A SIMPLE Iterative Method for Linear and Semidefinite Programming

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primal and dual SDPs

$$
\text{(PSDP)} \qquad \begin{array}{rll} p^* := & \max & \operatorname{trace} CX \\ & \text{s.t.} & \mathcal{A}X = b \\ & & X \succeq 0. \end{array}
$$

$$
\text{(DSDP)} \qquad \begin{array}{rll} d^* := & \min & b^T y \\ & \text{s.t.} & \mathcal{A}^* y - Z = C \\ & & Z \succeq 0, \end{array}
$$

$\mathcal{S}^n$ space of $n \times n$ real symmetric matrices; $X, Z, C \in \mathcal{S}^n$;
$\succ 0$ ($\succeq 0$) denotes positive (semi)definiteness
(i.e. Loewner partial order);
$\mathcal{A} : \mathcal{S}^n \to \Re^m$ linear operator; $\mathcal{A}^*$ adjoint operator

Motivation/Background:
Many applications; polynomial time algorithms; difficulty with large sparse problems.

SDP (cone optimization) studied back in 1960's;
Engineering applications in 60's;
matrix completion problems in 80's;
polynomial time algorithms Nesterov-Nemirovski 80's;
combinatorial applications and p-d i-p algorithms (explosion) in 90's.

SDP looks like a linear program (similar development)

For p-d i-p path following approach, use e.g. log-barrier on dual-problem to perturbed optimality conditions:

$$F_\mu(X, y, Z) := \begin{pmatrix} \mathcal{A}^* y - Z - C \\ \mathcal{A}X - b \\ X - \mu Z^{-1} \end{pmatrix} = 0.$$

square system: $F_\mu : \mathcal{S}^n \times \Re^m \times \mathcal{S}^n \to \mathcal{S}^n \times \Re^m \times \mathcal{S}^n$

So can apply Newton's method;

But: highly nonlinear; slow convergence; condition number of $F'_\mu \to \infty$ as $\mu \to 0$.

Replace perturbed complementary slackness conditions with

$$ZX - \mu I = 0.$$

Z, X diagonal in LP case, so ZX is diagonal. But ZX not necessarily symmetric in SDP case, so new
$F_\mu : \mathcal{S}^n \times \Re^m \times \mathcal{S}^n \rightarrow \mathcal{S}^n \times \Re^m \times \mathcal{M}^n$, i.e.

overdetermined nonlinear system in SDP case

current approach (follows successful LP development; however, nonsquare system resulted in symmetrization approaches);

Premultiplication by Z (a form of preconditioning) leads to a less nonlinear system; avoids the ill-conditioning;

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Z\cdot \end{pmatrix} \begin{pmatrix} \mathcal{A}^* y - Z - C \\ \mathcal{A} X - b \\ X - \mu Z^{-1} \end{pmatrix} = \begin{pmatrix} \mathcal{A}^* y - Z - C \\ \mathcal{A} X - b \\ ZX - \mu I \end{pmatrix} = 0.$$

We now precondition a second time using the linear operator $\mathcal{P}$ with symmetrization operator $\mathcal{S} : \mathcal{M}^n \to \mathcal{S}^n$.

$$\begin{aligned}
\mathcal{P}F_\mu(x, y, Z) &= \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathcal{S} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Z\cdot \end{pmatrix} \begin{pmatrix} \mathcal{A}^* y - Z - C \\ \mathcal{A}X - b \\ X - \mu Z^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{A}^* y - Z - C \\ \mathcal{A}X - b \\ \mathcal{S}(ZX) - \mu I \end{pmatrix} = 0
\end{aligned}$$

However, symmetrizations, in general, reverse previous process, ill-conditioning problem returns;

e.g. AHO, HKM, NT, GT, MTW directions

For many different symmetrizations, e.g. Monteiro-Zhang family, for given P nonsingular (scale-symmetrize):
$\mathcal{S}(M) = \frac{1}{2}\left(PMP^{-1} + P^{-T}M^TP^T\right)$

$$\mathcal{Z}(\cdot) := \mathcal{S}(Z\cdot); \quad \mathcal{X}(\cdot) := \mathcal{S}(\cdot X)$$

use equation 1; eliminate ΔZ:

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\mathcal{X} & 0 & I \end{pmatrix} \begin{pmatrix} 0 & \mathcal{A}^* & I \\ \mathcal{A} & 0 & 0 \\ \mathcal{Z} & 0 & \mathcal{X} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{A}^* & I \\ \mathcal{A} & 0 & 0 \\ \mathcal{Z} & -\mathcal{X}\mathcal{A}^* & 0 \end{pmatrix}.$$

use equation 3; eliminate ΔX:

