Linear Matrix Inequality Formulation of Spectral Mask Constraints*

Timothy N. Davidson†‡, Zhi-Quan (Tom) Luo‡ and Jos F. Sturm§

19 September 2000

Abstract

The design of a finite impulse response filter often involves a spectral 'mask' which the magnitude spectrum must satisfy. This constraint can be awkward because it is semi-infinite, since it yields two inequality constraints for each frequency point. In current practice, spectral masks are often approximated by discretization, but in this paper we will show that piecewise constant masks can be precisely enforced in a finite and convex manner via linear matrix inequalities. This facilitates the formulation of a diverse class of filter and beamformer design problems as semidefinite programmes. These optimization problems can be efficiently solved using recently developed interior point methods. Our results can be considered as extensions to the well-known Positive-Real and Bounded-Real Lemmas from the systems and control literature.

Keywords: FIR digital filter design; beamforming; spectral masks; optimization
SP-EDICS: 2–FILT

*This research was supported in part by the Natural Sciences and Engineering Research Council of Canada, Grant No. OPG0090391. The third author was also supported by travel grant R46-415 of the Netherlands Organization for Scientific Research (NWO).

†Corresponding author. Department of Electrical and Computer Engineering, McMaster University, 1280 Main Street West, Hamilton, Ontario, L8S 4K1, Canada. Tel: +1-905-525-9140, Ext. 27352. Fax: +1-905-521-2922. Email: `davidson@ece.mcmaster.ca`

‡T. N. Davidson and Z.-Q. Luo are with the Department of Electrical and Computer Engineering, McMaster University, Hamilton, Ontario, L8S 4K1, Canada.

§J. F. Sturm is with the Department of Econometrics, University of Maastricht, 620 MD, Maastricht, The Netherlands.

1 Introduction

In the design of finite impulse response (FIR) filters, one often encounters a spectral mask constraint on the magnitude of the frequency response of the filter (e.g., [1–4]). That is, for given $L(e^{j\omega})$ and $U(e^{j\omega})$, constrain the (possibly complex) filter coefficients g_k so that

$$L(e^{j\omega}) \leq |G(e^{j\omega})| \leq U(e^{j\omega}) \qquad \text{for all } 0 \leq \omega < 2\pi, \tag{1.1}$$

or determine that the constraint cannot be satisfied. Here, $j = \sqrt{-1}$ and $G(e^{j\omega}) = \sum_k g_k e^{-j\omega k}$ is the frequency response of the filter. A spectral mask constraint can be rather awkward to accommodate into general optimization-based filter design techniques for two reasons. First, it is semi-infinite in the sense that there are two inequality constraints for every $\omega \in [0, 2\pi)$. Second, the set of feasible filter coefficients is in general non-convex due to the lower bound on $|G(e^{j\omega})|$. In order to efficiently solve filter design problems employing such constraints, we must find a way in which (1.1) can be represented in a finite and convex manner.

There are two established approaches [1] to deal with the problem of non-convexity of (1.1). The first is to enforce additional constraints on the parameters g_k so that $G(e^{j\omega})$ has 'linear phase'. In that case $|G(e^{j\omega})|$ becomes a linear function of approximately half the g_k's (the rest are determined via the linear phase constraint), and hence (1.1) reduces to two semi-infinite linear (and hence convex) constraints. However, phase linearity may be an excessively restrictive constraint in some applications [5]. The second approach to deal with non-convexity is to reformulate (1.1) in terms of the autocorrelation of the filter [5–10]. In particular, if $r_m = \sum_k g_k g_{k-m}$ represents the autocorrelation of the filter, then $R(e^{j\omega}) = |G(e^{j\omega})|^2$, and hence (1.1) is equivalent to

$$L(e^{j\omega})^2 \leq R(e^{j\omega}) \leq U(e^{j\omega})^2 \qquad \text{for all } 0 \leq \omega < 2\pi, \tag{1.2}$$

which amounts to two semi-infinite linear constraints on r_m. Hence, by reformulating the mask constraint in terms of r_m, $m \geq 0$, we obtain convex constraints. Note that the constraint that $R(e^{j\omega}) \geq L(e^{j\omega})^2 \geq 0$ is sufficient to ensure that a filter g_k can be extracted (though not uniquely) from a designed autocorrelation r_m via spectral factorization [9, 11]. A different convex model for nonlinear phase filter design was proposed by Coleman and Scholnik [12]. Their model does not use the autocorrelation coefficients r_k as variables, but involves a semi-infinite quadratic constraint in the filter coefficients g_k. However, the choice of the autocorrelation coefficients r_k as variables facilitates the formulation of not only spectral mask constraints, but also other properties of a filter, such as self-orthogonality; see Section 5.

The problem of representing (1.1) or (1.2) in a finite manner is more challenging. (For simplicity we will phrase our discussion in terms of (1.2).) One heuristic approach is to approximate the constraints by discretizing them uniformly in frequency and enforcing

$$L(e^{\mathrm{j}\omega_i})^2 + \epsilon \le R(e^{\mathrm{j}\omega_i}) \le U(e^{\mathrm{j}\omega_i})^2 - \epsilon \qquad \text{for } \omega_i = 2\pi i/N,\ i = 0, 1, \ldots, N-1, \tag{1.3}$$

where ϵ and N are chosen heuristically. Unfortunately, as N is increased so that ϵ can be reduced, the resulting formulation can become prone to numerical difficulties. (Other discretization techniques are also available [13, 14].) Algorithms of the exchange type [1, 3, 4] employ a non-uniform discretization of (1.2) at each stage of the algorithm, where the sample points are determined by the stationary points of the current estimate of the optimal $R(e^{\mathrm{j}\omega})$, and any points of discontinuity in the mask. (In practice, the stationary points are often approximated using fine uniform discretization [4].) At each stage of the algorithm an optimization problem is solved subject to appropriate equality constraints derived from (1.2) at those sample points. Although exchange methods often work well for the design of low-pass filters, substantial effort is required to guarantee the algorithm's convergence [4]. Furthermore, the algorithms may require substantial 're-tailoring' in order to incorporate additional constraints on the filter coefficients (e.g., [15]). Certain dual parameterization methods lead to precise finite representations of (1.2), but may result in non-convex design problems [16].

In this paper, we will show that piecewise constant masks can be precisely represented in a convex and finite manner via a set of linear matrix inequalities (LMIs). As a result, these masks can be efficiently incorporated, without discretization, into the diverse class of filter and beamformer design algorithms based on convex optimization [9], and in particular semidefinite programming [8, 10]. We will provide examples which show how this result leads to effective design algorithms for standard linear-phase and nonlinear-phase FIR filters, for robust 'chip' waveforms for digital wireless communication systems based on code division multiple access, and for narrow-band beamformers for linear antenna arrays.

When specialized to the case of a constant lower mask constraint, our results imply a new LMI formulation of the Positive Real Lemma [17, 18] (and the closely related Kalman-Yakubovich-Popov [KYP] Lemma) for FIR systems. This new formulation states that for r_m, $-M+1 \le m \le M-1$, with $r_{-m} = \bar{r}_m$, $R(e^{\mathrm{j}\omega}) \ge 0$ for all $\omega \in [0, 2\pi]$ if and only if there exists an $M \times M$ positive semidefinite Hermitian matrix $\boldsymbol{X}$ such that $\mathrm{tr}(\boldsymbol{X}) = r_0$ and $\sum_{\ell=1}^{M-\ell} [\boldsymbol{X}]_{\ell+m,\ell} = r_m$, for $1 \le m \le$

$M-1$. The more general results in this paper are LMI formulations of constraints of the form $R(e^{\mathrm{j}\omega}) \geq L^2$ for all $\omega \in [\alpha, \beta]$, or for all $\omega \in [0, 2\pi) \setminus (\alpha, \beta)$, and therefore they can be considered as generalizations of the Positive Real Lemma. Simple manipulation of those results generates LMI formulations of constraints of the form $R(e^{\mathrm{j}\omega}) \leq U^2$ for all $\omega \in [\alpha, \beta]$, or for all $\omega \in [0, 2\pi) \setminus (\alpha, \beta)$, which can be considered as generalizations of the Bounded Real Lemma [19].

Our notational conventions are as follows: Vectors and matrices will be represented by bold lowercase and uppercase letters, respectively. The elements of these structures will be denoted by medium weight lower case letters with appropriate subscripts; e.g., $g_k = [\boldsymbol{g}]_k$ and $x_{ij} = [\boldsymbol{X}]_{ij}$. Operators will be represented by bold uppercase letters in a sans-serif font. In order to illuminate the connections between the results for polynomials on the real line and trigonometric polynomials on the unit circle, we define

$$\boldsymbol{u}(t;n) := \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix}^{\mathrm{T}}, \quad \boldsymbol{v}(\theta;n) := \begin{bmatrix} 1 & e^{\mathrm{j}\theta} & e^{\mathrm{j}2\theta} & \cdots & e^{\mathrm{j}n\theta} \end{bmatrix}^{\mathrm{T}}, \tag{1.4}$$

where the superscript "T" denotes the transpose (without conjugation). Thus, the components of $\boldsymbol{u}(t;n)$ form a basis of the (real) function space of polynomials of degree n on the real line, whereas the components of $\boldsymbol{v}(\theta;n)$ form a basis of the (complex) function space of trigonometric polynomials of degree n on $[0, 2\pi)$. Consequently, an nth order polynomial $p(t) = \sum_{k=0}^{n} p_k t^k$ can be written as $p(t) = \boldsymbol{u}(t;n)^{\mathrm{T}}\boldsymbol{p}$, where $[\boldsymbol{p}]_k = p_k$. Similarly, if the sequence g_k denotes the impulse response of a causal FIR filter, and if $[\boldsymbol{g}]_k = g_k$, then the frequency response $G(e^{\mathrm{j}\theta}) = \sum_{k=0}^{n} g_k e^{-\mathrm{j}k\theta} = \boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{g}$, where the superscript "H" denotes the Hermitian transpose. The (complex valued) inner product between two complex matrices $\boldsymbol{X}$ and $\boldsymbol{Z}$ is defined as

$$\boldsymbol{X} \bullet \boldsymbol{Z} = \operatorname{tr} \boldsymbol{X}^{\mathrm{H}}\boldsymbol{Z}. \tag{1.5}$$

We shall denote by $\mathcal{H}_+^{n\times n}$ the set of $n \times n$ positive semidefinite (complex) Hermitian matrices, and by $\mathcal{S}_+^{n\times n}$ the subset of $\mathcal{H}_+^{n\times n}$ consisting of the real symmetric positive semidefinite matrices. For a complex number $x \in \mathbb{C}$, we denote the polar coordinates as $(|x|, \arg x) \in \mathbb{R}_+ \times [0, 2\pi)$, i.e. $x = |x|e^{\mathrm{j}\arg x}$, with $\arg x \in [0, 2\pi)$.

2 Transformation of Polynomial Basis

In this section, we establish a one-to-one correspondence between polynomials of degree $2n$ on the real line, and trigonometric polynomials of degree n on $(0, 2\pi)$, where the coefficients of the trigonometric polynomials may be complex numbers. The main result of this section is stated in Theorem 2.1, but we first state some intermediate results.

By a result from classical complex analysis [20], the complex exponential function $e^{\mathrm{j}\theta}$ can be represented as a (complex) rational function of t over the real line:

$$e^{\mathrm{j}\theta} = \frac{t + \mathrm{j}}{t - \mathrm{j}} = \frac{(t + \mathrm{j})^2}{1 + t^2}.$$

This mapping from $t \in (-\infty, \infty)$ to $\theta \in (0, 2\pi)$ is one-to-one. In fact, it is a conformal mapping and is closely related to the 'bilinear transform' which is used to map the left half plane to the unit disc in the transformation of analog filter designs into the discrete-time domain [21]. This mapping provides the basis for relating the polynomials over the real line with trigonometric polynomials over the unit circle. The following lemma further relates an arbitrary power of $e^{\mathrm{j}\theta}$ to a rational function of t.

Lemma 2.1 *Let $\theta \in (0, 2\pi)$ and $t \in \mathbb{R}$ be related by*

$$e^{\mathrm{j}\theta} = \frac{(t + \mathrm{j})^2}{1 + t^2}.$$

Then for any positive integer $j \geq 1$, we have

$$e^{\mathrm{j}j\theta} = \frac{1}{(1 + t^2)^j} \left(\sum_{k=0}^{j} (-1)^k \binom{2j}{2k} t^{2(j-k)} + \mathrm{j} \sum_{k=0}^{j-1} (-1)^k \binom{2j}{2k+1} t^{2(j-k)-1} \right).$$

Proof. This is a simple application of Newton's binomial formula. **Q.E.D.**

Let us define a lower triangular matrix $\boldsymbol{G}(n)$ of size $(n + 1) \times (n + 1)$ whose (i, j)-th entry is given by

$$g_{ij}(n) := \begin{cases} 0, & \text{for } 0 \leq i \leq j - 1, \\ \binom{n-j}{n-i}, & \text{for } j \leq i \leq n. \end{cases} \tag{2.1}$$

Notice that the diagonal entries of $\boldsymbol{G}(n)$ are equal to 1, hence, it is invertible. We shall denote the columns of $\boldsymbol{G}(n)$ by $\boldsymbol{g}_j(n)$, $j = 0, 1, ..., n$. In addition, for each $0 \le k \le n$, we define a $(n+1)\times(n+1)$ matrix $\boldsymbol{H}(k;n)$ whose (i,j)-th entry is given by

$$h_{ij}(k;n) := \begin{cases} \binom{n-j}{n-i-k}, & \text{for } k \le j \le n,\ j-k \le i \le n-k, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

We shall denote the j-th column of this matrix by $\boldsymbol{h}_j(k;n)$. Obviously, we have $\boldsymbol{H}(0;n) = \boldsymbol{G}(n)$. We remark that

$$\boldsymbol{u}(x;n)^{\mathrm{T}}\boldsymbol{h}_j(k;n) = \sum_{i=j-k}^{n-k}\binom{n-j}{n-i-k}x^i = x^{j-k}\sum_{\ell=0}^{n-j}\binom{n-j}{n-j-\ell}x^\ell = x^{j-k}(1+x)^{n-j}. \tag{2.3}$$

Based on (2.3), we can now restate Lemma 2.1 as follows.

