exists with A K-nonnegative.

new: extensions to cones in $\mathcal{S}^n$

Ellipsoidal Cones, K

$Q = Q^T$, $n \times n$ with exactly one negative eigenvalue

$$\lambda_n < 0 < \lambda_{n-1} \leq \ldots \leq \lambda_1$$

with corresponding normalized eigenvectors u^i

$$K := \left\{ x \in \mathbb{R}^n : x^T Q x \leq 0, x^T u^n \geq 0 \right\} \quad \text{ellipsoidal cone}$$

- $H := \{u^n\}^\perp$ is supporting hyperplane to K and to $-K$.
- $u^n \in \text{int } K$
- For $\alpha > 0$, $S_\alpha := \left\{ x \in K : x^T u^n = \alpha \right\}$ is an ellipsoid in the hyperplane $H_\alpha := \left\{ x \in \mathbb{R}^n : x^T u^n = \alpha \right\}$.
- Each $z \in \partial K$, has unique supporting hyperplane $H_z = (Qz)^\perp$.
- $H_z \cap K$ is an (exposed) extreme ray.

Ice-Cream (Lorentz, 2-nd order) Cone

PROPOSIITON: Suppose K is ellipsoidal cone. Then

$$K = \left\{ x \in \mathbb{R}^n : x^T Q x \leq 0, v^T x \geq 0 \right\},$$

if v satisfies $v^T u^n \geq 0$ and $\{v^T x = 0, x^T Q x \leq 0\} \Rightarrow x = 0.$ $\blacksquare$

$$K_n := \left\{ x \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n-1} x_i^2} \leq x_n \right\} \qquad \text{ice-cream cone}$$

where $Q = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$ and $u^n = e^n$

Ellipsoidal $\leftrightarrow$ Ice-Cream Cone

PROPOSIITON: K is an ellipsoidal cone if and only if $K = T(K_n)$, for some nonsingular T. In particular, $T = UD$, where D is diagonal with elements $D_{ii} = \sqrt{\frac{1}{|\lambda_i|}}, i = 1, \ldots, n$, and $\Lambda = U^T QU$ is the orthogonal diagonalization of Q. ■

COROLLARY: K ellipsoidal, T nonsingular, implies $T(K)$ ellipsoidal.

Real Canonical Form

A real $n \times n$, $C(A)$ is the Real Canonical Form; real similarity of A, unique up to order of blocks;

$$B(\lambda; 1) := (\,\lambda\,)\,, \qquad \text{order-}1 \text{ block}$$

$$B(\lambda; k) := \begin{pmatrix} \lambda & & & & & \\ 1 & \lambda & & & & \\ & 1 & \lambda & & & \\ & & \cdots & \cdots & & \\ & & & \cdots & \cdots & \\ & & & & 1 & \lambda \end{pmatrix}, \qquad \text{order-}k \text{ block}$$

complex eigenvalue $\lambda = a + ib$, $b > 0$, and conjugate $\bar{\lambda}$,

$$\lambda \leftarrow B(a, b; 2) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, 1 \leftarrow I_2 \qquad \text{order-}2 \text{ blocks}$$

Real Canonical Form cont...

$\sigma(A)$ - spectrum; $\rho(A)$ - spectral radius
for $\lambda \in \sigma(A)$ (spectrum): larger below smaller blocks
for $|\lambda| > |\mu|$ blocks for λ below those for μ.
degree of λ, $d(\lambda)$, size of largest block in Jordan canonical form

LEMMA:

1. $\|B(a;1)\|^2 = a^2$
2. $\|B(a;2)\|^2 \leq a^2 + |a| + 1$
3. $\|B(a;k)\|^2 \leq a^2 + 2|a| + 1$, for $k \geq 3$
4. $\|B(a,b;2)\|^2 = a^2 + b^2$
5. $\|B(a,b;4)\|^2 \leq a^2 + b^2 + |a| + |b| + 1$
6. $\|B(a,b;2k)\|^2 \leq a^2 + b^2 + 2(|a| + |b|) + 1$, for $k \geq 3$

