A New Lower Bound via Projection for the Quadratic Assignment Problem

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Abstract

New lower bounds for the quadratic assignment problem QAP are presented. These bounds are based on the orthogonal relaxation of QAP. The additional improvement is obtained by making efficient use of a tractable representation of orthogonal matrices having constant row and column sums. The new bound is easy to implement and often provides high quality bounds under an acceptable computational effort.

Key Words: quadratic assignment problem, lower bounds, relaxations, orthogonal projection, eigenvalue bounds.

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1 Introduction

The *Quadratic Assignment Problem* QAP is a generic model for various problems arising e.g. in location theory, VLSI design, facility layout, keyboard design and many other areas, see [1] for a recent survey on the QAP. Formally the QAP consists of minimizing

$$f(X) = tr(AXB^t + C)X^t$$

over the set of permutation matrices. A, B and C are given (real) matrices defining the QAP. (Throughout, M^t denotes the transpose of M, while trM is the trace of the square matrix M.)

If the quadratic part of f is zero, i.e. A or B is zero, the resulting problem of minimizing

$$trCX^t$$

over permutation matrices X is called the *Linear Assignment Problem* LAP. While LAP can be solved in $O(n^3)$ worst case running time, it is well known that QAP belongs to the class of NP-hard combinatorial optimization problems. Therefore the computational effort to solve QAP is very likely to grow exponentially with the problem size. There are several Branch and Bound based solution procedures described in the literature to solve the QAP, see [2, 10, 12]. All these approaches seem to break down on problems of sizes around $n = 15$, because the number of nodes in the branching tree becomes excessively large. All these approaches use, as a basic bounding procedure for the QAP, a technique proposed by Gilmore and Lawler in the early sixties, [6, 8]. This bounding technique is combinatorial in nature and requires the solution of a LAP, besides sorting the rows of both A and B.

Table 5.1 contains the Gilmore-Lawler-Bound, denoted GLB, for QAPs of various sizes published in the literature as well as for randomly generated problems. It turns out that with growing problem size the relative gap between a feasible solution and GLB gets larger as well. This indicates that progress to solve larger QAPs is unlikely unless stronger bounding rules are used.

Recently a lower bound for symmetric QAPs was introduced based on the eigenvalues of A and B, see [4, 11]. The basic idea to derive this bound consists in minimizing $f(X)$ over orthogonal rather than just permutation matrices.

In the present paper this "orthogonal relaxation" of the QAP will be further improved. We do this by constraining orthogonal matrices to have constant row and column sums, just as permutation matrices do. This leads to new bounds for the symmetric QAP. We also discuss perturbation techniques that allow further improvements of a given bound. The paper is organized as follows. In Section 2 we review the orthogonal relaxation for symmetric QAPs along with necessary definitions and preliminaries. In Section 3 we describe how the constraints on the row and column sums of X can be modelled so as to preserve the trace structure of $f(X)$ and the orthogonality of X. Various new bounds are presented in Section 4. These bounds will basically be derived from a smaller dimensional problem, equivalent to QAP, where the constraints on the row and column sums of X are automatically satisfied. Finally we conclude with some computational experiments on published as well as randomly generated data. These experiments indicate that for many problem instances, the new bounds are competitive with all the existing bounds, if one considers a performance measure that incorporates not only the quality of a bound, but also the computation time to obtain it.

2 Orthogonal relaxation of QAP

A QAP is called *symmetric*, if both matrices A and B are symmetric. From now on we consider only symmetric QAPs and observe that it is possible to transform an arbitrary QAP into an Hermitian QAP with (complex) Hermitian A and B, see [7]. For the sake of simplicity of presentation we consider just the real symmetric case, even though the results carry over also to the Hermitian case, but become more complicated. The following notation will be used throughout.