Lemma 2.2 *Let $\theta \in (0, 2\pi)$ and $t \in \mathbb{R}$ be related by*

$$e^{\mathrm{j}\theta} = \frac{(t+\mathrm{j})^2}{1+t^2}.$$

Then

$$1 = e^{\mathrm{j}0\theta} = \frac{\boldsymbol{u}(t^2;n)^{\mathrm{T}}\boldsymbol{f}_0}{(1+t^2)^n},$$

and

$$e^{\mathrm{j}j\theta} = \frac{\boldsymbol{u}(t^2;n)^{\mathrm{T}}\boldsymbol{f}_j + \mathrm{j}t(\boldsymbol{u}(t^2;n-1)^{\mathrm{T}}\tilde{\boldsymbol{f}}_{j-1})}{(1+t^2)^n}, \qquad j = 1, 2, \ldots, n,$$

where

$$\boldsymbol{f}_j := \sum_{k=0}^{j}(-1)^k\binom{2j}{2k}\boldsymbol{h}_j(k;n), \qquad j = 0, 1, \ldots, n, \tag{2.4}$$

and

$$\tilde{\boldsymbol{f}}_j := \sum_{k=0}^{j}(-1)^k\binom{2(j+1)}{2k+1}\boldsymbol{h}_j(k;n-1), \qquad j = 0, 1, \ldots, n-1. \tag{2.5}$$

Proof. The proof follows directly from Lemma 2.1 and the definitions of $\boldsymbol{f}_j$ and $\tilde{\boldsymbol{f}}_j$. **Q.E.D.**

Now, consider the elementary identities

$$\binom{n-1}{m} = \frac{n-m}{n}\binom{n}{m}, \quad \binom{n-1}{m-1} = \frac{m}{n}\binom{n}{m}, \quad \binom{n-1}{n-1} = \binom{n}{n} = 1,$$

which are valid for any $m = 0, 1, \ldots, n-1$. An immediate consequence of the above identities and (2.2) is that

$$\boldsymbol{h}_j(k;n) + \boldsymbol{h}_j(k-1;n) = \boldsymbol{h}_{j-1}(k-1;n). \tag{2.6}$$

Alternatively, the above identity can be established from (2.3). Since $\boldsymbol{h}_j(0;n) = \boldsymbol{g}_j(n)$, a simple induction argument shows that

$$\boldsymbol{h}_j(k;n) \in \text{ span } \{\boldsymbol{g}_0(n), \boldsymbol{g}_1(n), \ldots, \boldsymbol{g}_j(n)\}. \tag{2.7}$$

The following lemma further strengthens the above relation.

Lemma 2.3 *There holds that*

$$(-1)^k \boldsymbol{h}_j(k;n) - \boldsymbol{g}_j(n) \in \text{ span } \{\boldsymbol{g}_0(n), \boldsymbol{g}_1(n), \ldots, \boldsymbol{g}_{j-1}(n)\}. \tag{2.8}$$

Proof. For $k = 0$, we have $(-1)^k \boldsymbol{h}_j(k;n) - \boldsymbol{g}_j(n) = \boldsymbol{g}_j(n) - \boldsymbol{g}_j(n) = 0$, so (2.8) holds trivially. Consider now $k \in \{1, 2, \ldots, j\}$. We have

$$\begin{aligned} (-1)^k \boldsymbol{h}_j(k;n) - \boldsymbol{g}_j(n) &= (-1)^k \big(\boldsymbol{h}_j(k;n) + \boldsymbol{h}_j(k-1;n)\big) + (-1)^{k-1} \boldsymbol{h}_j(k-1;n) - \boldsymbol{g}_j(n) \\ &= (-1)^k \boldsymbol{h}_{j-1}(k-1;n) + (-1)^{k-1} \boldsymbol{h}_j(k-1;n) - \boldsymbol{g}_j(n). \end{aligned}$$

By (2.7), we know that

$$\boldsymbol{h}_{j-1}(k-1;n) \in \text{ span } \{\boldsymbol{g}_0(n), \boldsymbol{g}_1(n), \ldots, \boldsymbol{g}_{j-1}(n)\},$$

and by induction we have

$$(-1)^{k-1} \boldsymbol{h}_j(k-1;n) - \boldsymbol{g}_j(n) \in \text{ span } \{\boldsymbol{g}_0(n), \boldsymbol{g}_1(n), \ldots, \boldsymbol{g}_{j-1}(n)\}.$$

Combining the above three relations proves (2.8). **Q.E.D.**

Now we can substitute (2.8) into the expressions (2.4)–(2.5) to obtain the following relations:

$$\boldsymbol{f}_j - \left(\sum_{k=0}^{j} \binom{2j}{2k}\right) \boldsymbol{g}_j(n) \in \text{ span } \{\boldsymbol{g}_0(n), \boldsymbol{g}_1(n), \ldots, \boldsymbol{g}_{j-1}(n)\},$$

and

$$\tilde{\boldsymbol{f}}_j - \left(\sum_{k=0}^{j} \binom{2(j+1)}{2k+1}\right) \boldsymbol{g}_j(n-1) \in \text{ span }\{\boldsymbol{g}_0(n-1), \boldsymbol{g}_1(n-1), \ldots, \boldsymbol{g}_{j-1}(n-1)\}.$$

As a result, we have that

$$\text{span }\{\boldsymbol{f}_0, \boldsymbol{f}_1, \ldots, \boldsymbol{f}_j\} = \text{ span }\{\boldsymbol{g}_0(n), \boldsymbol{g}_1(n), \ldots, \boldsymbol{g}_j(n)\}, \quad \text{for } j = 0, 1, \ldots, n,$$

and

$$\text{span }\{\tilde{\boldsymbol{f}}_0, \tilde{\boldsymbol{f}}_1, \ldots, \tilde{\boldsymbol{f}}_j\} = \text{ span }\{\boldsymbol{g}_0(n-1), \boldsymbol{g}_1(n-1), \ldots, \boldsymbol{g}_j(n-1)\}, \quad \text{for } j = 0, 1, \ldots, n-1.$$

This implies that the matrices

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{f}_0, & \boldsymbol{f}_1, & \ldots, & \boldsymbol{f}_n \end{bmatrix} \quad \text{and} \quad \tilde{\boldsymbol{F}} = \begin{bmatrix} \tilde{\boldsymbol{f}}_0, & \tilde{\boldsymbol{f}}_1, & \ldots, & \tilde{\boldsymbol{f}}_{n-1} \end{bmatrix}$$

must be invertible. In light of Lemma 2.2, this establishes an one-to-one correspondence between polynomials of degree $2n$ on the real line, and trigonometric polynomials of degree n on $(0, 2\pi)$. We summarize the result in the following theorem.

Theorem 2.1 *The mapping $\eta(t) := \arg((t + \mathrm{j})^2/(1 + t^2))$ is a bijection between $\mathbb{R}$ and $(0, 2\pi)$. In particular, the inverse function $\eta^{-1}(\theta)$ for $\theta \in (0, 2\pi)$ is given by*

$$\eta^{-1}(\theta) = \frac{\sin\theta}{1 - \cos\theta}.$$

Furthermore, for any vector $\boldsymbol{p} \in \mathbb{R}^{2n+1}$ there exists a vector $\boldsymbol{c} \in \mathbb{R}^{2n+1}$ such that

$$\frac{\sum_{i=0}^{2n} p_i t^i}{(1+t^2)^n} = \mathrm{Re}\left(c_0 + \sum_{k=1}^{n} (c_{2k} - \mathrm{j}c_{2k-1}) e^{\mathrm{j}k\eta(t)}\right), \quad \textit{for all } t \in \mathbb{R}.$$

Conversely, for any vector $\boldsymbol{c} \in \mathbb{R}^{2n+1}$ there exists a vector $\boldsymbol{p} \in \mathbb{R}^{2n+1}$ such that

$$\frac{\sum_{i=0}^{2n} p_i (\eta^{-1}(\theta))^i}{(1 + (\eta^{-1}(\theta))^2)^n} = \mathrm{Re}\left(c_0 + \sum_{k=1}^{n} (c_{2k} - \mathrm{j}c_{2k-1}) e^{\mathrm{j}k\theta}\right), \quad \textit{for all } \theta \in (0, 2\pi).$$

Proof. The bijectivity of $\eta(t)$ follows from simple calculus. The remaining part of the theorem is due to the invertibility of $\boldsymbol{F}$ and $\tilde{\boldsymbol{F}}$. **Q.E.D.**

3 Characterization of Nonnegative Polynomials on a Segment

In this section we characterize the set of trigonometric polynomials which are non-negative over a segment of the unit circle. The main result will be stated in Theorem 3.1, but first we state some preliminary results. We begin with a review of a well known characterization [22,23] of non-negative polynomials over a line segment in $\mathbb{R}$. We refer to Powers and Reznick [24] for a recent survey on characterizations for polynomials that are non-negative on an interval.

Proposition 3.1 (Markov-Lukacs) *Let $\boldsymbol{p} \in \mathbb{R}^{2n+1}$ and $a, b \in \mathbb{R}$, $a < b$. Then*

$$\boldsymbol{u}(t; 2n)^{\mathrm{T}} \boldsymbol{p} \geq 0 \textit{ for all } t \in [a, b]$$

if and only if there exist $\boldsymbol{q} \in \mathbb{R}^{n+1}$ and $\boldsymbol{r} \in \mathbb{R}^{n}$ such that

$$\boldsymbol{u}(t; 2n)^{\mathrm{T}} \boldsymbol{p} = \big(\boldsymbol{u}(t; n)^{\mathrm{T}} \boldsymbol{q}\big)^2 + (t-a)(b-t)\big(\boldsymbol{u}(t; n-1)^{\mathrm{T}} \boldsymbol{r}\big)^2.$$

Moreover, it holds that

$$\boldsymbol{u}(t; 2n)^{\mathrm{T}} \boldsymbol{p} \geq 0 \textit{ for all } t \in [a, \infty)$$

if and only if there exist $\boldsymbol{q} \in \mathbb{R}^{n+1}$ and $\boldsymbol{r} \in \mathbb{R}^{n}$ such that

$$\boldsymbol{u}(t; 2n)^{\mathrm{T}} \boldsymbol{p} = \big(\boldsymbol{u}(t; n)^{\mathrm{T}} \boldsymbol{q}\big)^2 + (t-a)\big(\boldsymbol{u}(t; n-1)^{\mathrm{T}} \boldsymbol{r}\big)^2.$$

The following corollary of Proposition 3.1 provides a characterization for polynomials which are non-negative over the complement of a finite symmetric interval in $\mathbb{R}$.

Corollary 3.1 *Let $\boldsymbol{p} \in \mathbb{R}^{2n+1}$ and let $a > 0$ be a given positive number. Then*

$$\boldsymbol{u}(t; 2n)^{\mathrm{T}} \boldsymbol{p} \geq 0 \textit{ for all } t \notin (-a, a)$$

if and only if there exist $\boldsymbol{q} \in \mathbb{R}^{n+1}$ and $\boldsymbol{r} \in \mathbb{R}^{n}$ such that

$$\boldsymbol{u}(t; 2n)^{\mathrm{T}} \boldsymbol{p} = \big(\boldsymbol{u}(t; n)^{\mathrm{T}} \boldsymbol{q}\big)^2 + (t^2 - a^2)\big(\boldsymbol{u}(t; n-1)^{\mathrm{T}} \boldsymbol{r}\big)^2.$$

Proof. Let $\tilde{p}_i := p_{2n-i}$ for $i = 0, 1, \ldots, 2n$. Then for $t \neq 0$ we have

$$\boldsymbol{u}(t;2n)^{\mathrm{T}}\boldsymbol{p} = t^{2n}\big(\boldsymbol{u}(1/t;2n)^{\mathrm{T}}\tilde{\boldsymbol{p}}\big).$$

Since $t^2 \geq a^2 > 0$ for all $t \notin (-a,a)$, it follows that

$$\boldsymbol{u}(t;2n)^{\mathrm{T}}\boldsymbol{p} \geq 0 \quad \text{for all } t \notin (-a,a)$$

if and only if

$$\boldsymbol{u}(s;2n)^{\mathrm{T}}\tilde{\boldsymbol{p}} \geq 0 \quad \text{for all } s \in [-1/a, 1/a].$$

By Proposition 3.1, the above relation holds if and only if there exist $\tilde{\boldsymbol{q}} \in \mathbb{R}^{n+1}$ and $\tilde{\boldsymbol{r}} \in \mathbb{R}^n$ such that

$$\boldsymbol{u}(s;2n)^{\mathrm{T}}\tilde{\boldsymbol{p}} = \big(\boldsymbol{u}(s;n)^{\mathrm{T}}\tilde{\boldsymbol{q}}\big)^2 + \frac{1-a^2s^2}{a^2}\big(\boldsymbol{u}(s;n-1)^{\mathrm{T}}\tilde{\boldsymbol{r}}\big)^2.$$

Letting

$$q_i := \tilde{q}_{n-i} \;\text{ and }\; r_i := \tilde{r}_{n-1-i}/a, \qquad \text{for } i = 0,1,\ldots,n,$$

proves the result. **Q.E.D.**

The next corollary further extends Corollary 3.1 to the case where the interval is non-symmetric.