$$
\begin{aligned}
F_n := P_n K \quad &:= \begin{pmatrix} I & 0 & 0 \\ 0 & I & -\mathcal{A}\mathcal{Z}^{-1} \\ 0 & 0 & \mathcal{Z}^{-1} \end{pmatrix} \begin{pmatrix} 0 & \mathcal{A}^* & I \\ \mathcal{A} & 0 & 0 \\ \mathcal{Z} & -\mathcal{X}\mathcal{A}^* & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathcal{A}^* & I_n \\ 0 & \mathcal{A}\mathcal{Z}^{-1}X\mathcal{A}^* & 0 \\ I_n & -\mathcal{Z}^{-1}X\mathcal{A}^* & 0 \end{pmatrix}
\end{aligned}
$$

algorithm for Newton search direction using the normal equations (need existence of $\mathcal{Z}^{-1}$):

use the second row (the operator $\mathcal{A}\mathcal{Z}^{-1}\mathcal{X}\mathcal{A}^*$) to solve for Δy;

back solve for ΔX using the third row ;

backsolve for ΔZ using the first row.

condition number for system found using matrix F_n, not just matrix $\mathcal{A}\mathcal{Z}^{-1}\mathcal{X}\mathcal{A}^T$. (Though, latter can have uniformly bounded condition number under standard neighbourhood assumptions)

$$F_n^* F_n = \begin{pmatrix} I_n & -\mathcal{Z}^{-1}\mathcal{X}\mathcal{A}^* & 0 \\ -\mathcal{A}\mathcal{X}\mathcal{Z}^{-1} & (\mathcal{A}\mathcal{A}^* + (\mathcal{A}\mathcal{Z}^{-1}\mathcal{X}\mathcal{A}^*)^2 + \mathcal{A}\mathcal{Z}^{-2}\mathcal{X}^2\mathcal{A}^*) & \mathcal{A} \\ 0 & \mathcal{A}^* & I_n \end{pmatrix}.$$

using interlacing of eigenvalues, this operator becomes increasingly ill-conditioned

Simple/Stable Reduction

There are other choices for the above second step for LP, e.g. so-called augmented system or quasi-definite system.
Other choices for SDP usually involve *dual directions*.

New *second elimination for SDP*:

linear constraint $\mathcal{A}(X) = b$, (onto linear operator $\mathcal{A}$, full rank m);
range of linear operator $\mathcal{N} : \Re^{\binom{n+1}{2}-m} \to \mathcal{S}^n$ equals null space of $\mathcal{A}$;
$\mathcal{A}^\dagger$ generalized inverse; $A\hat{X} = b$:

$$
\begin{array}{lll}
\mathcal{A}(X) = b & \text{iff} & X = \mathcal{A}^\dagger b + (I - \mathcal{A}\mathcal{A}^\dagger)V, \quad \text{for some } V \in \mathcal{S}^n \\
 & \text{iff} & X = \hat{X} + \mathcal{N}(v), \quad \text{for some } v \in \Re^{\binom{n+1}{2}-m}.
\end{array}
$$

We partition the diagonal matrix Z, X using the vectors $z = \begin{pmatrix} z_m \\ z_v \end{pmatrix}, \quad x = \begin{pmatrix} x_m \\ x_v \end{pmatrix}$ with lengths $m, v = n - m$; and we define the matrices F_s, P_s with

$$
\begin{aligned}
F_s : &= P_s K = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & -Z_m & I_m & 0 \\ 0 & 0 & 0 & I_v \end{pmatrix} \begin{pmatrix} 0 & 0 & A^T & I_n \\ I_m & E & 0 & 0 \\ Z_m & 0 & -X_m & 0 \\ 0 & Z_v & -X_v E^T & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & A^T & I_n \\ I_m & E & 0 & 0 \\ 0 & -Z_m E & -X_m & 0 \\ 0 & Z_v & -X_v E^T & 0 \end{pmatrix}.
\end{aligned}
$$

Illustrate on SDP Relaxation of Max-Cut problem

MC:

> em Partition of the set of vertices of a given undirected graph with weights on the edges so that the sum of the weights of the edges cut by the partition is maximized.

This NP-hard discrete optimization problem can be formulated as a (quadratic) program (e.g. Q is a multiple of the Laplacian matrix of the graph).

$$(\mathrm{MC0}) \qquad \begin{array}{rll} \mu^* := & \max & v^T Q v \\ & \text{s.t.} & v_i^2 = 1, \quad i = 1, \ldots, n. \end{array}$$

Using e.g. Lagrangian relaxation, get SDP relaxation of (MC):

$$
(\mathrm{P}) \qquad \begin{array}{rll} \mu^* \leq \nu^* := & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X) = e \\ & & X \succeq 0, X \in \mathcal{S}^n, \end{array}
$$

diag is vector from the diagonal elements;
e (column) vector of ones;
wolg $\operatorname{diag}(Q) = 0$.