$\lambda_j(M)$ denotes the j-smallest eigenvalue of the symmetric matrix M, and

$$\lambda(M) := (\lambda_1(M), \ldots, \lambda_n(M))^t \in \Re^n.$$

The $n-$vector of all ones is $u := (1, \ldots, 1)^t \in \Re^n$. The vector of row sums of the $n \times n-$matrix M is denoted by $r(M)$, while the sum of all the entries of M is $s(M)$, i.e.

$$r(M) := Mu, \quad s(M) := u^t M u.$$

We will use the following sets of $n \times n$-matrices.

$$\mathcal{O} := \{X : X^t X = I\}$$

denotes the set of orthogonal matrices. Sometimes we also write $\mathcal{O}_n$ to indicate the size.

$$\mathcal{E} := \{X : Xu = X^t u = u\}$$

contains all matrices having row and column sums equal to one. Finally

$$\mathcal{N} := \{X : X \geq 0\}$$

denotes the set of matrices with nonnegative coefficients.

It is well known that the set Π of permutation matrices satisfies

$$\Pi = \mathcal{O} \cap \mathcal{E} \cap \mathcal{N}. \tag{2.1}$$

If $a, b \in \Re^n$ we denote by $< a, b >_-$ the *minimal scalar product* of a and b, which is defined by

$$< a, b >_- := \min\{\sum_i a_i b_{\pi(i)} : \pi \ \ permutation\}.$$

We observe that

$$< a, b >_- \quad = \quad a^t b$$

if a is in nonincreasing and b in nondecreasing order, respectively. We will make extensive use of this fact. The *maximal scalar product* $< a, b >_+$ is defined similarly.

The orthogonal relaxation of symmetric QAPs was introduced in [4, 11] and makes use of the following result.

Theorem 2.1 *[11] Let A and B be real symmetric $n \times n$ matrices. Then*

$$< \lambda(A), \lambda(B) >_- \quad \leq \quad tr\ AXB^tX^t \quad \leq \quad < \lambda(A), \lambda(B) >_+ \qquad \forall X \in \mathcal{O}.$$

Moreover, the lower bound is attained for $X = PQ^t$, where $P, Q \in \mathcal{O}$ contain the eigenvectors of A and B in the order prescribed by the minimal scalar product of the eigenvalues.

We will use Theorem 2.1 later on as a basic tool to bound the quadratic part of $f(X)$. It seems rather difficult to extend Theorem 2.1 to include also the linear term present in $f(X)$. This difficulty can be overcome by separating $f(X)$ into a quadratic term

$$q(X) := tr\ AXB^tX^t$$

and a linear term

$$l(X) := tr\ CX^t.$$

The quadratic part can be bounded using Theorem 2.1 while the linear part is solved independently as a LAP. We let QAP(A,B,C) denote the optimal objective function value of the QAP defined by matrices A, B, C and LAP(C) denote the optimal value of the LAP defined by C. The following eigenvalue related bound was proposed in [4].

$$QAP(A, B, C) \quad \geq \quad < \lambda(A), \lambda(B) >_- \quad + \ LAP(C). \tag{2.2}$$

In Table 5.1 the column EVB contains the lower bounds obtained by (2.2). Noting that all input matrices for these problems belong to $\mathcal{N}$, a trivial lower bound is of course 0, thus (2.2) does not seem to be of any advantage.

To improve the bound in (2.2), transformations are applied to A, B and C that leave $f(X)$ unchanged over Π. Two types of transformations are known to have this property.

a) adding a constant to A or B either row or column-wise and appropriately modifiying C, or

b) changing the main diagonal of A or B and appropriately modifying C.