Nonnegativity Definitions for K Proper (ellipsoidal) Cone

name	notation	definition	set	ellips.
A K-positive	$A \succ^K 0$	$A(K \backslash \{0\}) \subset \operatorname{int} K$	Π^P	Π_e^P
A K-nonnegative	$A \succeq^K 0$	$A(K) \subset K$	Π^N	Π_e^N
A K-irreducible	$A \succeq^{KI} 0$	$A(K) \subset K$ no eigenv. in ∂K	Π^I	Π_e^I
A strongly K-nonnegative	$A \succeq^{SK} 0$	$A(K) \subset K$ eigenv. in $\operatorname{int} K$	Π^{SN}	Π_e^{SN}

A K-**positive,** $A \succ^K 0$, $A(K \backslash \{0\}) \subset \operatorname{int} K$

$\Pi^P = \{A : A \succ^K 0,$ for some proper cone$K\}$,
$\Pi^P_e = \{A : A \succ^K 0,$ for some ellipsoidal cone$K\}$

THEOREM (V.-E.)
$A \in \Pi^P$
iff
$\rho(A)$ is a simple eigenvalue $>$ modulus of other eigenvalues ■

(K is not ellipsoidal but rather polyhedral in proof.)

THEOREM (S.W.)
For fixed n, $\Pi^P = \Pi^P_e$. ■

Proof Outline

Suppose $\rho(A)$ is a simple eigenvalue $>$ modulus of other eigenvalues. We need only find appropriate K ellipsoidal. Wlog A is in real canonical form. (Will show $A \succ^{K_n} 0$.) A is made up of diagonal blocks corresp. to m distinct eigs:

$$A = \text{Diag}\left(Q_m, \ldots, Q_2, \rho(A) = Q_n\right).$$

Case (i): Suppose $\lambda_j = a_j + ib_j$ and
$(*)\ \ a_j^2 + b_j^2 + 2(|a_j| + |b_j|) + 1 < \rho(A)^2,\ j = 2, \ldots, m.$
For partitioned $0 \neq x \in K_n$ (ice-cream cone) $\Rightarrow$
$x_n \neq 0$ and $\|Q_j\| < \rho(A),\ j = 2, \ldots, m \Rightarrow$
$Ax \in \text{int}\, K_n$ from $(*)$ and

$$\sum_2^m \|Q_j x^j\|^2 \leq \sum_2^m \|Q_j\|^2 \|x^j\|^2 < \rho(A)^2 (x_n)^2.$$

Proof Outline Cont...

Case (i): If $(*)$ fails, since $\rho(A) > |\lambda|$,

$$\lim_{\gamma\to\infty} \frac{(\gamma\rho(A))^2}{(\gamma a_j)^2 + (\gamma b_j)^2 + 2(|\gamma a_j| + |\gamma b_j|) + 1} = \frac{(\rho(A))^2}{(a_j)^2 + (b_j)^2} > 1.$$

Now use $\gamma A \in \Pi_e^P$ iff $A \in \Pi_e^P$. $\blacksquare$

A K-irreducible, $A \succeq^{KI} 0$; $A \succeq^{K} 0$ and no $v \in \partial K$

$\Pi^I = \{A : A \succ^{KI} 0,$ for some proper cone$K\}$,
$\Pi^I_e = \{A : A \succ^{KI} 0,$ for some ellipsoidal cone$K\}$

THEOREM (V.-E.)
$A \in \Pi^I$
iff
$\rho(A)$ is a simple eigenvalue; $|\lambda| = \rho(A)$ implies $d(\lambda) = 1$. ■

THEOREM (R.S.)
For fixed n, $\Pi^I = \Pi^I_e$. ■

Proof Outline

Same proof as above shows that $A \succeq^K 0$.
Wolg A in real canonical form. Then only eigenvector of $\rho(A)$ is unit vector $e^n \in \operatorname{int} K_n$.
If v is eigenvector corresponding to $|\lambda| = \rho(A)$, then v is linear combination of unit vectors not containing e^n, so $v \notin K_n$. $\blacksquare$

A Strongly K-nonnegative, $A \succeq^{SK} 0$; $A \succeq^{K} 0$ and $v \in \operatorname{int} K$

$\Pi^{SN} = \{A : A \succeq^{SK} 0,$ for some proper cone $K\},$
$\Pi_e^{SN} = \{A : A \succeq^{SK} 0,$ for some ellipsoidal cone $K\}$

THEOREM (V.-E.)
$A \in \Pi^{SN}$
iff
$\rho(A) \in \sigma(A)$; $|\lambda| = \rho(A)$ implies $d(\lambda) = 1$. ∎

THEOREM (R.S.)
For fixed n, $\Pi^{SN} = \Pi_e^{SN}$. ∎

Proof Outline

Similar proof; eigenvector in ∂K possible since multiple 1×1 blocks for $\rho(A)$ is possible, e.g. $e^n + e^{n-1}$ gives a boundary eigenvector. ■

REMARK The three results hold for any class of cones which contain ellipsoidal cones, e.g. rotund cones.