Corollary 3.2 *Let* $\boldsymbol{p} \in \mathbb{R}^{2n+1}$ *and* $a, b \in \mathbb{R}$, $a < b$. *Then*

$$\boldsymbol{u}(t;2n)^{\mathrm{T}}\boldsymbol{p} \geq 0 \textit{ for all } t \notin (a,b) \tag{3.1}$$

if and only if there exist $\boldsymbol{q} \in \mathbb{R}^{n+1}$ *and* $\boldsymbol{r} \in \mathbb{R}^n$ *such that*

$$\boldsymbol{u}(t;2n)^{\mathrm{T}}\boldsymbol{p} = \big(\boldsymbol{u}(t;n)^{\mathrm{T}}\boldsymbol{q}\big)^2 + (t-a)(t-b)\big(\boldsymbol{u}(t;n-1)^{\mathrm{T}}\boldsymbol{r}\big)^2. \tag{3.2}$$

Proof. It is obvious that (3.2) implies (3.1). Suppose now that (3.1) holds. Let us define

$$s := 2t - (a+b), \qquad \tilde{a} := b - a.$$

It is clear (e.g. from Newton's binomial formula) that there exists $\tilde{\boldsymbol{p}} \in \mathbb{R}^{n+1}$ such that

$$\boldsymbol{u}(t;2n)^{\mathrm{T}}\boldsymbol{p} = \boldsymbol{u}(2t-(a+b);2n)^{\mathrm{T}}\tilde{\boldsymbol{p}} = \boldsymbol{u}(s;2n)^{\mathrm{T}}\tilde{\boldsymbol{p}}.$$

Since $(s^2 - \tilde{a}^2) = 4(t-a)(t-b)$, Corollary 3.1 implies that (3.2) holds for some $\boldsymbol{q} \in \mathbb{R}^{n+1}$ and $\boldsymbol{r} \in \mathbb{R}^n$. **Q.E.D.**

The next lemma determines a simple trigonometric polynomial which is non-negative over a given segment of the unit circle but is non-positive over its complement. In particular, given $\alpha,\ \beta \in [0, 2\pi)$, we define a vector $\boldsymbol{d}(\alpha, \beta) \in \mathbb{R} \times \mathbb{C}$ as follows:

$$\boldsymbol{d}(\alpha,\beta) := \begin{cases} \begin{bmatrix} \cos\alpha + \cos\beta - \cos(\beta-\alpha) - 1 \\ (1-e^{\mathrm{j}\alpha})(e^{\mathrm{j}\beta}-1) \end{bmatrix} \text{ if } \alpha > 0 \\ \begin{bmatrix} -\sin\beta \\ \mathrm{j}(1-e^{\mathrm{j}\beta}) \end{bmatrix} \text{ if } \alpha = 0. \end{cases} \tag{3.3}$$

We remark that

$$\boldsymbol{d}(0,\beta) = \lim_{\alpha \downarrow 0} \frac{1}{\sin\alpha} \boldsymbol{d}(\alpha,\beta), \tag{3.4}$$

and

$$\boldsymbol{d}(\alpha, 2\pi-\alpha) = 2(1-\cos\alpha) \begin{bmatrix} \cos\alpha \\ -1 \end{bmatrix}. \tag{3.5}$$

Lemma 3.1 *Let* $0 \le \alpha < \beta < 2\pi$ *and let* $\boldsymbol{d}(\alpha,\beta)$ *be defined as in* (3.3)*. Then the trigonometric polynomial* $\mathrm{Re}\, \boldsymbol{v}(\theta;1)^{\mathrm{H}} \boldsymbol{d}(\alpha,\beta)$ *satisfies the following properties:*

$$\begin{cases} \mathrm{Re}\, \boldsymbol{v}(\theta;1)^{\mathrm{H}} \boldsymbol{d}(\alpha,\beta) > 0 \ \ \textit{for all } \theta \in (\alpha,\beta), \\ \mathrm{Re}\, \boldsymbol{v}(\theta;1)^{\mathrm{H}} \boldsymbol{d}(\alpha,\beta) < 0 \ \ \textit{for all } \theta \in [0,2\pi) \setminus [\alpha,\beta]. \end{cases}$$

Finally, we also need the following well known representation result for trigonometric polynomials which are non-negative over the entire unit circle.

Proposition 3.2 (Riesz-Féjer) *Let* $\boldsymbol{p} \in \mathbb{R} \times \mathbb{C}^n$*. Then*

$$\mathrm{Re}\, \boldsymbol{v}(\theta;n)^{\mathrm{H}} \boldsymbol{p} \ge 0 \textit{ for all } \theta \in [0,2\pi)$$

if and only if there exists $\boldsymbol{q} \in \mathbb{R} \times \mathbb{C}^n$ *such that*

$$\mathrm{Re}\, \boldsymbol{v}(\theta;n)^{\mathrm{H}} \boldsymbol{p} = \left| \boldsymbol{v}(\theta;n)^{\mathrm{H}} \boldsymbol{q} \right|^2.$$

We are now ready to establish the main result of this section.

Theorem 3.1 *Let $\boldsymbol{p} \in \mathbb{R} \times \mathbb{C}^n$, $0 \le \alpha < \beta < 2\pi$, and let $\boldsymbol{d}(\alpha,\beta)$ be given by (3.3). Then*

$$\operatorname{Re} \boldsymbol{v}(\theta; n)^{\mathrm{H}} \boldsymbol{p} \ge 0 \quad \text{for all} \quad \theta \in [\alpha, \beta] \tag{3.6}$$

if and only if there exist $\boldsymbol{q} \in \mathbb{R} \times \mathbb{C}^n$ and $\boldsymbol{r} \in \mathbb{R} \times \mathbb{C}^{n-1}$ such that

$$\operatorname{Re} \boldsymbol{v}(\theta; n)^{\mathrm{H}} \boldsymbol{p} = \left|\boldsymbol{v}(\theta; n)^{\mathrm{H}} \boldsymbol{q}\right|^2 + \left(\operatorname{Re} \boldsymbol{v}(\theta; 1)^{\mathrm{H}} \boldsymbol{d}(\alpha,\beta)\right) \left|\boldsymbol{v}(\theta; n-1)^{\mathrm{H}} \boldsymbol{r}\right|^2. \tag{3.7}$$

Moreover,

$$\operatorname{Re} \boldsymbol{v}(\theta; n)^{\mathrm{H}} \boldsymbol{p} \ge 0 \quad \text{for all} \quad \theta \in [0, 2\pi) \setminus (\alpha, \beta) \tag{3.8}$$

if and only if there exist $\boldsymbol{q} \in \mathbb{R} \times \mathbb{C}^n$ and $\boldsymbol{r} \in \mathbb{R} \times \mathbb{C}^{n-1}$ such that

$$\operatorname{Re} \boldsymbol{v}(\theta; n)^{\mathrm{H}} \boldsymbol{p} = \left|\boldsymbol{v}(\theta; n)^{\mathrm{H}} \boldsymbol{q}\right|^2 - \left(\operatorname{Re} \boldsymbol{v}(\theta; 1)^{\mathrm{H}} \boldsymbol{d}(\alpha,\beta)\right) \left|\boldsymbol{v}(\theta; n-1)^{\mathrm{H}} \boldsymbol{r}\right|^2. \tag{3.9}$$

Proof. In light of Lemma 3.1, it is obvious that (3.7) implies (3.6), and similarly (3.9) implies (3.8). We now establish the converse relations. Fix some $\boldsymbol{p} \in \mathbb{R} \times \mathbb{C}^n$ such that (3.6) holds. We introduce a vector $\tilde{\boldsymbol{p}} \in \mathbb{R}^{2n+1}$ as follows:

$$\begin{cases} \tilde{p}_{2i} = \operatorname{Re} p_i, & \text{for } i = 0, 1, \ldots, n, \\ \tilde{p}_{2i-1} = \operatorname{Im} p_i, & \text{for } i = 1, 2, \ldots, n. \end{cases}$$

Using Theorem 2.1 and its notations, we have

$$\operatorname{Re} \boldsymbol{v}(\theta; n)^{\mathrm{H}} \boldsymbol{p} = \frac{\boldsymbol{u}(\eta^{-1}(\theta); 2n)^{\mathrm{T}} \tilde{\boldsymbol{p}}}{(1 + (\eta^{-1}(\theta))^2)^2} \quad \text{for all} \quad \theta \in (0, 2\pi). \tag{3.10}$$

Recall from Theorem 2.1 that $\eta(t) := \arg((t+\mathrm{j})^2/(1+t^2))$ is a bijection between $\mathbb{R}$ and $(0, 2\pi)$. For convenience, we let $\eta(\infty) := 0$ to obtain a bijection between $\mathbb{R} \cup \{\infty\}$ and $[0, 2\pi)$. Let $a := \eta^{-1}(\alpha)$ and $b := \eta^{-1}(\beta)$. Since η is a bijection between $\mathbb{R} \cup \{\infty\}$ and $[0, 2\pi)$ and η is a decreasing function, it follows that

$$\eta([b, \infty)) = [0, \beta], \quad [b, \infty) = \eta^{-1}([0, \beta]).$$

and for $\alpha > 0$ that

$$\eta([b,a]) = [\alpha,\beta], \quad [b,a] = \eta^{-1}([\alpha,\beta]).$$

Since (3.6) holds, it follows from (3.10) that $\boldsymbol{u}(t;2n)^{\mathrm{T}}\tilde{\boldsymbol{p}} \geq 0$ for all $t \in [a,b]$. As a result, we may apply Proposition 3.1 to conclude that for given α and β there exist $\tilde{\boldsymbol{q}} \in \mathbb{R}^{n+1}$ and $\tilde{\boldsymbol{r}} \in \mathbb{R}^n$ such that

$$\boldsymbol{u}(t;2n)^{\mathrm{T}}\tilde{\boldsymbol{p}} = \big(\boldsymbol{u}(t;n)^{\mathrm{T}}\tilde{\boldsymbol{q}}\big)^2 + (t-b)(a-t)\big(\boldsymbol{u}(t;n-1)^{\mathrm{T}}\tilde{\boldsymbol{r}}\big)^2,$$

if $\alpha > 0$, or

$$\boldsymbol{u}(t;2n)^{\mathrm{T}}\tilde{\boldsymbol{p}} = \big(\boldsymbol{u}(t;n)^{\mathrm{T}}\tilde{\boldsymbol{q}}\big)^2 + (t-b)\big(\boldsymbol{u}(t;n-1)^{\mathrm{T}}\tilde{\boldsymbol{r}}\big)^2,$$

if $\alpha = 0$. Due to Theorem 2.1 and Proposition 3.2, there must exist a $\boldsymbol{q} \in \mathbb{R}\times\mathbb{C}^n$ and a $\hat{\boldsymbol{r}} \in \mathbb{R}\times\mathbb{C}^{n-1}$ such that

$$\frac{\big(\boldsymbol{u}(t;n)^{\mathrm{T}}\tilde{\boldsymbol{q}}\big)^2}{(1+t^2)^n} = \big|\boldsymbol{v}(\eta(t);n)^{\mathrm{H}}\boldsymbol{q}\big|^2,$$

and

$$\frac{\big(\boldsymbol{u}(t;n-1)^{\mathrm{T}}\tilde{\boldsymbol{r}}\big)^2}{(1+t^2)^{n-1}} = \big|\boldsymbol{v}(\eta(t);n-1)^{\mathrm{H}}\hat{\boldsymbol{r}}\big|^2.$$

Furthermore, we know from Theorem 2.1 that for given α and β there must exist some $\boldsymbol{c} \in \mathbb{R}\times\mathbb{C}$ such that

$$\frac{(t-b)(a-t)}{1+t^2} = \operatorname{Re}\,\boldsymbol{v}(\eta(t);1)^{\mathrm{H}}\boldsymbol{c}, \tag{3.11}$$

if $\alpha > 0$, or

$$\frac{t-b}{1+t^2} = \operatorname{Re}\,\boldsymbol{v}(\eta(t);1)^{\mathrm{H}}\boldsymbol{c}, \tag{3.12}$$

if $\alpha = 0$. We have now shown that (3.6) implies the existence of vectors $\boldsymbol{q}$, $\hat{\boldsymbol{r}}$ and $\boldsymbol{c}$ such that

$$\operatorname{Re}\,\boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{p} = \big|\boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{q}\big|^2 + \big(\operatorname{Re}\,\boldsymbol{v}(\theta;1)^{\mathrm{H}}\boldsymbol{c}\big)\,\big|\boldsymbol{v}(\theta;n-1)^{\mathrm{H}}\hat{\boldsymbol{r}}\big|^2.$$

With an analogous argument, using Corollary 3.2, Theorem 2.1 and Proposition 3.2, we can show that if (3.8) holds, then

$$\operatorname{Re}\,\boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{p} = \big|\boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{q}\big|^2 - \big(\operatorname{Re}\,\boldsymbol{v}(\theta;1)^{\mathrm{H}}\boldsymbol{c}\big)\,\big|\boldsymbol{v}(\theta;n-1)^{\mathrm{H}}\hat{\boldsymbol{r}}\big|^2.$$

It remains to show that $\boldsymbol{c}$ is a positive multiple of $\boldsymbol{d}(\alpha,\beta)$.

