Positive Definite Completions of Partial Hermitian Matrices*

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ABSTRACT

The question of which partial Hermitian matrices (some entries specified, some free) may be completed to positive definite matrices is addressed. It is shown that if the diagonal entries are specified and principal minors, composed of specified entries, are positive, then, if the undirected graph of the specified entries is chordal, a positive definite completion necessarily exists. Furthermore, if this graph is not chordal, then examples exist without positive definite completions. In case a positive definite completion exists, there is a unique matrix, in the class of all positive definite completions, whose determinant is maximal, and this matrix is the unique one whose inverse has zeros in those positions corresponding to unspecified entries in the original

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partial Hermitian matrix. Additional observations regarding positive definite completions are made.

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1. INTRODUCTION

Suppose $A = [a_{ij}]$ is an n-by-n positive semidefinite Hermitian matrix (we will use the notation $A \geqslant 0$). The well-known inequality of Hadamard is the following:

$$\det(A) \leqslant \prod_{i=1}^{n} a_{ii}.$$

It further states that if $a_{ii} > 0$ for all $i = 1,\dots,n$, then equality holds if and only if A is diagonal. This result can be restated as the solution to the following maximization problem. Suppose a_{ii}, $i = 1,\dots,n$, are fixed positive scalars, and we define

$$\mathscr{D}_0 = \{ B \mid B \geqslant 0, \operatorname{diag}(B) = (a_{11},\dots,a_{nn}) \}.$$

Then

$$\max\{ \det(B) | B \in \mathscr{D}_0 \}$$

occurs at $B \in \mathscr{D}_0$ exactly when B^{-1} (and hence B) is diagonal.

Thus rephrased, the following result of Dym and Gohberg [1] becomes an extension. Suppose $2k+1$ bands are specified in A. That is, there are fixed scalars a_{ij} for all $|i-j| \leqslant k$. Suppose in addition that the scalars satisfy the following condition:

Any principal $k+1$-by-$k+1$ submatrix contained within the $2k+1$ bands is positive definite.

Then the following theorem holds.

THEOREM 1 [1].

(i) *If* $\mathscr{D}_k = \{ B \mid B \geqslant 0,\ b_{ij} = a_{ij},$ *for all* $|i-j| < k \}$, *then* $\mathscr{D}_k \neq \varnothing$.

(ii) $\max\{\det(B) \mid B \in \mathscr{D}_k\}$ *occurs at the unique matrix* $B \in \mathscr{D}_k$ *having the property that for all* $|i-j| \geqslant k+1$, *the* (i, j)*-entry of* B^{-1} *equals* 0.

In this paper we extend this result to other sets of entries of A other than consecutive bands. In Section 2 we show that if one specifies an arbitrary subset of entries in such a way that positive definite completions exist, then the maximum determinant among the set of completions is attained at a unique matrix whose inverse has entries identically zero outside the specified subset of entries (see Theorem 2). In Section 3, we consider other kinds of constraints on the entries than specifying a fixed set of entries. In Section 5 we consider the problem of when a completion exists. That is, we characterize those sets of indices, Ω, having the property that whenever a set of scalars, $\{a_{ij} | (i, j) \in \Omega\}$, is specified in such a way that $a_{ij} = \overline{a_{ji}}$, and any principal submatrix which is included entirely in positions Ω is specified in a positive definite fashion, then a positive definite matrix B exists which satisfies

$$b_{ij} = a_{ij} \qquad \text{for all} \quad (i, j) \in \Omega.$$

The characterization (Theorem 7) is in terms of the graph corresponding to Ω.

2. PRELIMINARIES

We will denote by $G = (V, E)$ a finite undirected graph. That is, the set V of *vertices* is finite, and the set E of *edges* is a subset of the set $\{\{x, y\} : x, y \in V\}$. We allow that F may contain *loops*, i.e. that x may equal y for an edge $\{x, y\} \in E$. We assume that $V = \{1, 2, \ldots, n\}$.

Define a *G-partial matrix* as a set of complex numbers, denoted by $[a_{ij}]_G$ or $A(G)$, where a_{ij} is defined if and only if $\{i, j\} \in E$ (so a_{ij} is defined if and only if a_{ji} is).

A *completion* of $A(G) = [a_{ij}]_G$ is an $n \times n$ matrix $M = [m_{ij}]$ which satisfies $m_{ij} = a_{ij}$ for all $\{i, j\} \in E$. We say that M is a *positive completion* (a *nonnegative completion*) of $A(G)$ if and only if M is a completion of $A(G)$ and M is positive definite (positive semidefinite).