Nonnegative Case; $A \succeq^K 0$; $\Pi^N \neq \Pi_e^N$ for $n \geq 3$

$\Pi^N = \{A : A \succeq^K 0,$ for some proper cone$K\}$,
$\Pi_e^N = \{A : A \succeq^K 0,$ for some ellipsoidal cone$K\}$

THEOREM (V.-E.)

$A \in \Pi^N$
iff
$\rho(A) \in \sigma(A)$; $|\lambda| = \rho(A)$ implies $d(\lambda) \leq d(\rho(A))$.

Furthermore, $A \succeq^K 0$ implies K contains an eigenvector corresponding to $\rho(A)$. ■

THEOREM (R.S.)
$\Pi^N = \Pi_e^N$ iff $n \leq 2$. ($n = 1, 2$: all cones are rotund) ■

Geometrical LEMMA 1

K ellipsoidal; $0 \neq z \in \partial K$; $H = (Qz)^{\perp}$. Then:

1. H is the unique supporting hyperplane at z; $v = Qz$ is an outward normal to K at z ($v^T x \leq 0, \forall x \in K$).
2. $H \cap K = \{\alpha z : \alpha \geq 0\}$
3. $z - \epsilon v \in \operatorname{int} K$ for all suff. small $\epsilon > 0$

Geometrical LEMMA 2

$A \succeq^K 0$; K ellipsoidal; eigenvector $z \in \partial K$. Then:
$v = Qz$ is a left eigenvector of A corresp. to nonneg. eigenvalue.

Geometrical LEMMA 3

A in Jordan canonical form; B a block corresp. $\lambda \in \sigma(A)$; eigenvector v assoc. with λ corresponds to B provided all its nonzero entries correspond to positions occupied by B in A.
A has only one eigenvector, v, in K; $v \in \partial K$ and corresp. to the maximal-order block for $\rho(A)$.

Geometrical LEMMA 3
$A \succeq^K 0$; K ellipsoidal; $d(\rho(A)) > 1$. Then:
Jordan canonical form of A has only one maximal-order block for $\rho(A)$. eigenvector $z \in \partial K$.

Exmpl of A with NO Ellipsoidal $\succeq^K 0$

Let

$$A := B(0;3) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (d(a) = 3 > 1)$$

Suppose K ellipsoidal exists for $A \succeq^K 0$.

Then Lemma 3 implies eigenvector $z = \begin{pmatrix} 0 & 0 & \beta \end{pmatrix}^T \in \partial K$;
Lemma 2 implies that the outward normal $v = \begin{pmatrix} \gamma & 0 & 0 \end{pmatrix}$ is a left eigenvector of A;
Lemma 1 yields $\epsilon > 0$ small so that $p = z - \epsilon v \in \operatorname{int} K$.

Then $Ap = \begin{pmatrix} 0 & -\epsilon & 0 \end{pmatrix}^T \in (H \cap K)$ (ray), where $H = v^\perp$. But, $Ap \neq \alpha z$, for any α.
Contradicts Lemma 1. ∎

An Example in $\bar{\Pi}_e^N$

Let $A \in \bar{\Pi}_e^N$ denotes $AK = K$.
Then

$$\lambda > 0 \Rightarrow A = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} \in \bar{\Pi}_e^N,$$

i.e. wlog $\lambda = 1$; use $\mu = 1$

$$Q = \begin{pmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and note $A^T QA - \mu Q = 0, \mu > 0$.
(A characterization for $\pm A \in \bar{\Pi}_e^N$.)

More on Nonnegative Case

Suppose K ellipsoidal with given Q.

$$x \in K \Rightarrow Ax \in K$$

iff

$$x^T Q x \leq 0 \Rightarrow (Ax)^T Q (Ax) \leq 0$$

iff

$$0 = \max x^T (A^T Q A) x \text{ subject to } x^T Q x \leq 0 \quad \text{generalized TRS}$$

S.W. Extension to ellips. of Nec. cond. in Loewy-Schneider/75:

$A^T Q A - \mu Q \preceq 0,$ for some $\mu \geq 0$

(suff. if rank of A is > 1.)