To be more specific, suppose $e, f, r, s \in \Re^n$. We define

$$A(e, r) := A + eu^t + ue^t + diag(r),$$

$$B(f, s) := B + fu^t + uf^t + diag(s),$$

$$C(e, f, r, s) := C + 2Auf^t + 2eu^tB - 2nef^t - 2\sum_k e_k uf^t + diag(A)s^t + r\ diag(B)^t - 2es^t - 2rf^t - rs^t.$$

Then it can easily be verifed, see [4, 5], that

$$tr\ (AXB^t + C)X^t = tr\ (A(e, r)XB^t(f, s) + C(e, f, r, s))X^t \quad \forall e, f, r, s \in \Re^n, \ \forall X \in \Pi. \tag{2.3}$$

We point out that we add the same constant to the rows and columns to preserve symmetry of A and B.

Relation (2.3) shows that we may choose any transformation $d := (e, f, r, s) \in \Re^{n \times 4}$ to derive bounds for QAP. We now describe two strategies proposed in the literature to make "reasonable"choices for d.

The first strategy is based on the observation that a small "variance" of the spectrum of A and B results in small fluctuations for the quadratic part $q(X)$ by Theorem 2.1. The *variance* of the spectrum of a symmetric matrix $A, var(A)$, can be defined as

$$var(A) := \sqrt{\frac{tr\ A^2}{n} - \frac{1}{n^2}(tr\ A)^2}.$$

In [4, 11] it is shown that

$$\min\{var(A(e,r)) : e, r \in \Re^n\}$$

is attained for

$$e_k = \frac{1}{n-2}[r_k(A) - a_{kk} - \frac{1}{2n-2}(s(A) - tr\ A)], \quad \forall k; \quad r_k = a_{kk} - 2e_k, \quad \forall k. \tag{2.4}$$

Using these transformations for A and similarly (and independently) f and s for B we obtain the lower bound EVB1 contained in Table 5.1. We note that EVB1 requires essentially the same computational effort as EVB. The dominating part is the eigenvalue computation for A and B. Moreover, the resulting bound has considerably improved, as compared to EVB. Since e and r is chosen independently of f and s, and independently of LAP(C(e,f,r,s)), it is reasonable to expect further improvements of the bound by a more careful choice of the transformation d.

As a second strategy to choose d, an iterative improvement technique is proposed in [11] that tries to choose d so that the lower bound (2.2), as a function of d, is as large as possible. This leads to a nonlinear, nonsmooth, nonconcave maximization problem, so that to find the best possible choice for d seems at least as difficult as solving the original QAP. The improvement strategy described in [11] applied to our test data produced the bounds in column EVB2 of Table 5.1 . This bound constitutes a further improvement over EVB1 and often yields the best bound available for QAP. We point out however, that the computational effort to obtain EVB2 is incomparably higher than for all the previously described bounds. (EVB2 is obtained after up to 70 iterations where each iteration requires the eigenvalue computations and the solution of a LAP plus some overhead to carry out the iteration.) This makes EVB2 a computationally expensive candidate as a bounding rule in a Branch and Bound scheme to solve QAPs.

It is the purpose of the present paper to describe alternate ways of making the bound (2.2) work, that are computationally more tractable.

3 Projection of QAP

In this section we eliminate the constraints defining $\mathcal{E}$ by providing a tractable representation of the linear manifold spanned by $\mathcal{E}$. This is a standard way of treating equality constraints in optimization, often referred to as gradient projection or reduced gradient methods. More precisely, if the linear equality constraints which define $\mathcal{E}$ are described by the linear operator L, then we need to find the linear operator T such that $X = T(Y)$, for some $Y \in \Re^k$, if and only if $L(X) = 0$. In this case

$$\mathcal{E} = \{X : X = \hat{X} + T(Y),\ Y \in \Re^k\},$$

where $\hat{X}$ is some matrix in $\mathcal{E}$. We can then substitute for X in the objective function and eliminate the constraints that define $\mathcal{E}$. However, there is no guarantee that the permutation matrices are mapped to the orthogonals by the substitution. Moreover, the operator T must have special structure to preserve the trace structure of f. Our way of doing the projection preserves orthogonality of X as well as the important trace formulation of f. The technique is based on a simple characterization of permutation matrices given below.