And its (strong) dual SDP:

$$
(D) \qquad \begin{array}{rll} \nu^* = & \min & e^T y \\ & \text{subject to} & \operatorname{Diag}(y) - Z = Q \\ & & Z \succeq 0. \end{array}
$$

Diag is diagonal matrix from vector

Preprocess:

for $X \in \mathcal{S}^n$, u2svec columnwise, strict upper triangular part of X; $\sqrt{2}$ for isometry:

$$x = \operatorname{u2svec}(X) := \sqrt{2}\begin{pmatrix} X_{12} \\ X_{13} \\ X_{23} \\ X_{14} \\ \ldots \end{pmatrix} \in \Re^{\binom{n}{2}}$$

$$\operatorname{u2sMat} := \operatorname{u2svec}^{\dagger} = \operatorname{u2svec}^{*} : \Re^{\binom{n}{2}} \to \mathcal{S}^n$$

$$\boxed{\operatorname{diag}(X) = e \quad \text{iff} \quad X := I + \operatorname{u2sMat}(x)} \quad \text{for some } x \in \Re^{\binom{n}{2}}$$

get single (overdetermined) perturbed optimality condition

$$F_\mu(x, y) := (\mathrm{Diag}\,(y) - Q)(\mathrm{u2sMat}\,(x) + I) - \mu I = 0$$

$$F_\mu(x, y) : \Re^{\binom{n}{2}} \times \Re^n \to \mathcal{M}^n.$$

$$\mathrm{Diag}\,(y) - Q \succeq 0, \quad \mathrm{u2sMat}\,(x) + I \succeq 0$$

linearization (or Gauss-Newton equation) for search direction $\Delta x, \Delta y$ is (vec is understood)

$$\begin{aligned}
-F_\mu(x,y) &= F'_\mu(x,y)\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}\\
&= \mathcal{Z}(\Delta x) + \mathcal{X}(\Delta y)
\end{aligned}$$

i.e.

$$(\operatorname{Diag}(y) - Q)\operatorname{u2sMat}(\Delta x) + \operatorname{Diag}(\Delta y)(\operatorname{u2sMat}(x) + I) = -F_\mu(x,y)$$

which defines the two linear operators

$$\mathcal{Z}, \quad \mathcal{X}$$

This is a linear, full rank, overdetermined system and we find the least squares solution.

The first part $\mathcal{Z}(\Delta x)$ is the large part of the system since it has $n(n-1)/2$ variables. However, the operator $\mathcal{Z}$ is sparse if the data Q is sparse. The second part is the small part since it only has n variables, though the operator $\mathcal{X}$ is usually dense. This is the size of the normal equations system that arises in the standard approaches for SDP.

The above system is full rank at each iteration, and, in addition, it has the strong property of being full rank at optimality, i.e. it does not necessarily get ill-conditioned as μ approaches 0.

START Algorithm (p-d; GN with PCG)

- **Initialization:**

  •• **Input data:** a real symmetric $n \times n$ matrix Q (set $\operatorname{diag}(Q) = 0$)

  •• **Positive tolerances:** δ_1 (relative duality gap), δ_2 (accuracy for lss)

  •• **Find initial feasible points:**

$$\text{set:} \qquad x^0 = 0; \quad y^0 = 2\operatorname{Diag}(q) - Q,$$

  where $q_j = \|Q_{:,j}\|_2$;

  (guarantees: both

  $X := (\operatorname{u2sMat}(x) + I), Z := (\operatorname{Diag}(y) - Q) \succ 0$.

  •• **Set initial parameters:**

$$\text{gap} = \operatorname{trace} ZX; \quad \mu = \text{gap}/n; \quad \text{optval} = \operatorname{trace} QX; \quad k = 0.$$

- **while** $\{\frac{\text{gap}}{|\text{optval}|+1}\} > \delta_1$

  •• **solve for the search direction** (to tolerance $\max\left\{\frac{\delta_2}{10}, \frac{\min\{\mu^{\frac{1}{3}},1\}}{10^6}\right\}$, in a least squares sense with PCG, using k-th iterate $(x,y)=(x^k,y^k)$)

$$F'_{\sigma\mu}(x,y)\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = -F_{\sigma\mu}(x,y),$$

  where σ_k centering,
  $\mu_k = \text{trace}\,(\text{Diag}\,(y)-Q)(\text{u2sMat}\,(x)+I)/n$

$$x^{k+1} = x+\alpha_k\Delta x, \quad y^{k+1} = y+\alpha_k\Delta y; \quad \alpha_k > 0,$$

  so that both
  $(X := \text{u2sMat}\,(x^{k+1})+I), (Z := \text{Diag}\,(y^{k+1})-Q) \succ 0$,
  before the crossover; so that $\alpha_k = 1$ after the crossover.

•• **update**

$$k \leftarrow k+1 \quad \text{and then update}$$

$$\mu_k, \sigma_k \quad (\sigma_k = 0 \text{ after the crossover}).$$

• **end (while)**.