Let $\mathscr{S}$ denote the cone of $n \times n$ positive semidefinite Hermitian matrices, and let $\mathscr{S}^0$ be the interior (consisting of the positive definite matrices). Given $A(G) = [a_{ij}]_G$, we denote by $\mathscr{S}(A(G))$ [by $\mathscr{S}^0(A(G))$] the set of nonnegative completions [positive completions] of $A(G)$. That is,

$$\mathscr{S}(A(G)) = \{M \in \mathscr{S} : m_{ij} = a_{ij} \text{ for all } \{i, j\} \in E\}.$$

Let $Z = [z_{kl}]$ denote an $n \times n$ Hermitian matrix. So $z_{kl} = x_{kl} + iy_{kl}$, where x_{kl} and y_{kl} are real variables, $x_{kl} = x_{lk}$, $y_{kl} = -y_{lk}$. If we restrict Z to run over $\mathscr{S}(A(G))$, then $z_{kl} = a_{kl}$, i.e., the variables x_{kl} and y_{kl} will be fixed, for $\{k, l\}$ in E. Accordingly, we denote by

$$\nabla_{G'} \det(Z)$$

the gradient of $\det(Z)$ as a function of the real variables x_{rs} and y_{rs} for $\{r, s\}$ not in E.

For completeness, we give short proofs of two simple, possibly known lemmas.

LEMMA 1. *The function* $f(Z) = \log\det(Z)$ *is strictly concave on* $\mathscr{S}^0$.

Proof. Suppose $A, B \in \mathscr{S}^0$, and let P be a nonsingular matrix such that

$$P^*AP = D_1 \quad \text{and} \quad P^*BP = D_2$$

are both diagonal and $\det(P) = 1$. Now, for any $0 < t < 1$ we have by the strict concavity of $\log x$ that:

$$\begin{aligned}
\log\det(tA + (1-t)B) &= \log\det(P^*[tA + (1-t)B]P) \\
&= \log\det(tD_1 + (1-t)D_2) \\
&> t\log\det(D_1) + (1-t)\log\det(D_2) \\
&= t\log\det(A) + (1-t)\log\det(B).
\end{aligned}$$

■

LEMMA 2. *Suppose* $Z = [z_{rs}]$ *is* $n \times n$ *Hermitian and* $z_{rs} = x_{rs} + iy_{rs}$. *Then*

$$\frac{\partial}{\partial x_{kk}} \det(Z) = Z(k|k),$$

$$\frac{\partial}{\partial x_{kl}} \det(Z) = 2(-1)^{k+l}\operatorname{Re}(Z(k|l)),$$

$$\frac{\partial}{\partial y_{kl}} \det(Z) = -2(-1)^{k+l}\operatorname{Im}(Z(k|l))$$

for $k \neq l$, *where* Re *and* Im *denote real and imaginary parts respectively and* $Z(k|l)$ *is the* (k, l)*-minor.*

Proof. We just sketch the proof of the last formula:

$$\frac{\partial}{\partial y_{kl}}\det(Z)=\frac{\partial}{\partial z_{kl}}\det(Z)\cdot\frac{\partial z_{kl}}{\partial y_{kl}}+\frac{\partial}{\partial z_{lk}}\det(Z)\cdot\frac{\partial z_{lk}}{\partial y_{kl}}$$

$$=(-1)^{k+l}(Z(k\,|\,l)\cdot i+(-1)^{k+l}Z(l\,|\,k)\cdot(-i)$$

$$=i(-1)^{k+l}(Z(k\,|\,l)-\overline{Z(l\,|\,k)})$$

$$=-2(-1)^{k+l}\mathrm{Im}(Z(k\,|\,l)).$$ ∎

3. GENERAL REMARKS

Let $g: U\to R$ be a function defined on an open subset U of R^p and let K be such that

(1) K is a nonempty nonsingleton, convex, compact subset of U;
(2) $g(x)\geqslant 0$, for x in K, and g is *strictly log-concave* on K, that is, $\log g(x)$ is a strictly concave function on K (we put $\log 0=-\infty$).

Then it is easy to see that some well-known properties of strictly concave functions can be extended to strictly log-concave functions, namely:

(i) There exists a unique $b\in K$ such that

$$g(b)=\max\{g(x): x\in K\}.$$

(ii) If g is differentiable, this maximizing point b is the unique element of K satisfying $g(b)>0$ and

$$\langle \nabla g(b), x-b\rangle\leqslant 0 \qquad \text{for all} \quad x\in K, \tag{3.1}$$

where ∇g denotes the gradient of g and $\langle\ ,\ \rangle$ is the standard inner product in R^p.