Let the $n \times (n-1)-$matrix V be such that

$$V^t u = 0; \quad V^t V = I_{n-1}.$$

The columns of V therefore constitute an orthonormal basis of $\{u\}^\perp$. Further, let

$$v := \frac{u}{||u||}; \quad P = [v \,\vdots\, V] \in \mathcal{O}.$$

Thus $Q := VV^t = I - vv^t$ describes the orthogonal projection on $\{u\}^\perp$.

Lemma 3.1 *Let X be $n \times n$ and Y be $(n-1) \times (n-1)$. Suppose that X and Y satisfy*

$$X = P \begin{bmatrix} 1 & 0 \\ 0 & Y \end{bmatrix} P^t. \tag{3.1}$$

Then

$$X \in \mathcal{E},$$

$$X \in \mathcal{N} \Longleftrightarrow VYV^t \geq -vv^t,$$

$$X \in \mathcal{O}_n \Longleftrightarrow Y \in \mathcal{O}_{n-1}.$$

Conversely, if $X \in \mathcal{E}$, then there is a Y such that (3.1) holds.

Proof. Expanding (3.1) we get the equivalent relation

$$X = vv^t + VYV^t. \tag{3.2}$$

Thus $Xu = vv^tu = u$, and similarly $u^tX = u^t$. Next note that by (3.2)

$$X \geq 0 \Longleftrightarrow VYV^t \geq -vv^t.$$

Moreover

$$X \in \mathcal{O}_n \Longleftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{O}_n \Longleftrightarrow Y \in \mathcal{O}_{n-1}.$$

Finally, if $X \in \mathcal{E}$ then v is a right and left singular vector of X, corresponding to the singular value 1, because $X^tXv = v,\ v^tX^tX = v^t$. Therefore X can be written in the form (3.1). □

The representation in (3.2) illustrates the projection that we are using.The matrix $\hat{X} = vv^t \in \mathcal{E}$, while the linear operator $T(Y) = VYV^t$ yields the null space of the linear operator L which describes the manifold $\mathcal{E}$. As a consequence we have

$$\mathcal{E} = \{vv^t + VYV^t : Y \in \Re^{(n-1)\times(n-1)}\}.$$

Substitution for X in f using this representation allows us to maintain the trace structure of f as well as guarantee that each permutation matrix X corresponds to an orthogonal matrix Y. We get

$$\begin{aligned} f(X) &= tr[A(vv^t + VYV^t)B^t + C](vv^t + VY^tV^t) \\ &= tr\{Avv^tB^tvv^t + Avv^tB^tVY^tV^t + AVYV^tB^tvv^t + AVYV^tB^tVY^tV^t + Cvv^t + \\ &\quad CVY^tV^t\} \\ &= \tfrac{s(A)s(B)}{n^2} + \tfrac{s(C)}{n} + tr\{(V^tAV)Y(V^tB^tV) + V^tCV + \tfrac{2}{n}V^tr(A)r^t(B)V\}Y^t. \end{aligned}$$

Let $\hat{A} := V^tAV,\ \hat{B} := V^tBV, \hat{C} := V^tCV$ and $\hat{D} := \frac{2}{n}V^tr(A)r^t(B)V + \hat{C}$. We define the "projected" problem PQAP by minimizing

$$tr[\hat{A}Y\hat{B}^t + \hat{D})Y^t] + \frac{s(A)s(B)}{n^2} + \frac{s(C)}{n}$$

such that

$$Y \in \mathcal{O}_{n-1}, \quad VYV^t \geq -vv^t.$$

In view of the Lemma and the characterization of Π from (2.1), we have proved the following Theorem.

Theorem 3.1 *Let X and Y be related by (3.1). Then X solves QAP $\Longleftrightarrow$ Y solves PQAP.*

We note that PQAP has a structure similar to QAP, but is smaller in dimension. Moreover, there is a linear term in PQAP due to the projection, even if $C = 0$ in QAP.