• **Conclusion:** optimal X is approx. $\mathrm{u2sMat}\,(x) + I$

END ALGORITHM

Optimal Diagonal Column Preconditioning

for $A, m \times n, m \geq n$ a full rank matrix,
solution of optimization problem
$\min \omega((AD)^T(AD))$, over all positive diagonal matrices D,
condition number $\omega(K) = \frac{\mathrm{trace}\,(K)/n}{\det(K)^{\frac{1}{n}}}$

$$\text{optimum: } D_{rr} = \frac{1}{\|A_{:r}\|}.$$

columns of the operator $F'(x,y)(\cdot,\cdot) = \mathcal{Z}(\cdot) + \mathcal{X}(\cdot)$
$(F' = [\mathcal{Z} \mid \mathcal{X}])$
ordered using $c = 1, 2, \ldots,$ c represents $(k,j),\ 1 \le k < j \le n$ for the upper triangular part of X; then represents $k = 1, \ldots, n$ for the elements of y

$$
\begin{aligned}
\mathcal{Z}(e_c) &= \operatorname{vec}(\operatorname{Diag}(y) - Q)\operatorname{u2sMat}(e_c) \\
&= \frac{1}{\sqrt{2}}\operatorname{vec}(\operatorname{Diag}(y) - Q)\left(e_k e_l^T + e_l e_k^T\right) \\
&= \frac{1}{\sqrt{2}}\operatorname{vec}\left\{\left(y_k e_k e_l^T + y_l e_l e_k^T\right) - \left(Q_{:k} e_l^T + Q_{:l} e_k^T\right)\right\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|\mathcal{Z}(e_c)\|_F^2 &= \frac{1}{2}\left\{\|y_k e_k - Q_{:k}\|^2 + \|y_l e_l - Q_{:l}\|^2\right\} \\
&= \frac{1}{2}\left\{\|Q_{:k}\|^2 + \|Q_{:l}\|^2 + y_k^2 + y_l^2\right\},
\end{aligned}
$$

since $\operatorname{diag}(Q) = 0$ by assumption.

$$\mathcal{X}(e_i) = \mathrm{vec}\ (\mathrm{Diag}\,(e_i)(\mathrm{u2sMat}\,(x) + I))\,.$$

Therefore

$$\|\mathcal{X}(e_i)\|_F^2 = \|X_{i,:}\|^2.$$

Partial (Block) Cholesky Preconditioning

Jacobian $F' = [\mathcal{Z} \mid \mathcal{X}]$

$$(F')^* F' = \begin{pmatrix} \mathcal{Z}^*\mathcal{Z} & \mathcal{Z}^*\mathcal{X} \\ \mathcal{X}^*\mathcal{Z} & \mathcal{X}^*\mathcal{X} \end{pmatrix}$$

We find a partial Cholesky factorization of $(F')^* F'$ using incomplete Cholesky factorizations of the two diagonal blocks.

Bottom right block $\mathcal{X}^*\mathcal{X}$ is a positive definite diagonal matrix with diagonal elements $\|X_{i,:}\|^2$. Therefore, cannot improve on the diagonal scaling.

We use the transformation between indices

$$c \cong (k,l), \quad c = \frac{(l-1)(l-2)}{2} + k, \quad k \leq c.$$

The columns of top left block:

$$\begin{array}{rcl} \mathcal{Z}^* Z(e_c) & = & \frac{1}{2\sqrt{2}}\mathrm{u2svec}\left\{Z^2\left(e_k e_l^T + e_l e_k^T\right) + \left(e_k e_l^T + e_l e_k^T\right)Z^2\right\} \\ & = & \frac{1}{2\sqrt{2}}\mathrm{u2svec}\begin{pmatrix} \text{in row } k & \left(Z^2\right)_{l:} \\ \text{in row } l & \left(Z^2\right)_{k:} \\ \begin{pmatrix}\text{in col } k \\ \left(Z^2\right)_{:l}\end{pmatrix} & \begin{pmatrix}\text{in col } l \\ \left(Z^2\right)_{:k}\end{pmatrix} \end{pmatrix} \end{array}$$

For $i \neq j$, $E_{ij} = \frac{1}{\sqrt{2}}\left(e_i e_j^T + e_j e_i^T\right)$ denotes orthonormal basis for the symmetric matrix space. (When $i = j$, we use $e_i e_i^T$.) δ_{ij} denotes the *Kronecker delta*.

Therefore, element in row $r \cong (i, j)$ and column $c \cong (k, l)$ of $\mathcal{Z}^*\mathcal{Z}$ is:

$$
\begin{aligned}
&\langle \operatorname{u2svec}(E_{ij}), \mathcal{Z}^*\mathcal{Z}(\operatorname{u2svec}(E_{kl}))\rangle = \\
&\quad = \tfrac{1}{2}\operatorname{u2svec}(E_{ij})^T \operatorname{u2svec}\left\{Z^2 E_{kl} + E_{kl} Z^2\right\} \\
&\quad = \tfrac{1}{2}\operatorname{trace}(E_{ij})\left\{Z^2 E_{kl} + E_{kl} Z^2\right\} \\
&\quad = \operatorname{trace} E_{ij} E_{kl} Z^2 \\
&\quad = \tfrac{1}{2}\left\{\delta_{jk}(Z^2)_{li} + \delta_{jl}(Z^2)_{ki} + \delta_{ik}(Z^2)_{lj} + \delta_{il}(Z^2)_{kj}\right\}.
\end{aligned}
\tag{0.0}
$$

Can exploit graph structure since sparsity patterns of Z, Q are the same.