The condition (3.1) summarizes the usual "first-order conditions" for a maximum. For example, if b is an interior point of K, (3.1) is equivalent to $\nabla g(b)=0$. If b is in the relative interior of K, then equality holds in (3.1). For another example, assume that b is a differentiable point of the boundary of K; so let $\phi(x)=0$ be the equation describing the part of the boundary of K, $\phi(b)=0$, and $\nabla\phi(b)\neq 0$. Then, the condition (3.1) implies the orthogonality condition:

$$\nabla g(b) = \lambda \nabla \phi(b) \tag{3.2}$$

for some Lagrange multiplier λ.

The above comments were designed to justify the uniqueness claims of the theorems of the next section. There, K will be a convex compact subset of the positive semidefinite matrices, and $g(x)$ will be the determinant function. According to Lemma 2, in our case (3.1) can be given the form

$$\sum_{k,l=1}^{n} (-1)^{k+l}\big[\mathrm{Re}(B(k\,|\,l))\cdot[x_{kl} - \mathrm{Re}(b_{kl})]$$
$$- \mathrm{Im}(B(k\,|\,l))\cdot[y_{kl} - \mathrm{Im}(b_{kl})]\big] \leqslant 0$$

for all $Z = [x_{kl} + iy_{kl}]$ in K, where B is a candidate for a maximizing matrix for the determinant. For $\det(B) > 0$ and $K \subset \mathscr{S}(A(G))$ this can be written as

$$\sum_{\{k,l\}\in E} \mathrm{Re}(c_{kl})\cdot[x_{kl} - \mathrm{Re}(b_{kl})] - \mathrm{Im}(c_{kl})\cdot[y_{kl} - \mathrm{Im}(b_{kl})] \leqslant 0 \qquad \text{for all} \quad Z \in K, \tag{3.3}$$

where $C = [c_{kl}]$ is the inverse of B.

4. MAXIMIZING THE DETERMINANT

We state and prove an extension of Theorem 1(ii).

THEOREM 2. *Assume G contains all loops, and let $A(G)$ be a G-partial matrix having a positive completion. Then there exists a unique positive completion of $A(G)$, say B, such that*

$$\det(B) = \max\{\det(Z) : Z \in \mathscr{S}(A(G))\}.$$

Furthermore, B is the unique positive completion of $A(G)$ whose inverse, $C = [c_{kl}]$, satisfies

$$c_{kl} = 0 \qquad \textit{for all} \quad \{k,l\} \notin E. \tag{4.1}$$

Proof. Let $K = \mathscr{S}(A(G))$. Then K is convex and, as E includes all loops, K is compact. The first part of the theorem follows from comment (i) in the previous section. As B is in the relative interior of K, then b is completely characterized by

$$\det(B) > 0 \quad \text{and} \quad \nabla_{G'} \det(B) = 0.$$

Therefore, $B(k \mid l) = 0$ for $\{k, l\} \notin E$. So the second part of the theorem follows. ■

Note that Theorem 1(ii) is an immediate corollary of Theorem 2 by letting $G = (V, E)$, $E = \{\{i, j\} : |i - j| \leqslant k\}$.

In the next theorem, for each $\{r, s\} \in E$, we let L_{rs} be a given straight line in the complex plane, and L_{sr} is the line conjugate to L_{rs}.

THEOREM 3. *Let G contain all loops, and assume there exists a positive completion Z of $A(G)$, such that $z_{rs} \in L_{rs}$ for all $\{r, s\} \in E$. Then, there exists a unique B in $\mathscr{S}(A(G))$ such that*

$$\det(B) = \max\{\det(Z) : Z \in \mathscr{S}(A(G)), z_{rs} \in L_{rs}\}.$$

This matrix B is the unique positive completion of $A(G)$ whose inverse, $C = [c_{ij}]$, satisfies

$$c_{rs} \text{ is orthogonal to } L_{rs} \qquad \text{for all} \quad \{r, s\} \notin E. \tag{4.2}$$

Here, orthogonality is with respect to the standard inner product on C identified with R^2.

Proof. Define $K = \{Z \in \mathscr{S}(A(G)) : z_{rs} \in L_{rs}, \text{ all } \{r, s\} \notin E\}$. K is obviously convex and compact. So the comments of Section 3 apply, because $\det(Z)$ is strictly log-concave. Therefore, we only need to prove that (4.2) is equivalent to (3.3).