(can also apply TRS opt. conditions of S.W./94 and More/93.)

Characterization of Π_e^N

THEOREM (R.S.) $A \in \Pi_e^N$ if and only if

1. $\rho(A) \in \sigma(A)$;
2. if $\lambda \in \sigma(A), |\lambda| = \rho(A)$, then $d(\lambda) \leq d(\rho(A))$;
3. $d(\rho(A)) \leq 3$ $(d(\rho(A)) \leq 2$ if $\rho(A) = 0)$;
4. Jordan canonical form of A has at most one block of order ≥ 2 corresponding to $|\lambda| = \rho(A)$.

Invariant SDP and EDM Cones

Operator Notation

$$x = \operatorname{svec}(X) \in \mathbb{R}^{\binom{n+1}{2}},$$

$\sqrt{2}$ times vector (columnwise) from upper-triang of X.
$\binom{n+1}{2} = n(n+1)/2$; $\sqrt{2}$ guarantees isometry.

$$\operatorname{sMat} := \operatorname{svec}^{-1}$$

adjoint transformation $\operatorname{sMat}^* = \operatorname{svec}$:

$$\begin{aligned} \langle \operatorname{sMat}(v), S \rangle &= \operatorname{trace} \operatorname{sMat}(v) S \\ &= v^T \operatorname{svec}(S) = \langle v, \operatorname{svec}(S) \rangle \end{aligned}$$

Cone of Euclidean Distance Matrices

$$\mathcal{E}^n := \{D = (d_{ij}) \in \mathcal{S}^n : d_{ij} = \|x_i - x_j\|^2, \text{ for some } x_i \in \Re^k\}$$

k is *embedding dimension*

Operator $\mathcal{L} : \mathcal{S}^{n-1} \to \mathcal{S}^n$

$$D \in \mathcal{E}^n \qquad (\subset \mathcal{S}^n)$$

$$\boxed{\textit{iff}}$$

$$D = \mathcal{L}(X) := \begin{pmatrix} 0 & \operatorname{diag}(X)^T \\ \operatorname{diag}(X) & \operatorname{diag}(X)e^T + e\operatorname{diag}(X)^T - 2X \end{pmatrix},$$

for some $X \succeq 0, X \in \mathcal{S}^{n-1}$

(e is vector of ones)

$$\mathcal{L} : \mathcal{S}^{n-1} \to \mathcal{S}^n, \quad \mathcal{L}(\mathcal{S}_+^{n-1}) = \mathcal{E}^n$$

Properties of $\mathcal{L}$

with partition:

$$D = \begin{bmatrix} \alpha & d^T \\ d & \bar{D} \end{bmatrix},$$

where $\alpha \in \mathbb{R}$

$$\mathcal{L}^*(D) = 2\left(\mathrm{Diag}\,(d) + \mathrm{Diag}\,(\bar{D}e) - \bar{D}\right)$$

$$\mathcal{L}^\dagger(D) = \frac{1}{2}\left(de^T + ed^T - \bar{D}\right)$$

$$\mathcal{L}^*, \mathcal{L}^\dagger : \mathcal{S}^n \to \mathcal{S}^{n-1}, \qquad \mathcal{L}^\dagger(\mathcal{E}^n) = \mathcal{S}^{n-1}_+$$

Invariance on SDP and $\mathcal{E}^n$ Cones

Let H be $n \times n$ symmetric matrix with nonnegative elements and 0 diagonal and with no zero row (or column).

$$X \succeq 0 \ \ (\text{resp. } \succ 0) \Rightarrow \mathcal{L}^*(H^{(2)} \circ \mathcal{L}(X)) \succeq 0 \ \ (\text{resp. } \succ 0),$$

i.e. $\mathcal{S}^n_+$ (and its interior) is invariant under the operator

$$\mathcal{W}(\cdot) := \mathcal{L}^*(H^{(2)} \circ \mathcal{L}(\cdot))$$

Perron Roots/Vectors

$\mathcal{W}_E(\cdot) := \mathcal{L}^*\mathcal{L}(\cdot)$ is strongly $\mathcal{S}^n_+$ nonnegative

The Perron root and eigenvector are:

$$\lambda = (2n+1) + \frac{\sqrt{(4n+2)^2 - 32}}{2} > 0, \qquad X = \alpha e e^T + \beta I \succ 0,$$

where $\alpha = -\frac{4\beta}{\lambda},\ \beta > 0.$

Perron Roots/Vectors cont...