Crossover Criteria

Advantage of GN approach: full rank and zero residual.

Therefore, there is a local neighbourhood of quadratic convergence for affine scaling with step lengths of one without backtracking to maintain positive definiteness.

$$
\begin{aligned}
\|F'(\Delta s)\|_F &= \|Z\,\mathrm{u2sMat}\,(\Delta x) + \mathrm{Diag}\,(\Delta y) X\|_F \\
&\leq \|Z\,\mathrm{u2sMat}\,(\Delta x)\|_F + \|\mathrm{Diag}\,(\Delta y) X\|_F \\
&\leq \|Z\|_F \|\mathrm{u2sMat}\,(\Delta x)\|_F + \|\mathrm{Diag}\,(\Delta y)\|_F \|X\|_F \\
&= \|Z\|_F \|\Delta x\|_2 + \|\Delta y\|_2 \|X\|_F \\
&\leq \sqrt{\|Z\|_F^2 + \|X\|_F^2}\,\|\Delta s\|_2, \ \text{(by C-S ineq.)}
\end{aligned}
$$

bound on the norm of the Jacobian is $\alpha = \sqrt{\|Z\|_F^2 + \|X\|_F^2}$.

$$
\begin{aligned}
&\|F'(s - \bar{s})(\Delta s)\|_F = \\
&\quad = \|(Z - \bar{Z})\mathrm{u2sMat}\,(\Delta x) + \mathrm{Diag}\,(\Delta y)(X - \bar{X})\|_F \\
&\quad \leq \|(Z - \bar{Z})\|_F \|\mathrm{u2sMat}\,(\Delta x)\|_F + \|\mathrm{Diag}\,(\Delta y)\|_F \|(X - \bar{X})\|_F \\
&\quad = \|(y - \bar{y})\|_2 \|\Delta x\|_2 + \|\Delta y\|_2 \|(x - \bar{x})\|_2.
\end{aligned}
$$

Lipschitz constant is $\gamma = 1$.

suppose optimum s^* is unique and the smallest singular value satisfies $\sigma_{\min}(F'(s)) \geq \sqrt{K}$, for all s in an ϵ_1 neighbourhood of s^*, for some constant $K > 0$. Then q-quadratic convergence is guaranteed once the current iterate is in the

$$\epsilon := \min\left\{\epsilon_1, \frac{1}{K\alpha\gamma}\right\} = \min\left\{\epsilon_1, \frac{1}{K\sqrt{\|Z^*\|_F^2 + \|X^*\|_F^2}}\right\}.$$

neighbourhood of the optimum.

Heuristic: start the crossover if $\sigma_{\min}(F'(s)) \geq 2\|ZX\|_2$. In our tests we started the crossover when the relative duality gap $\frac{\operatorname{trace} ZX}{|\operatorname{trace} QX|+1} < .1$; this simpler heuristic never failed to converge, q-quadratic convergence was detected empirically.

Numerical Tests

The crossover to using a steplength of 1, not maintaining interiority, and using affine scaling, has a significant effect on both the number of iterations and the conditioning of the linear system. The number of iterations are approximately halved. The best results were obtained by staying well-centered before the crossover.

The tests were done using MATLAB 6.0.0.88 (R12) on a SUNW, Ultra $5-10$ with one GB of RAM using SunOS 5.8 (denoted by SUN), as well as on a Toshiba Pentium II, 300 MHZ, with 128 MB of RAM (denoted by PII). 99% of the cputime was spent on finding the search direction.

The cost for the early iterations was very low, e.g. $21, 50$ CG iterations, $24, 58$ cpu seconds for the first two iterations for $n = 365$ on the SUN. This cost increased as the relative duality gap (and the requested tolerance for the least squares solution) decreased. Low accuracy solutions were obtained quickly, e.g. one decimal accuracy after 4 to 5 iterations. The cost per iteration increased steadily and then levelled off near the optimum.

Cross. tol. in rel. gap	nmbr iter	norm(ZX) at end	min. eig. pos. violation in Z, X
1.0e-1	11	3.6281e-12	-4.6478e-12
1.0e-3	14	1.9319e-13	-2.1580e-13
1.0e-5	16	2.0917e-11	-2.2839e-11
1.0e-7	18	1.8293e-13	-1.5546e-13
1.0e-9	20	7.2534e-10	6.7006e-12
1.0e-11	20	7.2544e-10	6.7004e-12

Table 0.1: Size of Q, $n = 25$; Requested Accuracy in relative duality gap: 1.0e-10; Density is $\frac{1}{25}$; Optimal Value is 5.7041e+01.