It is easily seen that in this case (3.3) is equivalent to the system of inequalities.

$$\operatorname{Re}(c_{rs}) \cdot [x_{rs} - \operatorname{Re}(b_{rs})] - \operatorname{Im}(c_{rs}) \cdot [y_{rs} - \operatorname{Im}(b_{rs})] \leqslant 0 \tag{4.3}$$

for all $\{r, s\} \in E$ [to see this, put $z_{kl} = b_{kl}$ in (3.3) for all $(k, l) \neq (r, s)$]. The inequality (4.3) holds for any $x_{rs} + iy_{rs}$ on an open segment of L_{rs} containing b_{rs}. Therefore (4.3) and (4.2) are equivalent. ■

If, for some k, the loop $\{k\}$ is not in E, then the set $\mathscr{S}(A(G))$ is not compact (whenever it is nonempty). To ensure that the set of matrices we are dealing with is compact, we will consider in the following two theorems the further constraints

$$\operatorname{tr}(Z) \leqslant T, \qquad \operatorname{tr}(Z^2) \leqslant L,$$

where T and L are given positive numbers.

THEOREM 4. *Suppose* $A(G)$ *has a positive completion* Z *subject to* $\operatorname{tr}(Z) \leqslant T$. *Then there is a unique such completion,* B, *with maximum determinant. If there exists* $\{k\} \notin E$, *we have* $\operatorname{tr}(B) = T$. *Furthermore,* B *is the unique matrix of* $\mathscr{S}^0(A(G))$ *such that* $\operatorname{tr}(B) \leqslant T$ *and whose inverse,* $C = [c_{ij}]$, *satisfies*

$$c_{kl} = 0 \qquad \text{for all} \quad \{k, l\} \notin E, \quad k \neq l, \tag{4.4}$$

$$c_{ii} = c_{jj} \qquad \text{for all} \quad \{i\}, \{j\} \notin E. \tag{4.5}$$

Proof. Let $K = \{Z \in \mathscr{S}(A(G)) : \operatorname{tr}(Z) \leqslant T\}$. This K is a nonempty, compact, convex set. By the comments of Section 3, the theorem will be proved if we prove that (4.4)–(4.5) are equivalent to (3.3). If G has all loops, then (4.5) is vacuous and the theorem is the same as Theorem 2.

Now assume there is $\{k\} \notin E$. Of course $\operatorname{tr}(B) = T$, because $\det(Z)$ is strictly increasing with respect to each free diagonal element. So we have a maximum at a differentiable point of the relative boundary of K. Using (3.2) (where we take into account just the "free" real parameters of Z), the first-order conditions are exactly

$$\nabla_{G'} \det(B) = \lambda \nabla_{G'} [\operatorname{tr}(Z) - T], \qquad \text{with} \quad \lambda \geqslant 0.$$

So, by Lemma 2, this is equivalent to

$$B(k \mid l) = 0 \qquad \text{for all} \quad \{k, l\} \notin E, \quad k \neq l,$$

$$B(k \mid k) = \lambda \qquad \text{for all} \quad \{k\} \notin E.$$

This is the same as (4.4)–(4.5), for $\det(B) > 0$. ∎

In this last result of this section, we characterize the maximizing matrix for the determinant over the set

$$K = \{Z \in \mathscr{S}(A(G)) : \operatorname{tr}(Z^2) \leqslant L\}.$$

THEOREM 5. *Suppose there is a positive* Z *in the above set* K.

(i) *If* G *has all loops and the matrix* B *satisfying* (4.1) *also satisfies* $\operatorname{tr}(B^2) \leqslant L$, *then this matrix has the maximum determinant on* K.

(ii) *Otherwise, the maximum occurs at* B *on* $\mathscr{S}^0(A(G))$ *uniquely determined by*

$$\operatorname{tr}(B^2) = L$$

and

$$c_{kl} = \gamma b_{kl} \qquad \textit{for all} \quad \{k, l\} \notin E, \tag{4.6}$$

where $C = [c_{ij}]$ *is the inverse of* B, *and* γ *is a positive constant.*

Proof. K is compact and convex. Therefore the proof may follow the same pattern as the others. We just sketch it.

Note that case (i) occurs if and only if at the maximizing matrix B we have $\nabla_{G'} \det(B) = 0$.