Notice that

$$\begin{aligned}\mathrm{Re}\ \boldsymbol{v}(\eta(t);1)^{\mathrm{H}}\boldsymbol{c} &= \mathrm{Re}\ \boldsymbol{c}^{\mathrm{H}}\boldsymbol{v}(\eta(t);1) = \mathrm{Re}\ (c_0 + \overline{c}_1 e^{\mathrm{j}\eta(t)}) \\ &= \mathrm{Re}\ (c_0 + \overline{c}_1(\mathrm{j}+t)^2/(1+t^2)) \\ &= \frac{1}{(1+t^2)}\left(c_0 - \mathrm{Re}\ c_1 + 2t\ \mathrm{Im}\ c_1 + (c_0 + \mathrm{Re}\ c_1)t^2\right).\end{aligned}$$

Comparing this with (3.11), we obtain for $\alpha > 0$ that

$$\frac{(t-b)(a-t)}{1+t^2} = \frac{1}{(1+t^2)}\left(c_0 - \mathrm{Re}\ c_1 + 2t\ \mathrm{Im}\ c_1 + (c_0 + \mathrm{Re}\ c_1)t^2\right).$$

This implies that

$$c_0 = -\frac{1+ab}{2}, \quad c_1 = \frac{ab-1+(a+b)\mathrm{j}}{2}.$$

For $\alpha = 0$, we have to obtain $\boldsymbol{c}$ from the equation

$$\frac{t-b}{1+t^2} = \frac{1}{(1+t^2)}\left(c_0 - \mathrm{Re}\ c_1 + 2t\ \mathrm{Im}\ c_1 + (c_0 + \mathrm{Re}\ c_1)t^2\right),$$

yielding $c_0 = -b/2$ and $c_1 = (b+\mathrm{j})/2$. It is easily verified that

$$\boldsymbol{d}(\alpha,\beta) = \begin{cases} 2(1-\cos\alpha)(1-\cos\beta) & \text{for } \alpha > 0 \\ 2(1-\cos\beta) & \text{for } \alpha = 0. \end{cases}$$

We complete the proof by setting $\boldsymbol{r} = 2(1-\cos\alpha)(1-\cos\beta)\hat{\boldsymbol{r}}$ for $\alpha > 0$, and $\boldsymbol{r} = 2(1-\cos\beta)\hat{\boldsymbol{r}}$ for $\alpha = 0$. **Q.E.D.**

Theorem 3.1 provides a complete analytical characterization of the set, say Ω, of trigonometric polynomials which are non-negative over a segment of the unit circle. In particular, it shows that any trigonometric polynomial in Ω can be written as a (non-negatively) weighted sum of two squared trigonometric polynomials (see (3.7)–(3.9)). This result will be used in next section to develop a linear matrix inequality representation of the polynomials in Ω.

4 Linear Matrix Inequality Formulation

In this section, we shall use a result of Nesterov [25] and Theorem 3.1 of Section 3 to develop a linear matrix inequality (LMI) representation for Ω, the set of trigonometric polynomials which are non-negative over a segment of the unit circle.

In reference [25], Nesterov showed how to obtain a linear matrix inequality representation for the cone of functions representable as a (weighted) sum of squares of functions in a given linear functional space V_n. In our case, the linear functional space V_n under consideration is the space of all trigonometric polynomials of degree at most n. Clearly, the components of $\boldsymbol{v}(\theta;n)$ form a basis of V_n. Below we paraphrase the representation result of Nesterov for our functional space V_n.

Proposition 4.1 *Let $\Delta \subseteq [0,2\pi)$ be a given subset. Let there be m given trigonometric polynomials which are non-negative over Δ:*

$$\mathrm{Re}\, \boldsymbol{v}(\theta;n_k)^{\mathrm{H}}\boldsymbol{w}_k \geq 0 \quad \textit{for all} \quad \theta \in \Delta,$$

where for each $k=1,2,\ldots,m$, $n_k \in \{0,1,...,n\}$ and $\boldsymbol{w}_k \in \mathbb{R}\times\mathbb{C}^{n_k}$. Let $\mathsf{L}_k(\cdot) : \mathbb{C}^{n+1} \mapsto \mathbb{C}^{(n+1)\times(n+1)}$ be a linear operator such that

$$\mathsf{L}_k\big(\boldsymbol{v}(\theta;n)\big) + \mathsf{L}_k\big(\boldsymbol{v}(\theta;n)\big)^{\mathrm{H}} = 2\big(\mathrm{Re}\, \boldsymbol{v}(\theta;n_k)^{\mathrm{H}}\boldsymbol{w}_k\big)\boldsymbol{v}(\theta;n-n_k)\boldsymbol{v}(\theta;n-n_k)^{\mathrm{H}}, \quad \forall \theta \in [0,2\pi).$$

Let $\ell \geq 1$ and $N \geq n+1$. Consider the cone

$$\begin{aligned}\mathcal{K} = \big\{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \mid \quad & \mathrm{Re}\, \boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{p} = \sum_{k=0}^{\ell}\big(\mathrm{Re}\, \boldsymbol{v}(\theta;n_k)^{\mathrm{H}}\boldsymbol{w}_k\big)\sum_{m=0}^{N}\big|\boldsymbol{v}(\theta;n-n_k)^{\mathrm{H}}\boldsymbol{q}_{m,k}\big|^2 \\ & \textit{for some } \boldsymbol{q}_{m,k} \in \mathbb{R}\times\mathbb{C}^{n-n_k}\big\}.\end{aligned} \tag{4.1}$$

Then we have the following alternative LMI description of $\mathcal{K}$:

$$\mathcal{K} = \left\{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \mid \boldsymbol{p} + \xi \mathrm{j}\boldsymbol{e}_0 = \sum_{k=1}^{m}\mathsf{L}_k^*(\boldsymbol{X}_k), \quad \textit{for some } \boldsymbol{X}_k \in \mathcal{H}_+^{(n+1-n_k)\times(n+1-n_k)},\ \xi \in \mathbb{R}\right\},$$

where $\boldsymbol{e}_0$ is the first column of the $(n+1)\times(n+1)$ identity matrix and $\mathsf{L}_k^* : \mathbb{C}^{(n+1)\times(n+1)} \mapsto \mathbb{C}^{n+1}$ is the adjoint linear operator of L_k.

Consider now the set of (the real part of) trigonometric polynomials of order n, i.e. functions $\mathrm{Re}\, \boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{p}$ in $\theta \in [0,2\pi)$, with coefficients $\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n$. Given $0 \leq \alpha < \beta < 2\pi$, we define the sets

$$\mathcal{K}(\alpha,\beta) := \{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \mid \mathrm{Re}\, \boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{p} \geq 0 \text{ for all } \theta \in [\alpha,\beta]\}, \tag{4.2}$$

and

$$\bar{\mathcal{K}}(\alpha,\beta) := \{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \mid \operatorname{Re} \boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{p} \geq 0 \text{ for all } \theta \in [0,2\pi) \setminus (\alpha,\beta)\}. \tag{4.3}$$

Thus, $\mathcal{K}(\alpha,\beta)$ and $\bar{\mathcal{K}}(\alpha,\beta)$ describe the sets of (the real parts of) trigonometric polynomials which are non-negative over a segment of the unit circle. We also let

$$\mathcal{K}(0,2\pi) := \{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \mid \operatorname{Re} \boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{p} \geq 0 \text{ for all } \theta \in [0,2\pi)\}, \tag{4.4}$$

describe the trigonometric polynomials that are nonnegative on the entire unit circle. We may interpret $\mathcal{K}(\alpha,\beta)$ and $\bar{\mathcal{K}}(\alpha,\beta)$ as convex cones in $\mathbb{R}^{2n+1}$. Since (the real part of) a trigonometric polynomial is non-negative on a closed segment if and only if it is non-negative on the corresponding open segment, these cones are invariant to the opening or closure of either end of the given segment. This fact simplifies the application of the LMI descriptions of these cones which we now develop. By Theorem 3.1, these cones can be equivalently described as

$$\begin{aligned}\mathcal{K}(\alpha,\beta) = \big\{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \;\big|\; &\operatorname{Re} \boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{p} = |\boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{q}|^2 + (\operatorname{Re} \boldsymbol{v}(\theta;1)^{\mathrm{H}}\boldsymbol{d}(\alpha,\beta))\; |\boldsymbol{v}(\theta;n-1)^{\mathrm{H}}\boldsymbol{r}|^2 \\ &\text{for some } \boldsymbol{q} \in \mathbb{R}\times\mathbb{C}^n \text{ and } \boldsymbol{r} \in \mathbb{R}\times\mathbb{C}^{n-1}\big\}\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\bar{\mathcal{K}}(\alpha,\beta) = \big\{\boldsymbol{p} \in \mathbb{R}\times\mathbb{C}^n \;\big|\; &\operatorname{Re} \boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{p} = |\boldsymbol{v}(\theta;n)^{\mathrm{H}}\boldsymbol{q}|^2 - (\operatorname{Re} \boldsymbol{v}(\theta;1)^{\mathrm{H}}\boldsymbol{d}(\alpha,\beta))\; |\boldsymbol{v}(\theta;n-1)^{\mathrm{H}}\boldsymbol{r}|^2 \\ &\text{for some } \boldsymbol{q} \in \mathbb{R}\times\mathbb{C}^n \text{ and } \boldsymbol{r} \in \mathbb{R}\times\mathbb{C}^{n-1}\big\}\end{aligned} \tag{4.6}$$

where $\boldsymbol{d}(\alpha,\beta)$ is given by (3.3). Notice that both $\mathcal{K}(\alpha,\beta)$ and $\bar{\mathcal{K}}(\alpha,\beta)$ are in the form of (4.1), since each element of $\mathcal{K}(\alpha,\beta)$ or $\bar{\mathcal{K}}(\alpha,\beta)$ can be written as a (non-negatively) weighted sum of squares. Therefore, Proposition 4.1 implies $\mathcal{K}(\alpha,\beta)$ and $\bar{\mathcal{K}}(\alpha,\beta)$ both possess an LMI description. To derive an explicit form of these LMI representations, we need to make precise the linear operators L_k. This is what we do next.

We define the unit lower triangular $(n+1)\times(n+1)$ Toeplitz matrices $\boldsymbol{T}_{0,n},\ \boldsymbol{T}_{1,n},\ \ldots,\ \boldsymbol{T}_{n,n}$ as

$$[\boldsymbol{T}_{k,n}]_{ij} = \begin{cases} 1, & \text{if } i = k+j, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{with } i,j \in \{0,1,\ldots,n\}. \tag{4.7}$$

Thus, $\boldsymbol{T}_{0,n} = \boldsymbol{I}$ and $\boldsymbol{T}_{k,n} \bullet \boldsymbol{X} = \sum_{\ell=1}^{n+1-k} \boldsymbol{X}_{\ell+k,\ell}$, for all $\boldsymbol{X} \in \mathbb{C}^{(n+1)\times(n+1)}$. That is, $\boldsymbol{T}_{k,n} \bullet \boldsymbol{X}$ is the sum of the elements on the kth off-diagonal of $\boldsymbol{X}$. Let us define a linear operator $\mathsf{L} : \mathbb{C}^{n+1} \mapsto$

$\mathbb{C}^{(n+1)\times(n+1)}$ as

$$\mathbf{L}(\boldsymbol{y}) = y_0\boldsymbol{T}_{0,n} + 2\sum_{i=1}^{n} y_i\boldsymbol{T}_{i,n}. \tag{4.8}$$

It can be checked that $\mathbf{L}(\boldsymbol{y})$ is lower triangular and

$$\mathbf{L}\big(\boldsymbol{v}(\theta;n)\big) + \mathbf{L}\big(\boldsymbol{v}(\theta;n)\big)^{\mathrm{H}} = 2\boldsymbol{v}(\theta;n)\boldsymbol{v}(\theta;n)^{\mathrm{H}}, \quad \text{for all } \theta \in [0,2\pi). \tag{4.9}$$