We will now express the linear term in PQAP through the original variable X. This representation will be used later on. Let

$$D := \frac{2}{n} r(A) r^t(B) + C.$$

Lemma 3.2

$$tr \hat{D} Y^t = tr D X^t - \frac{2 s(A) s(B)}{n^2} - \frac{s(C)}{n}. \tag{3.3}$$

Proof.

$$\begin{aligned} tr \hat{D} Y^t &= tr[\tfrac{2}{n} r(A) r^t(B) + C] V Y^t V^t \\ &= tr D (X^t - v v^t) \\ &= tr D X^t - \tfrac{2}{n} s(A) s(B) - \tfrac{1}{n} s(C). \qquad \square \end{aligned}$$

Therefore we can formulate PQAP in the following equivalent form

$$\textbf{PQAP} \quad \min\{tr \hat{A} Y \hat{B}^t Y^t + tr D[v v^t + V Y^t V^t] - \frac{1}{n^2} s(A) s(B) : Y \in \mathcal{O}, V Y V^t \geq -v v^t\} \tag{3.4}$$

We conclude this section with the following technical note. The matrix V representing an orthogonal basis of $\{u\}^\perp$ is not uniquely determined. We point out that PQAP is unaffected by the particular choice of V because any two representations V and V_1 are related by $V = V_1 U$ for some $U \in \mathcal{O}_{n-1}$. Noting that the linear map $U Y U^t$ for U fixed is an automorphism on $\mathcal{O}$ as well as on the set $V Y V^t \geq -v v^t$ shows the equivalence of PQAP under different choices for V. A very simple representation of V is given by

$$V = \begin{pmatrix} y & y & \dots & y \\ 1+x & x & \dots & x \\ x & 1+x & \dots & x \\ \vdots & & \ddots & \vdots \\ x & x & \dots & 1+x \end{pmatrix},$$

where $x = -1/(n + \sqrt{n}), y = -1/\sqrt{n}$.

4 Bounds derived from the projected program

The program PQAP, which we have shown to be equivalent to QAP, gives rise to new lower bounds for QAP. First we point out that the elimination of the constraints describing $\mathcal{E}$ introduces a linear term in the objective function of PQAP. A simple way to bound PQAP consists in bounding the quadratic part of PQAP using Theorem 2.1 and solving the linear term independently as a LAP. We get the following new bound.

Theorem 4.1 *Let a symmetric QAP with matrices A,B and C be given. Then, in the notation above*

$$QAP(A,B,C) \geq \quad <\lambda(\hat{A}),\lambda(\hat{B})>_- \quad + \quad LAP(D) \quad - \quad s(A)s(B)/n^2.$$

Proof. Since any feasible solution Y for PQAP is orthogonal, the quadratic part of (3.4) can be bounded using Theorem 2.1, contributing the first summand. The linear term is treated independently as a LAP in the original X-space. $\square$

Since we minimize two terms independently there will in general not be a matrix X for which the bound is actually attained. In the following special case however, we are able to treat the objective function of PQAP as a whole.

Corollary 4.1 *Under the conditions of Theorem 4.1 suppose that $C = 0$ and u is an eigenvector of A. Then*
$QAP(A,B,0) \geq \quad <\lambda(\hat{A}),\lambda(\hat{B})>_- \quad + \quad s(A)s(B)/n^2.$
Moreover the lower bound is attained for $X \in \mathcal{O} \cap \mathcal{E}$, where

$$X = vv^t + VPQ^tV^t.$$

Here $P,Q \in \mathcal{O}_{n-1}$ contain the eigenvectors of $\hat{A}$ and $\hat{B}$ in the order prescribed by the minimal scalar product of the eigenvalues.