Dim. of Q n	total cpu sec.	nmbr iter	norm(ZX) at end	violation of eig. pos. in Z, X
55	6.6000e+01	11	9.0884e-09	-1.7515e-08
65	1.1234e+02	11	7.2081e-10	-2.5326e-09
75	1.9134e+02	13	2.1361e-11	-1.7416e-10

Table 0.2: Requested Accuracy in relative duality gap: 1.0e-10; Density is $\frac{1}{n}$; crossover tolerance 1.0e-01; optimal values of order 150; SUN computer.

Dim. of Q n	total cpu sec	nmbr iter	norm(ZX) at end	violation of eig. pos. in Z, X
165	2635	14	7.7056e-14	-2.775e-14
175	6778.6	15	8.0369e-14	-2.5371e-14
185	6908.8	15	9.0987e-14	-4.011e-14
195	11397	15	1.7324e-12	-3.2036e-11
205	8958.5	15	1.0029e-13	-3.7858e-14

Table 0.3: Toler. relative duality gap: 1.0e-14; Density $\frac{1}{n}$; cross. tol. 1.0e-01; opt. values order 1000; SUN computer.

Dim Q n	total cpu sec	nmbr iter	norm(ZX) at end	violation of eig. pos. in Z, X
15	7.75	9	1.7535e-014	-3.0863e-015
25	17.36	10	1.9291e-013	-1.5217e-013
35	30.43	10	1.1044e-012	-8.4977e-013
45	59.87	11	2.7926e-011	-2.763e-011
55	86.89	12	4.0912e-011	-3.7568e-011
65	131.11	11	4.9924e-012	-5.0519e-012
75	275.89	12	1.0448e-008	-1.4504e-008
85	468.46	14	8.6208e-011	-3.4404e-010
95	414.03	11	7.4253e-011	-3.0935e-010
105	878.64	15	7.5993e-010	-1.4946e-008

Table 0.4: Tol. in rel. duality gap: 1.0e-10; Density is $\frac{3}{n}$; cross. toler. 1.0e-01; optimal values of order 150; PII computer.

Dim of Q n	total cpu sec	nmbr iter	norm(ZX) at end	violation of eig. pos. in Z, X
15	30.1	11	1.9039e-011	-3.5098e-012
25	134.13	15	1.9742e-009	-2.7465e-010
35	117.27	12	2.5534e-010	-8.8532e-011
45	174.34	11	1.3799e-008	-1.9154e-008
55	373.6	14	7.3741e-009	-3.1854e-009
65	1388.1	19	9.6109e-009	-2.7634e-009
75	507.46	13	3.5887e-009	-3.3886e-009
85	1856.1	16	6.4806e-011	-4.7964e-011

Table 0.5: **Without preconditioning**; Requested Accuracy in relative duality gap: 1.0e-10; Density is $\frac{3}{n}$; crossover tolerance 1.0e-01; optimal values of order 150; PII computer.

Dim of Q n	cpu sec	nmbr iter	norm(ZX) at end	violation of eig. pos. in Z, X
155	5493.6	14	6.068e-014	-4.3061e-013
165	4595.4	13	3.8625e-014	-2.8454e-013

Table 0.6: Requested Accuracy in relative duality gap: 1.0e-14; Density is $\frac{1}{n}$; crossover tolerance 1.0e-01; optimal values of order 150; PII computer.