To determine the adjoint operator $\mathbf{L}^*$, we note that

$$\mathbf{L}(\boldsymbol{y}) \bullet \boldsymbol{X} = y_0(\boldsymbol{T}_{0,n} \bullet \boldsymbol{X}) + 2\sum_{i=1}^{n} \overline{y}_i(\boldsymbol{T}_{i,n} \bullet \boldsymbol{X}) = \boldsymbol{y}^{\mathrm{H}}\mathbf{L}^*(\boldsymbol{X}), \quad \forall\ \boldsymbol{X} \in \mathbb{C}^{(n+1)\times(n+1)},$$

where the adjoint $\boldsymbol{q} = \mathbf{L}^*(\boldsymbol{X}) \in \mathbb{C}^{n+1}$ is given by

$$q_0 = \boldsymbol{T}_{0,n} \bullet \boldsymbol{X}, \quad q_i = 2(\boldsymbol{T}_{i,n} \bullet \boldsymbol{X}), \quad \text{for } i = 1,2,\ldots,n. \tag{4.10}$$

In addition, we need to define a family of operators $\boldsymbol{\Lambda}(\boldsymbol{y};\alpha,\beta) : \mathbb{C}^{n+1} \mapsto \mathbb{C}^{(n+1)\times(n+1)}$ which are linear in $\boldsymbol{y} \in \mathbb{C}^{n+1}$ and parameterized by $\alpha,\beta \in [0,2\pi]$. Let $\boldsymbol{d}(\alpha,\beta) = [d_0(\alpha,\beta), d_1(\alpha,\beta)]^{\mathrm{T}}$ be given by (3.3). Then

$$\begin{aligned}\boldsymbol{\Lambda}(\boldsymbol{y};\alpha,\beta) \;&:=\; d_0(\alpha,\beta)\Big(y_0\boldsymbol{T}_{0,n-1} + 2\sum_{k=1}^{n-1} y_k\boldsymbol{T}_{k,n-1}\Big)\\ &\quad + \overline{d_1(\alpha,\beta)}\Big(\sum_{k=1}^{n} y_k\boldsymbol{T}_{k-1,n-1}\Big) + d_1(\alpha,\beta)\Big(\sum_{k=0}^{n-2} y_k\boldsymbol{T}_{k+1,n-1}\Big).\end{aligned}$$

It can be checked that

$$\boldsymbol{\Lambda}\big(\boldsymbol{v}(\theta,n)\big) + \boldsymbol{\Lambda}\big(\boldsymbol{v}(\theta,n)\big)^{\mathrm{H}} = 2\big(\mathrm{Re}\ \boldsymbol{v}(\theta;1)^{\mathrm{H}}\boldsymbol{d}(\alpha,\beta)\big)\ \boldsymbol{v}(\theta;n-1)\boldsymbol{v}(\theta;n-1)^{\mathrm{H}}, \quad \forall\ \theta \in [0,2\pi).$$

To determine the adjoint operator of $\boldsymbol{\Lambda}$, we fix any $\boldsymbol{y} \in \mathbb{C}^{n+1}$ and any $\boldsymbol{X} \in \mathbb{C}^{n\times n}$ and consider

$$\begin{aligned}\boldsymbol{\Lambda}(\boldsymbol{y};\alpha,\beta) \bullet \boldsymbol{X} \;&=\; d_0(\alpha,\beta)\Big(\overline{y}_0(\boldsymbol{T}_{0,n-1} \bullet \boldsymbol{X}) + 2\sum_{k=1}^{n-1} \overline{y}_k(\boldsymbol{T}_{k,n-1} \bullet \boldsymbol{X})\Big)\\ &\quad + d_1(\alpha,\beta)\Big(\sum_{k=1}^{n} \overline{y}_k(\boldsymbol{T}_{k-1,n-1} \bullet \boldsymbol{X})\Big) + \overline{d_1(\alpha,\beta)}\Big(\sum_{k=0}^{n-2} \overline{y}_k(\boldsymbol{T}_{k+1,n-1} \bullet \boldsymbol{X})\Big)\\ &=\; \boldsymbol{y}^{\mathrm{H}}\boldsymbol{\Lambda}^*(\boldsymbol{X};\alpha,\beta).\end{aligned}$$

Thus, the adjoint $\boldsymbol{q} = \mathbf{\Lambda}^*(\boldsymbol{X}; \alpha, \beta)$ is given by

$$
\begin{cases}
q_0 = d_0(\alpha,\beta)(\boldsymbol{T}_{0,n-1} \bullet \boldsymbol{X}) + \overline{d_1(\alpha,\beta)}(\boldsymbol{T}_{1,n-1} \bullet \boldsymbol{X}), \\
q_k = 2d_0(\alpha,\beta)(\boldsymbol{T}_{k,n-1} \bullet \boldsymbol{X}) + d_1(\alpha,\beta)(\boldsymbol{T}_{k-1,n-1} \bullet \boldsymbol{X}) + \overline{d_1(\alpha,\beta)}(\boldsymbol{T}_{k+1,n-1} \bullet \boldsymbol{X}), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad k = 1, 2, ..., n-2, \\
q_{n-1} = 2d_0(\alpha,\beta)(\boldsymbol{T}_{n-1,n-1} \bullet \boldsymbol{X}) + d_1(\alpha,\beta)(\boldsymbol{T}_{n-2,n-1} \bullet \boldsymbol{X}), \\
q_n = d_1(\alpha,\beta)(\boldsymbol{T}_{n-1,n-1} \bullet \boldsymbol{X}).
\end{cases}
\tag{4.11}
$$

Combining this with (4.5), (4.6) and (4.10) and invoking Proposition 4.1, we obtain the following key result.

Theorem 4.1 *Let $0 \le \alpha < \beta < 2\pi$. Suppose $\mathcal{K}(\alpha,\beta)$ and $\bar{\mathcal{K}}(\alpha,\beta)$ are given by* (4.2) *and* (4.3)*. Let $\mathbf{L}^*$ and $\mathbf{\Lambda}^*$ be the adjoint operators defined by* (4.10) *and* (4.11)*. Then, the cones $\mathcal{K}(\alpha,\beta)$ and $\bar{\mathcal{K}}(\alpha,\beta)$ admit the following LMI description:*

$$
\begin{aligned}
\mathcal{K}(\alpha,\beta) = \big\{\boldsymbol{p} \,\big|\; & \boldsymbol{p} + \xi \mathrm{j}\boldsymbol{e}_0 = \mathbf{L}^*(\boldsymbol{X}) + \mathbf{\Lambda}^*(\boldsymbol{Z}; \alpha, \beta) \\
& \textit{for some } \boldsymbol{X} \in \mathcal{H}_+^{(n+1)\times(n+1)},\ \boldsymbol{Z} \in \mathcal{H}_+^{n\times n},\ \xi \in \mathbb{R}\big\},
\end{aligned}
\tag{4.12}
$$

and

$$
\begin{aligned}
\bar{\mathcal{K}}(\alpha,\beta) = \big\{\boldsymbol{p} \,\big|\; & \boldsymbol{p} + \xi \mathrm{j}\boldsymbol{e}_0 = \mathbf{L}^*(\boldsymbol{X}) - \mathbf{\Lambda}^*(\boldsymbol{Z}; \alpha, \beta) \\
& \textit{for some } \boldsymbol{X} \in \mathcal{H}_+^{(n+1)\times(n+1)},\ \boldsymbol{Z} \in \mathcal{H}_+^{n\times n},\ \xi \in \mathbb{R}\big\}.
\end{aligned}
\tag{4.13}
$$

Moreover,

$$
\mathcal{K}(0, 2\pi) = \big\{\mathbf{L}^*(\boldsymbol{X}) \,\big|\boldsymbol{X} \in \mathcal{H}_+^{(n+1)\times(n+1)}\big\}.
\tag{4.14}
$$

Theorem 4.1 provides an equivalent LMI description for a trigonometric polynomial which is non-negative over a given segment $[\alpha,\beta]$ (or its complement) of the unit circle. (Observe that $\mathcal{K}(\alpha, 2\pi) = \bar{\mathcal{K}}(0,\alpha)$.) As mentioned in Section 1 this LMI formulation is extremely useful as it allows a compact representation of the spectral mask constraints often encountered in the design of digital filters. This compact LMI representation makes it possible to efficiently solve many filter design problems with spectral mask constraints by the recently developed interior point methods for semidefinite programming [26]. We will give some detailed examples in Section 5.

Equation (4.14) is a new formulation of the Positive Real Lemma [17, 18] (and the closely related Kalman-Yakubovich-Popov [KYP] Lemma) for FIR systems (see also [25] and Section 3.2

in [27]). The new formulation is the dual of the standard formulation, and states that for r_m, $-M+1 \le m \le M-1$, with $r_{-m} = \bar{r}_m$, $R(e^{j\theta}) \ge 0$ for all $\theta \in [0, 2\pi)$ if and only if there exists an $\boldsymbol{X} \in \mathcal{H}_+^{M\times M}$ such that $\mathrm{tr}(\boldsymbol{X}) = r_0$ and $\sum_{\ell=1}^{M-\ell} [\boldsymbol{X}]_{\ell+m,\ell} = r_m$, for $1 \le m \le M-1$. Thus, Theorem 4.1 can be seen as an extension of Positive Real Lemma for FIR systems.

For real trigonometric polynomials of the form $\sum_{k=0}^{n} p_k \cos(k\theta)$, with coefficients $\boldsymbol{p} \in \mathbb{R}^{n+1}$, the results in Theorem 4.1 may be simplified. Since in this case $\cos(k\theta)$ is symmetric with respect to $\theta = \pi$, we only need to consider a segment $[\alpha, \beta]$ inside $[0, \pi]$. In this case, we get

$$
\begin{aligned}
\mathcal{K}_{\mathrm{real}}(\alpha,\beta) &= \left\{ \boldsymbol{p} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{k=0}^{n} p_k \cos(k\theta) \ge 0, \text{ for all } \theta \in [\alpha, \beta] \right\} \\
&= \left\{ \boldsymbol{p} \;\middle|\; \boldsymbol{p} = \mathsf{L}^*(\boldsymbol{X}) + \boldsymbol{\Lambda}^*(\boldsymbol{Z}; \alpha, \beta), \text{ for some } \boldsymbol{X} \in \mathcal{S}_+^{(n+1)\times(n+1)}, \boldsymbol{Z} \in \mathcal{S}_+^{n\times n} \right\}.
\end{aligned}
$$

Notice that $\boldsymbol{X}$ and $\boldsymbol{Z}$ in the above representation are real symmetric, rather than complex Hermitian. To see why we can restrict to real symmetric matrices, consider a Hermitian positive semidefinite matrix $\boldsymbol{X} = (\mathrm{Re}\, \boldsymbol{X}) + j(\mathrm{Im}\, \boldsymbol{X})$ given in the representation (4.12). Since $\boldsymbol{X} \succeq 0$, it follows in particular for real $\boldsymbol{q} \in \mathbb{R}^n$ that

$$
0 \le \boldsymbol{q}^{\mathrm{T}} \boldsymbol{X} \boldsymbol{q} = \boldsymbol{q}^{\mathrm{T}} (\mathrm{Re}\, \boldsymbol{X}) \boldsymbol{q} + j \boldsymbol{q}^{\mathrm{T}} (\mathrm{Im}\, \boldsymbol{X}) \boldsymbol{q} = \boldsymbol{q}^{\mathrm{T}} (\mathrm{Re}\, \boldsymbol{X}) \boldsymbol{q}.
$$

Hence, $\mathrm{Re}\, \boldsymbol{X} \in \mathcal{S}_+^{n\times n}$. Moreover, $\mathrm{Re}\, \boldsymbol{T}_{i,n} \bullet \boldsymbol{X} = \boldsymbol{T}_{i,n} \bullet (\mathrm{Re}\, \boldsymbol{X})$. Thus, if the imaginary parts of the coefficients p_i are restricted to zero, then we can replace $\boldsymbol{X}$ (obtained from (4.12) by $\mathrm{Re}\, \boldsymbol{X}$, and $\mathrm{Re}\, \boldsymbol{X}$ is still positive semidefinite.

If in addition to $\boldsymbol{p}$ being real, the segment $[\alpha, \beta]$ is of the form $[\alpha, \pi]$, then we can further simplify the above LMI representations as

$$
\begin{aligned}
\mathcal{K}_{\mathrm{real}}(\alpha) &= \left\{ \boldsymbol{p} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{k=0}^{n} p_k \cos(k\theta) \ge 0, \quad \text{for all } \theta \in [\alpha, \pi] \right\} \\
&= \left\{ \boldsymbol{p} \;\middle|\; \boldsymbol{p} = \mathsf{L}^*(\boldsymbol{X}) + \boldsymbol{\Lambda}^*(\boldsymbol{Z}; \alpha, 2\pi - \alpha), \text{ for some } \boldsymbol{X} \in \mathcal{S}_+^{(n+1)\times(n+1)}, \boldsymbol{Z} \in \mathcal{S}_+^{n\times n} \right\},
\end{aligned}
\tag{4.15}
$$

(for nonnegativity on $[\alpha, \pi]$) and

$$
\begin{aligned}
\bar{\mathcal{K}}_{\mathrm{real}}(\alpha) &= \left\{ \boldsymbol{p} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{k=0}^{n} p_k \cos(k\theta) \ge 0, \quad \text{for all } \theta \in [0, \alpha] \right\} \\
&= \left\{ \boldsymbol{p} \;\middle|\; \boldsymbol{p} = \mathsf{L}^*(\boldsymbol{X}) - \boldsymbol{\Lambda}^*(\boldsymbol{Z}; \alpha, 2\pi - \alpha), \text{ for some } \boldsymbol{X} \in \mathcal{S}_+^{(n+1)\times(n+1)}, \boldsymbol{Z} \in \mathcal{S}_+^{n\times n} \right\},
\end{aligned}
$$

(for nonnegativity on $[0,\alpha]$), where $\boldsymbol{\Lambda}^*(\boldsymbol{Z};\alpha,2\pi-\alpha)$ is given by (4.11) with $d_0(\alpha,2\pi-\alpha)$ and $d_1(\alpha,2\pi-\alpha)$ simplified to be

$$d_0(\alpha,2\pi-\alpha)=2\cos\alpha(1-\cos\alpha) \quad \text{and} \quad d_1(\alpha,2\pi-\alpha)=-2(1-\cos\alpha).$$

The latter is due to (3.5).