Proof. We show that in this case the linear term in PQAP is constant. We have, using $Au = \lambda u, V^t u = 0$

$$\begin{aligned} trD(vv^t + VY^tV^t) &= tr(\tfrac{2}{n}Auu^tB)(\tfrac{1}{n}uu^t + VY^tV^t) \\ &= \tfrac{2}{n^2}tru^tAuu^tBu \; + \; \tfrac{2}{n}tr\lambda V^tuu^tBVY^t \\ &= \tfrac{2}{n^2}s(A)s(B). \end{aligned}$$

Therefore the objective function of PQAP is purely quadratic and we can apply Theorem 2.1 to get the result. The minimizer is obtained using (3.2). □

So in this special case the only constraint possibly violated by X is the nonnegativity condition, due to (2.1).

We note that the minimizer X in Corollary 4.1 is not unique, even if all eigenvalues are distinct, because the unit eigenvectors could be multiplied individually by -1 without affecting the orthogonality properties.

The following special case allows a computational simplification of Theorem 4.1.

Corollary 4.2 *Under the assumptions of Theorem 4.1 assume that* $C = 0$*. Then*

$$QAP(A,B,0) \geq \; < \lambda(\hat{A}), \lambda(\hat{B}) >_- \; + \; \tfrac{2}{n} < r(A), r(B) >_- \; - \; s(A)s(B)/n^2.$$

Proof. Note that in this case the linear term simplifies to

$$trDX^t = \frac{2}{n} tr \; r(A) r^t(B) X^t.$$

Therefore the minimum over all $X \in \Pi$ is just the minimal scalar product of $r(A)$ and $r(B)$. □

In Table 5.1 the column PB contains the bound from Theorem 4.1. Comparing with the other bounds it turns out that this bound is competitive with all existing bounds except possibly EVB2. We have to note however, that the computational effort to obtain EVB2 exceeds by far the computation to get PB, which is of the same order of magnitude as the other bounds. Under the aspect of "quality over time" the new bound clearly is the favorite among all bounds presented so far.

We conclude this section with a discussion of various improvement techniques for the new bound PB that parallel those outlined in Section 2 for the eigenvalue bound (2.2). Let PB(A,B,C) denote the bound described in Theorem 4.1. We first observe that PB is invariant under constant row and column transformations given by e and f.

Lemma 4.1 $PB(A,B,C) = PB(A(e,0),B(f,0),C(e,f,0,0)) \quad \forall e, f \in \Re^n$.

Proof. First note that due to the properties of V we have

$$V^t(eu^t + ue^t)V = 0.$$

Therefore the quadratic term in the objective function of PQAP is unaffected by the transformations e and f. The remaining part of the objective function is bilinear in e and f, so its Hessian is indefinite. Since PB(A(e,0),B(f,0),C(e,f,0,0)) is bounded from above by QAP(A,B,C), this part of the objective function must also be independent of e and f. $\square$

To get further improvements of PB we therefore focus on choosing r and s appropriately. It seems tempting to select r and s so that the variance of $V^t(A + diag(r))V$ is minimized. It turns out however, that first minimizing the variance and then projecting leads to the same bound as first projecting and then minimizing the variance. The interested reader is invited to work out the details. In view of this observation we do not pursue this approach any further, because the corresponding transformation (2.4) was already investigated in [4, 11].

In general it is rather difficult to provide comparisons of different bounds. It is therefore interesting to see that there exists a transformation r, s such that PB(A(0,r), B(0,s), C(0,0,r,s)) is not worse than the bound EVB1, obtained through minimizing the spectral variance of A and B.