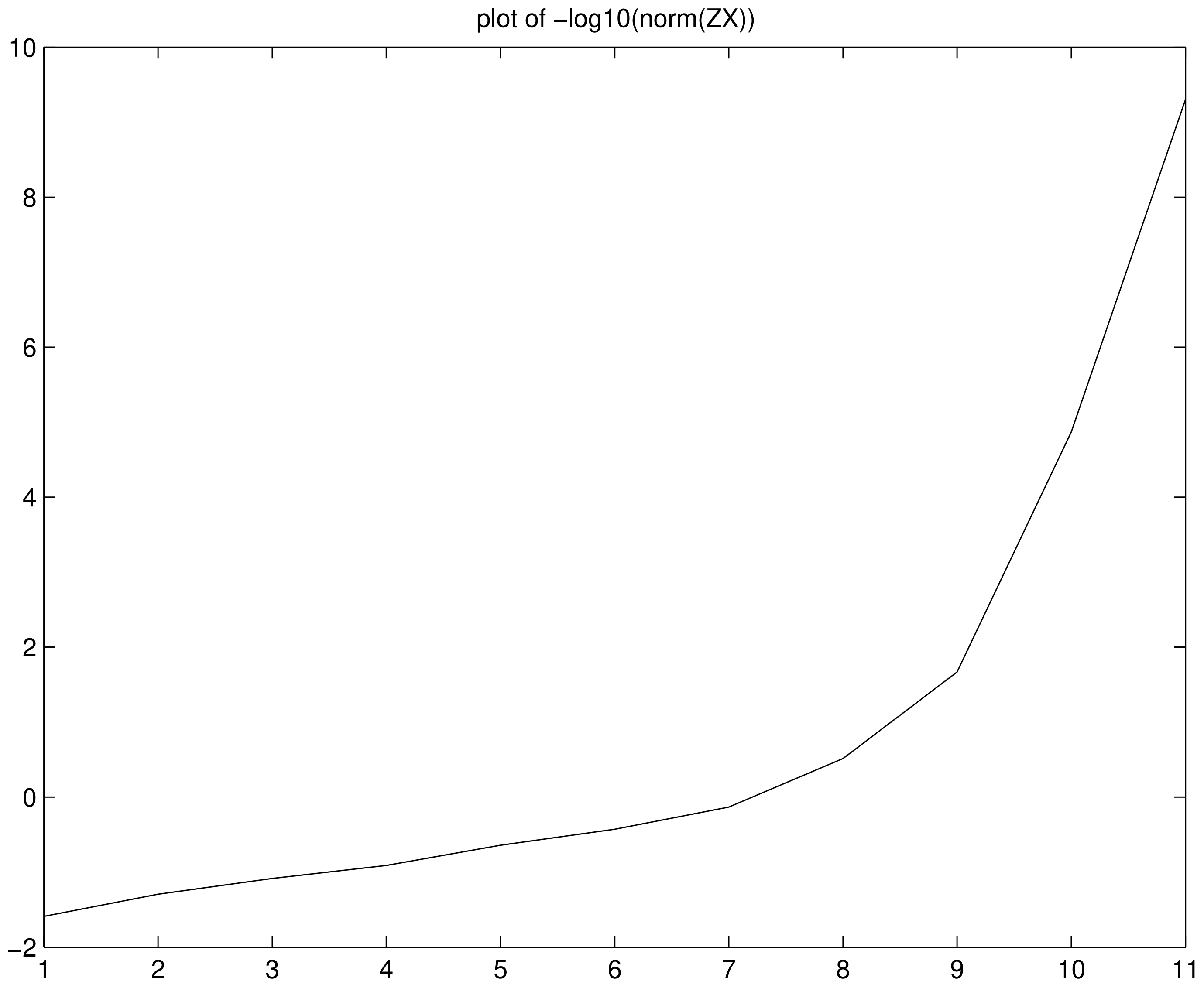

Figure 0.1: $n = 55$. Crossover at iteration 7.