5 Applications

We now show how the results of Section 4 can be applied to the design of FIR filters and to data-independent narrow-band beamformers.

5.1 FIR Filter Design

In optimization-based designs of (real-valued) FIR filters, one often encounters a (relative) spectral mask constraint of the form

$$\zeta\bar{L}(e^{\mathrm{j}\theta})\leq\left|G(e^{\mathrm{j}\theta})\right|\leq\zeta\bar{U}(e^{\mathrm{j}\theta}) \qquad \text{for all } \theta\in[0,\pi], \tag{5.1}$$

where $\zeta>0$, and a normalization constraint either on the filter coefficients or on ζ. As discussed in the introduction, the mask constraint can be made convex by constraining $G(e^{\mathrm{j}\theta})$ to have 'linear phase', or by reformulating the constraint in terms of the autocorrelation sequence $r_m=\sum_k g_k g_{k-m}$ as $\zeta^2\bar{L}(e^{\mathrm{j}\theta})^2\leq R(e^{\mathrm{j}\theta})\leq\zeta^2\bar{U}(e^{\mathrm{j}\theta})^2$. In both these cases, we can exploit the results of Section 4 to precisely transform the piecewise constant portions of the mask into pairs of LMIs. The first step is to write $G(e^{\mathrm{j}\theta})$ or $R(e^{\mathrm{j}\theta})$ in the form $\operatorname{Re}\boldsymbol{v}(\theta;\cdot)^{\mathrm{H}}\boldsymbol{p}$. To do so, we define $\boldsymbol{M}\in\mathbb{R}^{(2M-1)\times M}$ and $\tilde{\boldsymbol{I}}\in\mathbb{R}^{M\times M}$ such that

$$\boldsymbol{M}:=\begin{bmatrix}\mathbf{0} & \boldsymbol{J}\\ 1 & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}\end{bmatrix} \qquad \text{and} \qquad \tilde{\boldsymbol{I}}:=\begin{bmatrix}1 & \mathbf{0}\\ \mathbf{0} & 2\boldsymbol{I}\end{bmatrix}, \tag{5.2}$$

where $\boldsymbol{I}$ is the $(M-1)\times(M-1)$ identity matrix and $\boldsymbol{J}$ is the $(M-1)\times(M-1)$ matrix with ones on the anti-diagonal and zeros elsewhere. For a filter of length M, if we define $\tilde{\boldsymbol{r}}\in\mathbb{R}^M$ such that $[\tilde{\boldsymbol{r}}]_m=r_m$, $0\leq m\leq M-1$, then

$$R(e^{\mathrm{j}\theta})=e^{\mathrm{j}(M-1)\theta}\boldsymbol{v}(\theta;2M-1)^{\mathrm{H}}\boldsymbol{M}\tilde{\boldsymbol{r}}=\operatorname{Re}\boldsymbol{v}(\theta;M-1)^{\mathrm{H}}\tilde{\boldsymbol{I}}\tilde{\boldsymbol{r}}.$$

Similarly, for a filter of odd length $2M-1$ which is symmetric and centered at the origin, if we define $\tilde{\boldsymbol{g}} \in \mathbb{R}^M$ such that $[\tilde{\boldsymbol{g}}]_k = g_k$, $0 \leq k \leq M-1$ then, $G(e^{\mathrm{j}\theta}) = \mathrm{Re}\, \boldsymbol{v}(\theta; M-1)^{\mathrm{H}} \tilde{\boldsymbol{I}} \tilde{\boldsymbol{g}}$. The frequency response of other linear phase filters can be written in related ways, but for brevity we will consider only the odd-length symmetric case.

To further simplify our exposition, we will focus on the design of a simple low-pass filter. The extension to band-pass and multi-band filter is immediate and is implicit in the design in Section 5.2. For a simple low-pass filter, the spectral mask in (5.1) has

$$\bar{L}(e^{\mathrm{j}\theta}) = \begin{cases} \bar{L}_p & 0 \leq \theta \leq 2\pi f_p \\ \bar{L}_s & 2\pi f_p < \theta \leq \pi \end{cases} \qquad \text{and} \qquad \bar{U}(e^{\mathrm{j}\theta}) = \begin{cases} \bar{U}_p & 0 \leq \theta < 2\pi f_s \\ \bar{U}_s & 2\pi f_s \leq \theta \leq \pi, \end{cases} \tag{5.3}$$

with f_p and f_s denoting the normalized frequencies of the pass-band and stop-band edges, respectively, $0 \leq f_p < f_s \leq 1/2$, and $\bar{U}_s \leq \bar{U}_p$, $\bar{L}_p \leq \bar{U}_p$ and $\bar{L}_s \leq \bar{L}_p$. In the case of linear phase filters we set $\bar{L}_s = -\bar{U}_s$ and for autocorrelation designs we set $\bar{L}_s = 0$. By observing the common form of $G(e^{\mathrm{j}\theta})$ and $R(e^{\mathrm{j}\theta})$ above, and that $\bar{L}_s \leq \bar{L}_p$ and $\bar{U}_s \leq \bar{U}_p$, the spectral mask constraint can be re-written in a generic form as

$$\begin{cases} \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} - \zeta^q \bar{L}_p^q \boldsymbol{e}_0 \in \bar{\mathcal{K}}_{\mathrm{real}}(2\pi f_p) \\ \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} - \zeta^q \bar{L}_s^q \boldsymbol{e}_0 \in \mathcal{K}_{\mathrm{real}}(0) \end{cases} \qquad \text{and} \qquad \begin{cases} \zeta^q \bar{U}_p^q \boldsymbol{e}_0 - \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} \in \mathcal{K}_{\mathrm{real}}(0) \\ \zeta^q \bar{U}_s^q \boldsymbol{e}_0 - \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} \in \mathcal{K}_{\mathrm{real}}(2\pi f_s), \end{cases} \tag{5.4}$$

where $q = 1$ and $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{g}}$ when we design an odd-length symmetric filter, and $q = 2$ and $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{r}}$ for autocorrelation designs.

The constraints in (5.4) define the set of feasible filters. The question that remains is which of these filters is the 'best'. A large class of filter design objectives can be cast as the minimization of a convex quadratic function of the parameters. This class includes the weighted least-squares approximation of some desired magnitude response [3,4,9]. Filter design problems in this class take the following form: Given a positive semidefinite matrix $\boldsymbol{Q}$, a vector $\boldsymbol{l}$ and an integer $q \in \{1, 2\}$, find $\tilde{\boldsymbol{x}}$ achieving

$$\min_{\tilde{\boldsymbol{x}}} \tilde{\boldsymbol{x}}^{\mathrm{T}} \boldsymbol{Q} \tilde{\boldsymbol{x}} - 2\boldsymbol{l}^{\mathrm{T}} \tilde{\boldsymbol{x}}$$

subject to (5.4) and a linear normalization constraint on either $\tilde{\boldsymbol{x}}$ or ζ, or show that none exist. This generic design problem can be solved by solving the following convex optimization problem:

Problem 5.1 *Given $\boldsymbol{Q} = \boldsymbol{L}\boldsymbol{L}^{\mathrm{T}}$, $\boldsymbol{l}$, q, f_p, f_s, $\bar{L}_p$, $\bar{L}_s$, $\bar{U}_p$, $\bar{U}_s$, and M, find $\tilde{\boldsymbol{x}} \in \mathbb{R}^M$ achieving $\min \Gamma - 2\boldsymbol{l}^{\mathrm{T}}\tilde{\boldsymbol{x}}$ over $\tilde{\boldsymbol{x}}$, $\zeta > 0$, $\boldsymbol{X}^{(pu)}, \boldsymbol{X}^{(pl)}, \boldsymbol{X}^{(sl)}, \boldsymbol{X}^{(su)} \in \mathcal{S}_+^{M \times M}$ and $\boldsymbol{Z}^{(pl)}, \boldsymbol{Z}^{(su)} \in \mathcal{S}_+^{(M-1)\times(M-1)}$,* *subject to* $\|\boldsymbol{L}^{\mathrm{T}}\tilde{\boldsymbol{x}}\|_2^2 \leq \Gamma$,

$$\tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} - \zeta^q \bar{L}_p^q \boldsymbol{e}_0 = \mathsf{L}^*(\boldsymbol{X}^{(pl)}) - \boldsymbol{\Lambda}^*\big(\boldsymbol{Z}^{(pl)}; 2\pi f_p, 2\pi(1-f_p)\big), \tag{5.5}$$

$$\tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} - \zeta^q \bar{L}_s^q \boldsymbol{e}_0 = \mathsf{L}^*(\boldsymbol{X}^{(sl)}), \tag{5.6}$$

$$\zeta^q \bar{U}_p^q \boldsymbol{e}_0 - \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} = \mathsf{L}^*(\boldsymbol{X}^{(pu)}), \tag{5.7}$$

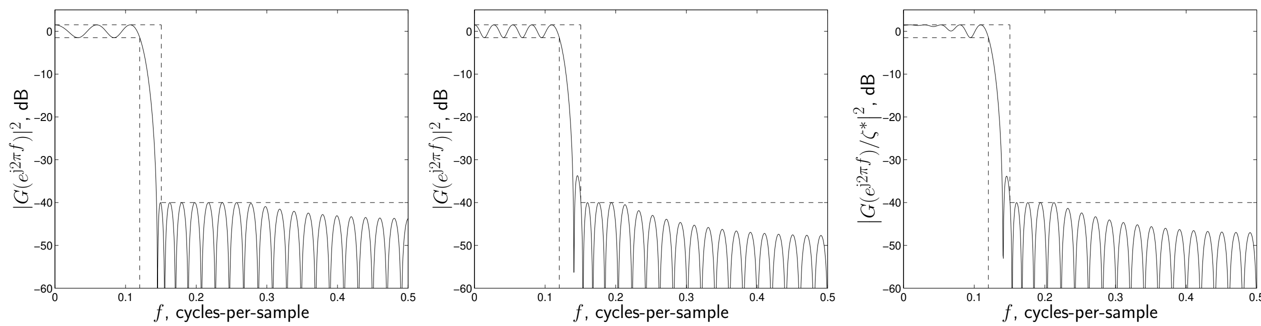
$$\zeta^q \bar{U}_s^q \boldsymbol{e}_0 - \tilde{\boldsymbol{I}}\tilde{\boldsymbol{x}} = \mathsf{L}^*(\boldsymbol{X}^{(su)}) + \boldsymbol{\Lambda}^*\big(\boldsymbol{Z}^{(su)}; 2\pi f_s, 2\pi(1-f_s)\big), \tag{5.8}$$

and one of the normalizations $\zeta = 1$ *or* $\boldsymbol{c}^{\mathrm{T}}\tilde{\boldsymbol{x}} = 1$*, for a given vector* $\boldsymbol{c}$*, or show that none exist.*

In Problem 5.1, Eqs (5.5) and (5.6) enforce the lower bound constraint of the spectral mask, and Eqs (5.7) and (5.8) enforce the upper bound constraint. Problem 5.1 consists of a linear objective, linear equality constraints [(5.6)–(5.8)], a linear inequality constraint on ζ, positive semi-definiteness constraints on the various $\boldsymbol{X}$ and $\boldsymbol{Z}$ matrices, and the constraint $\|\boldsymbol{L}^{\mathrm{T}}\tilde{\boldsymbol{x}}\|_2^2 \leq \Gamma$. The set of vectors $[\Gamma, (\boldsymbol{L}^{\mathrm{T}}\tilde{\boldsymbol{x}})^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{M+1}$ which satisfy this last constraint can be transformed to the intersection of a 'rotated' second-order cone in $\mathbb{R}^{M+2}$ and a hyperplane (e.g., [27]). Hence, Problem 5.1 is a convex symmetric cone programme [28,29], which can be efficiently solved using interior point methods [30]. Furthermore, infeasibility can be reliably detected. If $\tilde{\boldsymbol{x}}$ represents the autocorrelation sequence of the filter, then an optimal filter can be obtained from the solution of Problem 5.1 by spectral factorization [9, 11].

We now demonstrate the flexibility of this design method by solving a number of filter design problems using small variations on Problem 5.1.