Theorem 4.2 *There exists a transformation r,s such that*
PB(A(0,r), B(0,s), C(0,0,r,s)) $\geq$ *EVB1.*

Proof. We use the transformation (2.4). This transformation has the property that $A(e, r)$ and $B(f, s)$ each have row sums equal to 0, see [4]. Therefore the objective function of PQAP simplifies to

$$tr\hat{A}(e,r)Y\hat{B}(f,s)Y^t + trC(e,f,r,s)X^t.$$

In the $X-$space we have

$$trA(e,r)XB(f,s)X^t + trC(e,f,r,s)X^t.$$

Since the linear terms coincide we have to show that

$$< \lambda(\hat{A}), \lambda(\hat{B}) >_- \quad \geq \quad < \lambda(A), \lambda(B) >_- .$$

(For clarity of presentation we omit indicating the dependence on e, f, r, s.) Since $Au = 0$, as mentioned above, u is eigenvector of A corresponding to the eigenvalue 0. A being symmetric there exist orthonormal eigenvectors $w_1, \ldots, w_{n-1}$ of A, orthogonal to u with corresponding eigenvalues

$\lambda_1, \ldots, \lambda_{n-1}$. Therefore we have $w_i = V s_i$ for some nonzero $s_i \in \Re^{n-1}$. We conclude that $Aw_i = \lambda_i w_i$ implies

$$\hat{A} s_i = V^t A V s_i = \lambda_i V^t V s_i = \lambda_i s_i.$$

Thus we have proved that the eigenvalues of A are those of $\hat{A}$ and an additional eigenvalue 0. A similar argument applies to B. Therefore

$$< \lambda(\hat{A}), \lambda(\hat{B}) >_- \quad \geq \quad < \lambda(A), \lambda(B) >_-$$

with equality holding if and only if 0 is a k-largest eigenvalue of A and a k-smallest eigenvalue of B for some integer k. $\square$

Finally we observe that an improvement strategy similar to the one leading to EVB2 can also be applied to get "optimal" choices for the transformations r, s. In fact, the computational scheme to get EVB2 can be applied in a straightforward manner also to PB. We do not investigate this any further because the point that we want to make consists in the observation, that PB without any additional improvement strategy already yields a highly competitive bound at low computational cost.

5 Numerical Experiments and Discussion

We conclude with some numerical experiments comparing the new bound PB with several existing bounding strategies. The first group of QAP instances in Table 5.1 is taken from [9]. These data are commonly used in the literature. The second group of instances, all of size $n = 10$, is taken from [3]. According to the authors, these problems were generated as follows: A is symmetric with entries drawn uniformly from the integers $0, 1, \ldots, 10$; matrix C also has entries drawn at random from $0, 1, \ldots, 10$, while B represents the squared Euclidean distances of 10 points drawn randomly from the integer lattice $(0, 1, \ldots, 10)^2$. The last group of instances is randomly generated in the following way: $C = 0$. B is a symmetric matrix with entries drawn uniformly from $1, \ldots, 10$. Finally, A represents the l_1-distance of a connected cellular complex of n cells in the plane. These data therefore imitate the QAP as a floor planning problem.

The meaning of the columns is as follows. First comes the size n of the problem. Then we provide the objective function value of a good feasible solution. For problems with $n \leq 15$ this solution is provably optimal. Then we include the classical bound from Gilmore and Lawler in the column headed GLB. The subsequent three columns show eigenvalue related

Size	Opt.	GLB	EVB	EVB1	EVB2	PB
12	578	493	-909	446	498	472
15	1150	963	-1745	927	1002	973
20	2570	2057	-3198	2075	2286	2196
30	6124	4539	-7836	4982	5443	5266
10	4954	3586	-12791	2774	4541	4079
10	8082	6139	-13750	6365	7617	7211
10	8649	7030	-14647	6869	8233	7837
10	8843	6840	-13546	7314	8364	8006
10	9571	7627	-15748	8095	8987	8672
10	936	878	-777	885	895	887
12	1652	1536	-1407	1562	1589	1573
14	2724	2492	-2488	2574	2630	2609
16	3720	3358	-3271	3518	3594	3560
18	5358	4776	-4422	5035	5150	5104
20	6922	6166	-5785	6533	6678	6625

Table 5.1: Lower Bounds for QAP

bounds, based on the orthogonal relaxation of QAP. Column EVB contains the bound (2.2) applied to the original data. In EVB1 we first apply the transformations (2.4) to minimize the spectral variance of A and B. EVB2 is obtained as an iterative improvement of EVB and often yields the best bound available for symmetric QAPs. This bound however is computationally extremely involved and we provide it merely to show how far things can be pushed using nonlinear optimization. Finally the last column contains the bound PB, derived in the present paper.