Similarly: The cone $\mathcal{E}^n$ is strongly invariant under the linear operator $\mathbb{V} := \mathcal{L}\mathcal{L}^*$.

The Perron root and vector of $\mathcal{L}\mathcal{L}^*$ are

$$\lambda = (2n-1) + \sqrt{(2n-3)^2 + 8(n-2)} > 0$$

and

$$D = \begin{pmatrix} 0 & e^T \\ e & \alpha(E-I) \end{pmatrix}, \quad \alpha > 0,$$

where

$$\alpha = \frac{1}{2(n-2)} \left\{ (2n-3) + \sqrt{(2n-3)^2 + 8(n-2)} \right\},$$

D is nonsingular and $\mathcal{L}^\dagger(D) \succ 0$.

Ellipsoidal Cones in $\mathcal{S}^n$ $(\subset, \supset \mathcal{S}^n_+)$

LEMMA Let $\alpha \neq 0$.

$$
\begin{aligned}
K_\alpha &:= \left\{X \in \mathcal{S}^n : \alpha^2 \operatorname{trace} X^2 - \tfrac{\alpha^2+1}{n}(\operatorname{trace} X)^2 \leq 0, \operatorname{trace} X \geq 0\right\} \\
&= \left\{X \in \mathcal{S}^n : \frac{\operatorname{trace} X}{n} - |\alpha| \sqrt{\frac{\operatorname{trace} X^2}{n} - \left(\frac{\operatorname{trace} X}{n}\right)^2} \geq 0\right\}
\end{aligned}
$$

is an ellipsoidal cone.

And:

$$\alpha^2 \geq n-1 \quad \Rightarrow \quad K_\alpha \subset \mathcal{S}^n_+$$

$$\alpha^2 \leq \frac{1}{n-1} \quad \Rightarrow \quad K_\alpha \supset \mathcal{S}^n_+$$

Proof Outline

Consider the Rayleigh Principle (and Courant-Fisher):

$\min_X \quad \alpha^2 \operatorname{trace} X^2 - \frac{\alpha^2+1}{n}(\operatorname{trace} X)^2 = \langle \operatorname{svec}(X), Q\operatorname{svec}(X)\rangle$

with constraint $\operatorname{trace} X^2 = 1$, i.e.

$$0 = \nabla L(X,\lambda) = 2\alpha^2 X - 2\frac{\alpha^2+1}{n}(\operatorname{trace} X)I - 2\lambda X$$

implies the eigenvector $X = \frac{1}{\sqrt{n}} I$ and the eigenvalue

$\lambda_{\min} = \alpha^2 \operatorname{trace} X^2 - \frac{\alpha^2+1}{n}(\operatorname{trace} X)^2 = -1 < 0$.

And the second eigenvector $\langle X, I\rangle = 0, X \neq 0$ implies

$\operatorname{trace} X = 0, X \neq 0$

implies $\alpha^2 \operatorname{trace} X^2 - \frac{\alpha^2+1}{n}(\operatorname{trace} X)^2 = \alpha^2 \operatorname{trace} X^2 > 0$,

i.e. K_α is ellipsoidal.

Proof Outline Cont...

Suppose $X \in K_\alpha$, with eigs λ_i, i.e.
$(*)(n-1)\sum_i \lambda_i^2 - (\sum_i \lambda_i)^2 \leq 0.$
Now $\min \lambda_n$ subject to $*$.
Get (W.Styan80)

$$\lambda_n \geq \frac{\operatorname{trace} X}{n} - \sqrt{n-1}\sqrt{\frac{\operatorname{trace} X^2}{n} - \left(\frac{\operatorname{trace} X}{n}\right)} \geq 0 \Rightarrow X \succeq 0.$$

Similarly

$$X \succeq 0 \Rightarrow \frac{\operatorname{trace} X}{n} - \frac{1}{\sqrt{n-1}}\sqrt{\frac{\operatorname{trace} X^2}{n} - \left(\frac{\operatorname{trace} X}{n}\right)} \geq \lambda_n \geq 0$$

■