Example 1 Consider the design of a length 49 FIR filter which has the minimal 'stop-band energy', $E_s = (1/\pi)\int_{2\pi f_c}^{\pi} |G(e^{j\theta})|^2\, d\theta$, subject to the spectral mask in (5.3), with $f_p = 590/4915.2$, $f_s = 740/4915.2$, $\bar{L}_p^2 = 10^{-0.15}$, $\bar{U}_p^2 = 10^{0.15}$, $\bar{U}_s^2 = 10^{-4}$. For an odd-length symmetric filter, $E_s = \tilde{\boldsymbol{g}}^{\mathrm{T}}\boldsymbol{Q}\tilde{\boldsymbol{g}}$, where $\boldsymbol{Q} = \tilde{\boldsymbol{I}}\hat{\boldsymbol{Q}}\tilde{\boldsymbol{I}}$, $[\hat{\boldsymbol{Q}}]_{ij} = 2\big(\mathrm{sinc}(i+j) + \mathrm{sinc}(i-j)\big) - 4f_c\big(\mathrm{sinc}(2f_c(i+j)) + \mathrm{sinc}(2f_c(i-j))\big)$, for $0 \leq i, j \leq M-1$, and $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$ for $x \neq 0$ and 1 for $x = 0$. For a general filter, $E_s = \tilde{\boldsymbol{l}}^{\mathrm{T}}\tilde{\boldsymbol{r}}$, where $\tilde{l}_0 = 1/2 - f_c$ and $\tilde{l}_m = -2f_c\,\mathrm{sinc}(2f_c m)$, for $1 \leq m \leq M-1$. Therefore optimal filters can be designed using Problem 5.1 with the normalization constraint $\zeta = 1$. For $f_c = (f_p + f_s)/2$, the power spectrum of the optimal linear phase filter is shown in Figure 1(a) and that of an optimal nonlinear phase filter is shown in Figure 1(b). The linear-phase case was solved (using `SeDuMi` [27]) in 3.5 seconds on a 400 MHz PENTIUM II workstation, whilst the nonlinear phase case required 24 seconds. The sharper cut off and improved high-frequency decay of the nonlinear phase filter are clear from these figures. Although these filters minimize the stop-band energy, they



(a) Linear-phase filter minimizing stop-band energy.

(b) Nonlinear-phase filter minimizing stop-band energy.

(c) Nonlinear-phase filter minimizing proportion of energy in stop band.

Figure 1: Relative power spectra of length 49 filters from Example 1, along with the corresponding mask. In (c), ζ^* is the optimal value of ζ from Problem 5.1.

do not minimize the proportion of the total energy of the filter in the stop band. A nonlinear phase filter which does so can be found by removing the constraint $\zeta = 1$ from Problem 5.1 (and hence allowing the mask to 'float'), and replacing it with $r_0 = 1$. The resulting optimal autocorrelation was obtained in 25 seconds and the power spectrum of an optimal filter is shown in Figure 1(c). Observe that the flatter passband response in this case is achieved without greatly affecting the stop-band decay. □

In the transmission of digital data by pulse amplitude modulation (PAM), we often encounter design specifications in terms of a spectral mask of the form in Example 1. In fact, the mask in Example 1 is that specified for the 'chip' waveform in the IS95 digital cellular communication standard [31]. A simplified block diagram of a PAM scheme is shown in Figure 2. The transmitted power is normalized to unity; i.e., $r_0 = 1$. In order to control the intersymbol interference (ISI) in a distortionless channel we can enforce the constraint

$$2 \sum_{i>0} r_{Ki}^2 \leq \epsilon, \tag{5.9}$$

for some (small) $\epsilon \geq 0$, [32]. This term is the mean square error (MSE) in $\hat{d}_n$ in Figure 2 in the absence of noise and channel distortion. When $\epsilon = 0$ this constraint is equivalent to self-orthogonality (that is, to g_k being a 'root-Nyquist' filter), but when $\epsilon > 0$ it allows us to trade ISI for other system properties. One of these properties might be the sensitivity of the MSE in $\hat{d}_n$ to unknown channel distortion. If the unknown channel is modelled as $c_k = \delta_k + c_k^{(e)}$, where

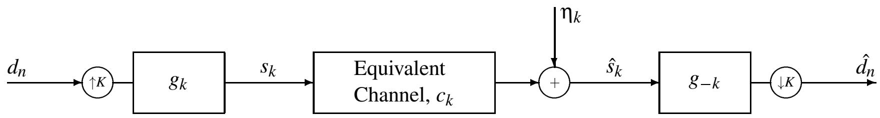


Figure 2: Equivalent discrete-time model of baseband PAM.

δ_k is the Kronecker delta (which is the impulse response of a distortionless channel), then an appropriate measure of the sensitivity of the PAM scheme is the worst-case MSE over a bounded set of $c_k^{(e)}$'s, [32]. For a given bound ϵ on the ISI, this sensitivity can be minimized by solving Problem 5.1 with $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{r}}$, $r_0 = 1$, $\boldsymbol{Q} = \tilde{\boldsymbol{I}}$, $\boldsymbol{l} = \boldsymbol{0}$, and the additional constraint in (5.9), [32].

Example 2 The filter specified for the synthesis of the chip waveform in IS95 has length 48 and $K = 4$ and satisfies the spectral mask specified in the standard (and illustrated in Figure 1), but it generates a large MSE in a distortionless channel. To efficiently determine whether this MSE can be reduced whilst simultaneously reducing the sensitivity to unknown channel distortion, the modified version of Problem 5.1 was solved for various values of ϵ. (Each solution was obtained in about 23 seconds.) The trade-off is shown in Figure 3, from which it is clear that the IS95 filter can be greatly improved upon. The spectra of the IS95 filter and a representative optimal filter are plotted in Figure 4. The robust filter provides a substantially lower 'chip error rate' than the IS95 filter in a slowly-varying frequency-selective Rician fading channel, as shown in Figure 5. (See [32] for the details.) □

In addition to tradeoffs between ISI and sensitivity, tradeoffs between ISI and bandwidth are also of interest in the design of PAM schemes. For a given level of ISI, a filter achieving the minimum bandwidth can be efficiently found using a bisection-based search on the stop-band edge for the feasibility boundary of a convex cone feasibility problem [32]. That feasibility problem is based on the modified version of Problem 5.1 used in Example 2. (This is a variation of the method used to find minimum bandwidth self-orthogonal filters in [10].) The resulting tradeoff for the IS95 spectral mask is plotted in Figure 6, from which is is clear that the IS95 filter is some distance from the optimal filters.

5.2 Beamformer Design

In standard narrow-band beamforming applications, the outputs of each antenna element at a given instant are linearly combined to form the array output at that instant [33]. If $x_k(n)$ denotes the

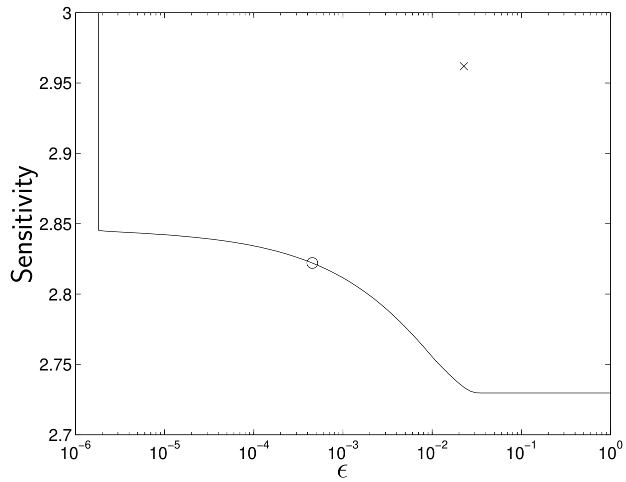
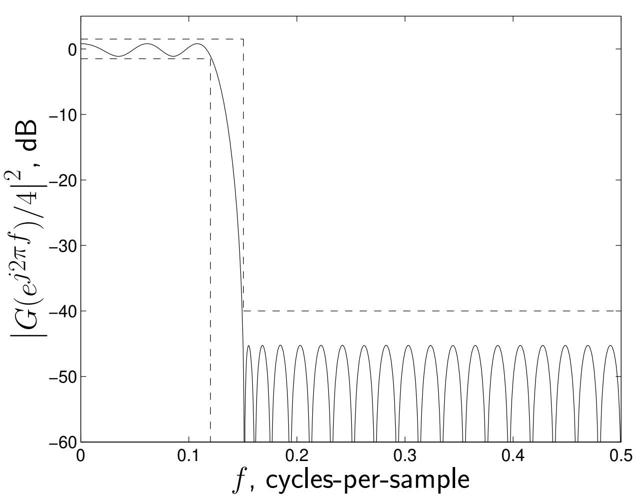
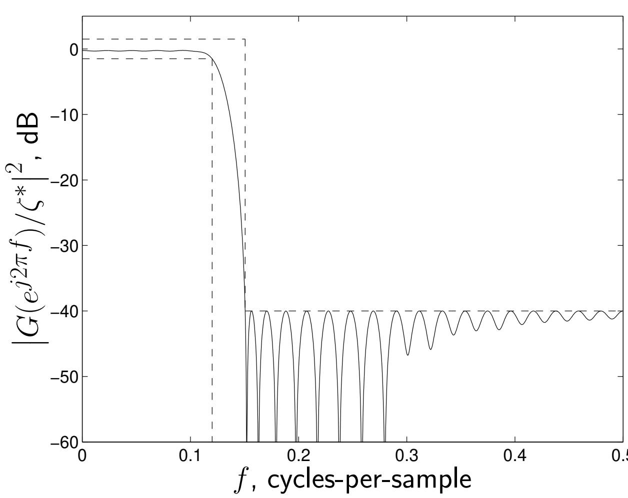


Figure 3: The trade-off between the sensitivity and ϵ for the IS95 standard for Example 2. The '$\times$' and '$\circ$' denote the positions achieved by the IS95 filter and the robust filter in Figure 4(b), respectively.



(a) IS95 filter, length 48.



(b) Robust filter, length 48.

Figure 4: Relative power spectra (in decibels) of the filters in Example 2, with the IS95 mask. Here, ζ^* is the optimal value of ζ from the modified version of Problem 5.1.

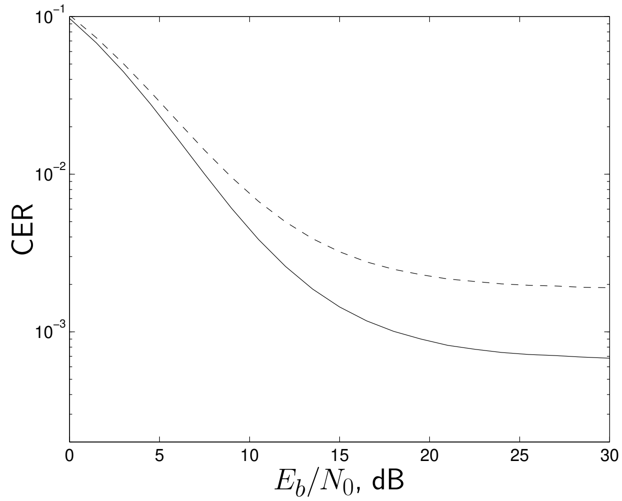


Figure 5: Simulated chip error rates (CER) in a slowly-varying frequency-selective Rician channel against signal-to-noise ratio for Example 2. Legend—Dashed: IS95 filter; Solid: robust filter.

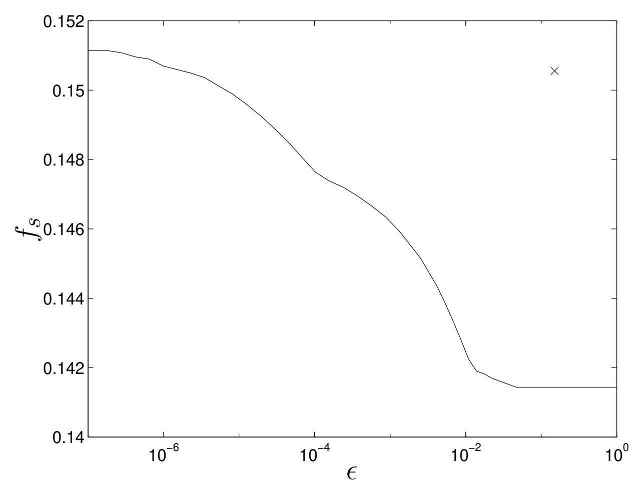


Figure 6: Minimal stop-band edge f_s against ISI bound ϵ for the IS95 mask. The '$\times$' denotes the position of the IS95 filter.

complex envelope of the output of the kth antenna element at the nth instant, and if $y(n)$ denotes the complex envelope of the array output at that instant, then

$$y(n) = \boldsymbol{x}(n)^{\mathrm{T}}\boldsymbol{w} = \boldsymbol{w}_c^{\mathrm{H}}\boldsymbol{x}(n),$$

where w_k is the 'weight' applied to the output of the kth antenna element, and $[\boldsymbol{w}_c]_k = \bar{w}_k$. It is well known [33] that if the array geometry is linear, with equi-spaced elements with separation d, and if the array operates on signals with wavelength λ, then the (complex) 'gain' of the array for a signal arriving at an angle ϕ to broadside (perpendicular to the array) is

$$\tilde{W}(\phi) = e^{\mathrm{j}\chi}W(e^{\mathrm{j}2\pi(d/\lambda)\sin\phi}),$$

where $W(e^{\mathrm{j}\theta})$ is the Fourier Transform of w_k, and $e^{\mathrm{j}\chi}$ determines the 'phase centre' of the array. For simplicity, we will focus on the standard case where $d = \lambda/2$.