The computational cost to obtain these bounds is as follows. GLB, EVB, EVB1 and PB each have worst case running time $O(n^3)$, but in practice GLB can be computed considerably faster than the eigenvalue bounds each of which essentially requires a spectral decomposition of two symmetric matrices. The number of iterations to get EVB2 was limited to 70, so this bound takes about 70 times longer than the other eigenvalue bounds. To put it more colorful, if PB takes a minute, than EVB2 takes more than a hour.

Let us now compare the different bounds. As mentioned already, EVB is of no practical use. We included this bound merely to show the drastic improvement obtained by additionally constraining X to $\mathcal{E}$ leading to PB. On these examples it turns out that GLB is outperformed by the eigenvalue bounds EVB1, EVB2 and PB. As we have shown in Theorem 4.2 we can always choose a transformation so that PB is not worse than EVB1. In all the problems considered, PB applied to the original data was already better than EVB1 so this transformation is not needed.

Finally we note that PB very often is just slightly worse than EVB2. This makes PB a promising candidate in a Branch and Bound scheme, where the computational cost to obtain the lower bounds plays a crucial role. Summarizing, the bound PB constitutes an improvement in the computation of lower bounds for symmetric QAPs because it often produces a high quality bound within reasonable computation times.

There is one more technical point to be addressed. To seriously apply PB when solving QAPs makes it necessary that the bound can also be applied in intermediate nodes of the branching tree. We point out that this does not cause any complications because fixing assignments leads to a QAP of smaller size that is symmetric, if the original problem is symmetric. Forbidding assignments, i.e. forcing $x_{ij} = 0$ for some i and j does not reduce the problem in size. However setting c_{ij} to a high number in the linear term effectively reflects the forbidden assignment. Thus the bound PB can readily be applied in a Branch and Bound scheme if the branching is done by either fixing or forbidding assignments.

Appendix

We include a MATLAB routine that computes the new bound PB of a symmetric QAP for the case that the linear term C=0, see Corollary 4.2. For the problem of size $n = 30$ in Table 5.1 this routine takes less than 15 seconds on an MS-DOS PC with a 80286 processor.

```
function lbd = pb(a,b)
% pb: projection bound of a symmetric QAP given by two
%     symmetric matrices a,b; the linear term c is assumed to be 0
% input arguments: a,b: symmetric n by n matrices
%
[m,n] = size(a);        % get the size n of the matrices
%
% sort row sums of a and b
%
```

```
ra = sum(  a)'; ra =  sort( ra); % ra is ordered nondecreasingly
rb = sum( -b)'; rb = -sort( rb); % rb is ordered nonincreasingly
%
% set up the projection matrix v
%
x = -1/(n + sqrt(n)); y = -1/sqrt(n);
v(1,   1:n-1) = y * ones(1,n-1);
v(2:n, 1:n-1) = x * ones(n-1) + eye(n-1);
%
% sort eigenvalues of v' * a * v and v' * b * v
%
a1 = v'* a* v; a1 = (a1 + a1')/2;   % make sure a1 is numerically symmetric
b1 = v'* b* v; b1 = (b1 + b1')/2;   % make sure b1 is numerically symmetric
l1 = eig(  a1 ); l1 =  sort( l1);
l2 = eig( -b1 ); l2 = -sort( l2);
%
lbd = l1' * l2 + ra' * rb * 2 / n - sum( sum( a)) * sum( sum( b)) / (n*n);
```

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