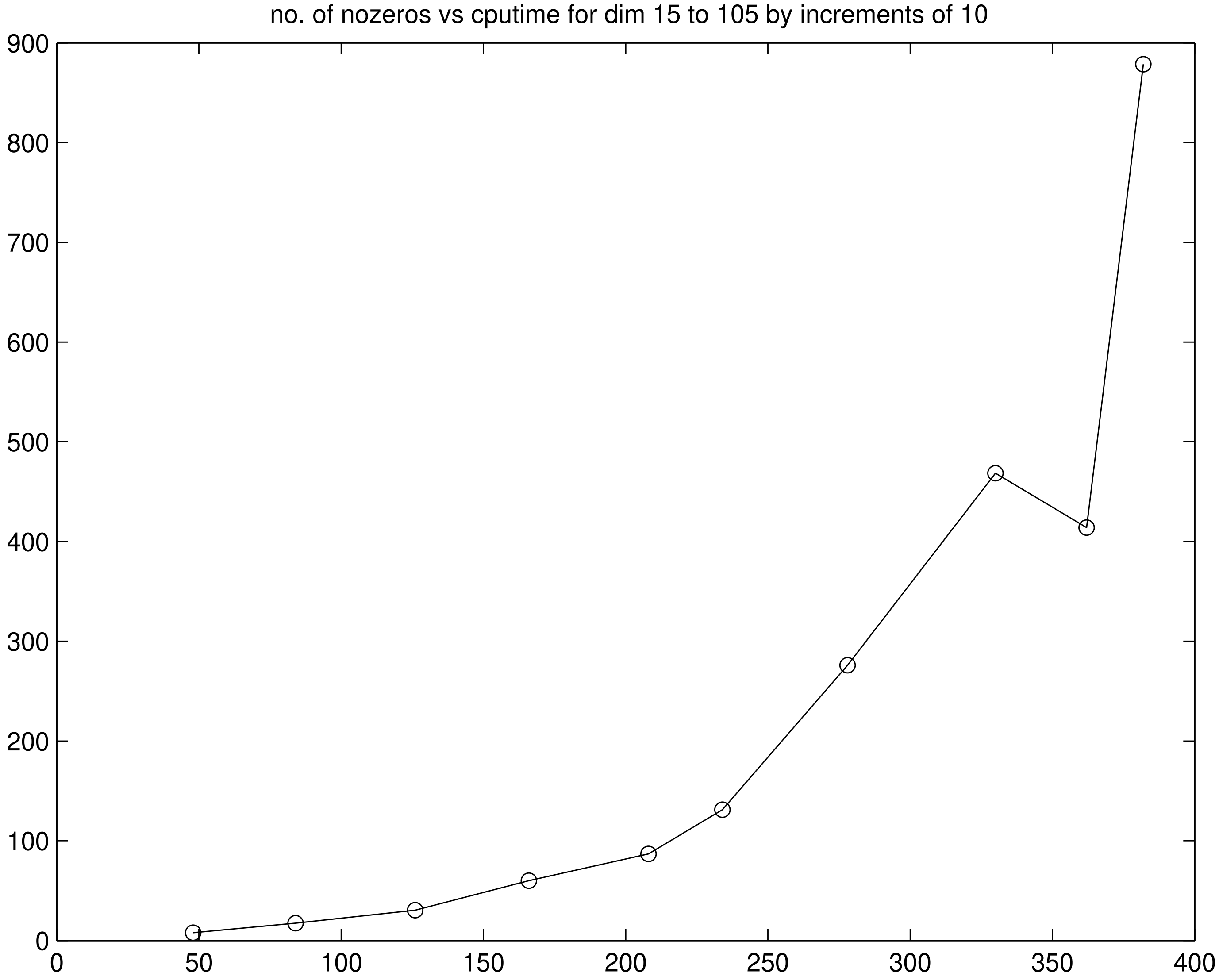

Figure 0.2: Nonzeros vs cputime. $n = 15 : 10 : 105$.

Dim n	cpu sec	iter.	opt. val. at end	rel. $\|ZX\|$ at end	rel. eig. viol. pos. Z, X	nnz in Q
165	1120	14	869.61	8.9414e-17	-3.3566e-17	476
175	1891	15	928.97	8.5746e-17	-3.2607e-17	504
185	3079.6	15	1041.8	8.9598e-17	-3.7512e-17	538
195	3251.9	14	1045.7	8.2733e-17	-4.2463e-17	572
205	5874.7	17	1207.7	1.0142e-16	-3.5923e-17	602

Table 0.7: Random problems; Requested Accuracy in relative duality gap: 1.0e-14; Density is $\frac{1}{n}$; crossover tolerance 1.0e-01; optimal values of order 1000; SUN computer.

Dim n	cpu sec 1 dec	cpu sec 7 dec	cpu sec 14 dec	iter.	opt. at end	rel. $\|ZX\|$ at end	releig. viol. pos. Z, X	nnz in Q
100	39.66	127.23	192.36	13	226.16	2.5926e-17	-5.0771e-17	638
124	70.93	351.73	502.57	14	141.99	3.3609e-17	-2.8961e-16	410
124	71.15	241.03	342.4	14	269.88	4.4706e-17	-1.0508e-16	760
124	56.32	392.78	547.51	14	467.75	2.3752e-17	-3.3513e-17	1364
124	82.79	370.25	577.79	14	864.41	1.6515e-16	-8.7386e-16	2666
250	648.9	2339.9	3087.2	15	317.26	2.1095e-16	-4.7591e-15	892
250	537.79	2384	2805.6	14	531.93	2.8552e-16	-5.5422e-16	1472
250	620.82	3844.2	5620	13	981.17	2.7276e-17	-2.6909e-16	2816
250	947.5	7834.3	10893	15	1682	6.5743e-16	-2.2708e-15	5092
500	6729.4	32472	42094	20	598.15	2.9931e-17	-1.5378e-16	1701
500	5422.5	38232	59391	18	1070.1	7.2114e-16	-9.7197e-15	2939
500	7282.3	42758	58360	16	1848	1.2577e-15	-2.0375e-14	5210
500	11548	80822	1.3137e+05	17	3566.7	6.1352e-15	-5.7 509e-14	10740

Table 0.8: Requested Accuracy in relative duality gap: 1.0e-14; SDPLIB test problems; crossover tolerance 1.0e-01; SUN computer.

Dim Q n	cpu sec	major iter.	rel. opt. at end	rel. norm(ZX) at end	rel. viol. eig. pos. Z, X	nnz in Q
100	96.34	12	226.16	1.2399e-13	-1.2062e-12	638
124	252.57	13	141.99	1.2238e-14	-2.3655e-13	410
124	171.99	13	269.88	2.0918e-11	-1.1324e-11	760
124	292.48	13	467.75	7.0013e-13	-5.9152e-13	1364
124	289.21	13	864.41	5.7229e-11	-1.035e-11	2666
250	1134.9	14	317.26	2.9025e-11	-5.5891e-11	892
250	1058.3	14	531.93	2.8552e-16	-5.5638e-16	1472
250	1633.6	12	981.17	8.9829e-12	-5.8956e-12	2816
250	3036	14	1682	3.481e-11	-2.1537e-11	5092
500	14669	19	598.15	4.12e-13	-2.9424e-12	1701
500	21489	17	1070.1	1.0229e-11	-2.7013e-11	2939
500	20691	15	1848	3.6924e-11	-1.364e-11	5210
500	46752	16	3566.7	3.3634e-11	-1.5e-11	10740

Table 0.9: Requested Accuracy in relative duality gap: 1.0e-10; SDPLIB test problems; crossover tolerance 1.0e-01; SGI (Irix6.5_64-Mips) computer.