In many applications, we would like to control the 'beam pattern' of the array, $|\tilde{W}(\phi)|^2$, but that results in non-convex constraints on w_k. Using the autocorrelation of the weights, $r_m = \sum_k w_k \bar{w}_{k-m}$, we have that $\tilde{R}(\phi) = \sum_m r_m e^{-\mathrm{j}m\pi\sin\phi} = |\tilde{W}(\phi)|^2$, and therefore bound constraints on $|\tilde{W}(\phi)|^2$ result in linear constraints on r_m. For an M-element array,

$$\tilde{R}(\phi) = \mathrm{Re}\ \boldsymbol{v}(\pi\sin\phi; M-1)^{\mathrm{H}}\tilde{\boldsymbol{I}}\tilde{\boldsymbol{r}},$$

where $[\tilde{\boldsymbol{r}}]_m = r_m$, $0 \le m \le M-1$ and $\tilde{\boldsymbol{I}}$ was defined in Section 5.1. Therefore, piecewise constant constraints on $|\tilde{W}(\phi)|^2$ can be compactly enforced in an analogous way to that for the spectral masks in Section 5.1, as we now demonstrate in a simple example derived from [33, Figure 2.5].

Example 3 Suppose that a desired signal impinges on a 16-element linear equi-spaced array with element separation $d = \lambda/2$ from an angle of $\phi_d = -18^\circ \pm 6^\circ$, and that interfering signals arrive from angles in the range $\phi_i = 21.5^\circ \pm 6^\circ$. An interesting data independent [33] beamforming problem is to minimize the response to (spatially) white noise (i.e., $\boldsymbol{w}^{\mathrm{H}}\boldsymbol{w} = r_0$), subject to the gain in the direction of the desired signal being within $\pm\Delta_d$ dB, and to the gain in the direction of the interferers being less than Δ_i dB. Furthermore, to guard against unexpected interferers from other directions, we would like to keep the sidelobes below Δ_s dB, and to constrain the main lobe (as determined by Δ_s) to be within $-18^\circ \pm 13^\circ$. In short, our objective is to minimize the white noise gain, subject to a mask of the shape in Figure 8, below. This problem can be cast in a similar way to Problem 5.1 with $\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{r}}$, $\zeta = 1$, $\boldsymbol{Q} = \boldsymbol{0}$, and $\boldsymbol{l} = -\boldsymbol{e}_0/2$, except that the vector $\tilde{\boldsymbol{r}}$ and

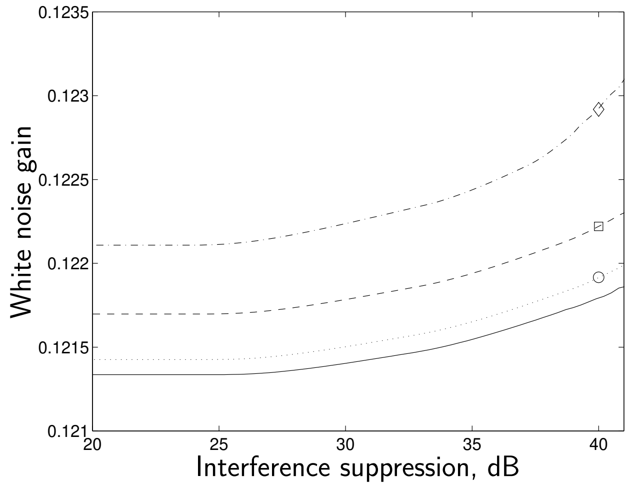
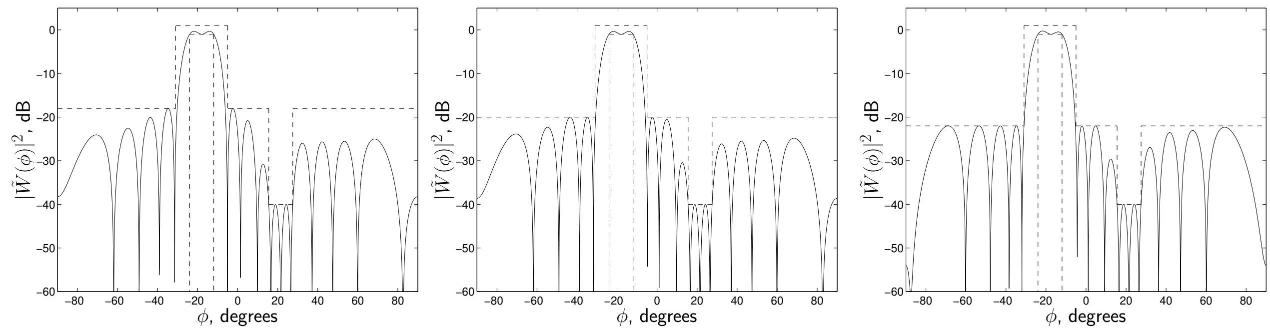


Figure 7: The trade-off between the minimum white noise gain $\boldsymbol{w}^{\mathrm{H}}\boldsymbol{w}$, and the interference suppression $|\Delta_i|$, for maximum sidelobe levels, Δ_s, of 0.1 dB (solid), -18 dB (dotted), -20 dB (dashed), and -22 dB (dot-dashed) for Example 3. The $\circ$, $\square$ and $\diamond$ denote the trade-offs achieved by the beamformers in Figure 8.

the various $\boldsymbol{X}$ and $\boldsymbol{Z}$ matrices may be complex. Therefore, the trade-offs between the white noise gain and the level of interference suppression, for different values of the maximum sidelobe level and for a 'look direction ripple' with $\Delta_d = 0.1$ dB can be efficiently found. They are presented in Figure 7. Examples of the resulting beam patterns are shown in Figure 8. (Each optimal $\tilde{\boldsymbol{r}}$ was computed in about seven seconds.) These examples clearly demonstrate the role which the sidelobe level constraint plays in determining the shape of the beam pattern. $\square$

6 Concluding Remarks

In this paper, we have provided a compact representation of piecewise constant spectral mask constraints via linear matrix inequalities. This representation is precise and does not require discretization of the mask. The representation is also convex, which allows it to be incorporated into efficient design algorithms for a diverse class of FIR filters and beamformers. For such models, (in)feasibility can be detected reliably, which is especially important if the linear matrix inequalities are solved iteratively in a binary search scheme (such as in minimal length filter design). In addition to these applications, generalizations of our results to rational filters (i.e., infinite impulse response filters) and to multidimensional filters are of interest in control theory, as well as signal and image processing, and are currently being pursued.



(a) $\Delta_s = -18$ dB. (b) $\Delta_s = -20$ dB. (c) $\Delta_s = -22$ dB.

Figure 8: Beam patterns with minimal white noise gain, subject to 40 dB interference suppression and different maximum sidelobe levels, Δ_s, for Example 3, along with the corresponding masks.

Acknowledgment. The third author would like to thank Professor Yu. Nesterov for sharing his ideas and for stimulating research in this area.

References

[1] T. W. Parks and C. S. Burrus, *Digital Filter Design*, J. Wiley & Sons, New York, 1987.

[2] K. Steiglitz, T. W. Parks, and J. F. Kaiser, "METEOR: A constraint-based FIR filter design program", *IEEE Trans. Signal Processing*, vol. 40, no. 8, pp. 1901–1909, Aug. 1992.

[3] I. W. Selesnick, M. Lang, and C. S. Burrus, "Constrained least square design for FIR filters without specified transition bands", *IEEE Trans. Signal Processing*, vol. 44, no. 8, pp. 1879–1892, Aug. 1996.

[4] J. W. Adams and J. L. Sullivan, "Peak-constrained least-squares optimization", *IEEE Trans. Signal Processing*, vol. 46, no. 2, pp. 306–321, Feb. 1998.

[5] P. Leister and T. W. Parks, "On the design of digital FIR filters with optimum magnitude and minimum phase", *Arch. El. Übertr.*, vol. 29, pp. 270–274, 1975.

[6] O. Herrmann and W. Schuessler, "Design of nonrecursive digital filters with minimum phase", *Electronics Letters*, vol. 6, no. 11, pp. 329–330, 28 June 1970.

[7] X. Chen and T. W. Parks, "Design of optimal minimum phase FIR filters by direct factorization", *Signal Processing*, vol. 10, pp. 369–383, June 1986.

[8] S.-P. Wu, S. Boyd, and L. Vandenberghe, "FIR filter design via semidefinite programming and spectral factorization", in *Proc. IEEE Conf. Decision Control*, 1996, pp. 271–276.

[9] S.-P. Wu, S. Boyd, and L. Vandenberghe, "FIR filter design via spectral factorization and convex optimization", in *Applied and Computational Control, Signals, and Circuits*, B. Datta, Ed., vol. 1. Birkhauser, Boston, May 1999, Also: `http://www.stanford.edu/~boyd/magdes.html`.

[10] T. N. Davidson, Z.-Q. Luo, and K. M. Wong, "Design of orthogonal pulse shapes for communications via semidefinite programming", *IEEE Trans. Signal Processing*, vol. 48, no. 5, pp. 1433–1445, May 2000.

[11] T. N. T. Goodman, C. A. Micchelli, G. Rodriguez, and S. Seatzu, "Spectral factorization of Laurent polynomials", *Adv. in Comp. Math.*, vol. 7, no. 4, pp. 429–454, 1997.

[12] J. O. Coleman and D. P. Scholnik "Design of nonlinear-phase FIR filters with second-order cone programming", *Proc. 1999 Midwest Symp. on Circuits and Systems*, Las Cruces, NM, Aug. 1999.

[13] R. Hettich and K. O. Kortanek, "Semi-infinite programming: Theory, methods and applications", *SIAM Review*, vol. 35, no. 3, pp. 380–429, Sept. 1993.

[14] P. Moulin, M. Anitescu, K. O. Kortanek, and F. A. Potra, "The role of linear semi-infinite programming in signal adapted QMF bank design", *IEEE Trans. Signal Processing*, vol. 45, no. 9, pp. 2160–2174, Sept. 1997.

[15] O. Rioul and P. Duhamel, "A Remez exchange algorithm for orthonormal wavelets", *IEEE Trans. Circuits Systems—II*, vol. 41, no. 8, pp. 550–560, Aug. 1994.

[16] H. H. Dam, K. L. Teo, S. Nordebo, and A. Cantoni, "The dual parameterization approach to optimal least square FIR filter design subject to maximum error constraints", *IEEE Trans. Signal Processing*, vol. 48, no. 8, pp. 2314–2320, Aug. 2000.

[17] B. D. O. Anderson, K. L. Hitz, and N. D. Diem, "Recursive algorithm for spectral factorization", *IEEE Trans. Circuits Systems*, vol. CAS–21, no. 6, pp. 742–750, Nov. 1974.

[18] B. Hassibi, A. H. Sayed, and T. Kailath, *Indefinite-Quadratic Estimation and Control: A Unified Approach to H^2 and H^∞ Theories*, SIAM, Philadelphia, 1999.

[19] P. P. Vaidyanathan, "The discrete-time Bounded-Real Lemma in digital filtering", *IEEE Trans. Circuits and Systems*, vol. CAS–32, no. 9, pp. 918–924, Sept. 1985.

[20] J. E. Marsden, *Basic Complex Variables*, W. H. Freeman and Company, San Francisco, 1973.

[21] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1989.

[22] A. A. Markov, "Lecture notes on functions with the least deviation from zero", 1906, Reprinted in *A. A. Markov, Selected Papers*, N. Achiezer, Ed., pp. 244–291, GosTechIzdat, Moscow, 1948 (in Russian).

[23] Lukacs, "Verscharfung der ersten mittelwertsatzes der integralrechnung fur rationale polynome", *Math. Zeitschrift*, vol. 2, pp. 229–305, 1918.

[24] V. Powers and B. Reznick, "Polynomials that are positive on an interval", Technical Report, University of Illinois at Urbana-Champaign, 1998. To appear in *Trans. Amer. Math. Soc.*.

[25] Y. Nesterov, "Squared functional systems and optimization problems", in *High Performance Optimization*, H. Frenk, K. Roos, T. Terlaky, and S. Zhang, Eds., pp. 405–440. Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[26] Y. Nesterov and A. Nemirovsky, *Interior Point Polynomial Algorithms in Convex Programming*, SIAM, Philadelphia, 1994.

[27] J. F. Sturm, Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653, 1999. Special issue on Interior Point Methods (CD supplement with software). Also: `http://www.unimaas.nl/~sturm/software/sedumi.html`.

[28] L. Vandenberghe and S. Boyd, "Semidefinite programming", *SIAM Review*, vol. 31, no. 1, pp. 49–95, Mar. 1996.

[29] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, "Applications of second-order cone programming", *Lin. Algebra Applics*, vol. 284, no. 1–3, pp. 193–228, Nov. 1998.

[30] Y. Ye, *Interior Point Algorithms: Theory and Analysis*, J. Wiley & Sons, New York, 1997.

[31] "Proposed EIA/TIA interim standard. Wideband spread spectrum digital cellular system dual-mode mobile station-base station compatibility standard", Technical Report TR45.5, QUALCOMM, Inc., San Diego, CA, Apr. 1992.

[32] T. N. Davidson, "Efficient design of waveforms for robust pulse amplitude modulation based on mean square error criteria", In *Proc. X European Signal Processing Conf.*, Tampere, Finland, Sept. 2000.

[33] B. D. Van Veen and K. M. Buckley, "Beamforming: A versatile approach to spatial filtering", *IEEE Acoust., Speech, Signal Processing Mag.*, pp. 4–24, Apr. 1988.