(i) is obvious. We prove (ii). As $\nabla_{G'} \det(B) \neq 0$, the maximum must occur on the relative boundary of K, i.e., we have $\operatorname{tr}(B^2) = L$. Therefore, as in (3.2), the first-order conditions for a maximum are equivalent to

$$\nabla_{G'} \det(B) = \lambda \nabla_{G'} \left[\operatorname{tr}(B^2) - L\right] \tag{4.7}$$

for some positive λ. Equation (4.7) may be written coordinatewise as

$$2\operatorname{Re}(B(k \mid l) = \lambda \cdot 4\operatorname{Re}(b_{kl}),$$
$$2\operatorname{Im}(B(k \mid l)) = \lambda \cdot 4\operatorname{Im}(b_{kl}),$$
$$B(i \mid i) = \lambda \cdot 2b_{ii}$$

for all $\{k, l\} \notin E$, $k \neq l$, all $\{i\} \notin E$. This is equivalent to (4.6), with $\gamma = 2\lambda / \det(B)$. ∎

5. COMPLETABLE GRAPHS

The previous notation and conventions on a graph $G = (V, E)$ remain here. If $e = \{x, y\}$ is not an edge of G, we will let $G + e$ denote the graph obtained by replacing E with $E \cup \{e\}$. A *clique* is a subset $C \subset V$ having the property that $\{x, y\} \in E$ for all $x, y \in C$. A *cycle* in G is a sequence of pairwise distinct vertices $\gamma = (v_1, \dots, v_s)$ having the property that

$\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{s-1}, v_s\}, \{v_s, v_1\} \in E$, and s is referred to as the *length* of the cycle. A *chord* of the cycle γ is an edge $\{v_i, v_j\} \in E$ where $1 \leqslant i < j \leqslant s$, $\{i, j\} \neq \{1, s\}$, and $|i - j| \geqslant 2$. The cycle γ is *minimal* if any other cycle in G has a vertex not in γ, or equivalently, γ has no chord. An *ordering* of G is a bijection $\alpha: V \to \{1,\ldots,n\}$, and y is said to *follow* x (with respect to the ordering α) if $\alpha(x) < \alpha(y)$. G is called a *band graph* if there exists an ordering α of G and an integer m, $2 \leqslant m \leqslant n$, such that

$$\{x, y\} \in E \quad \text{if and only if} \quad |\alpha(x) - \alpha(y)| \leqslant m - 1.$$

Assume $V = \{1,\ldots,n\}$, and let $A(G) = [a_{ij}]_G$ be a G-partial matrix. We say that $A(G)$ is a *G-partial positive* (*G-partial nonnegative*) *matrix* if

$$a_{ji} = \overline{a_{ij}} \qquad \text{for all} \quad \{i, j\} \in E$$

and for any clique C of G, this principal submatrix $[a_{ij}: i, j \in C]$ of $A(G)$ is positive definite (positive semidefinite).

Suppose $G = (V, E)$ is a subgraph of $J = (V, \bar{E})$ (that is, $E \subset \bar{E}$). A J-partial matrix $[b_{ij}]_J$ is said to *extend* a G-partial matrix $[a_{ij}]_G$ if $b_{ij} = a_{ij}$ for all $\{i, j\} \in E$. So a completion of $A(G)$ is an extension of $A(G)$ associated with the complete graph on V.

We say that the graph G is *completable* (*nonnegative-completable*) if and only if any G-partial positive (G-partial nonnegative) matrix has a positive (nonnegative) completion.

We conclude this section by showing that the terms "completable" and "nonnegative-completable" coincide.

Define L to be the set of vertices of G which have loops. That is: $x \in L$ if and only if $\{x, x\} \in E$. We assume without loss of generality that $L = \varnothing$ or $L = \{1,\ldots,k\}$ for some $1 \leqslant k \leqslant n$.

PROPOSITION 1. *G is completable (nonnegative-completable) if and only if the graph on L induced by G is completable (nonnegative-completable.)*

Proof (completable case). Let G' denote the graph induced on L. If G is completable, then G' is completable, since any clique must be contained in L. For the converse, it is sufficient to remark that if

$$A(x) = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & xI + H \end{bmatrix},$$

where A_{11} is positive definite k-by-k and H is $n-k$-by-$n-k$ Hermitian, then $A(x)$ is positive definite for sufficiently large positive x. ■

In view of Proposition 1, we will henceforth assume that $L=\{1,\dots,n\}$.

PROPOSITION 2. *G is completable if and only if G is nonnegative-completable.*

Proof. Assume that G is completable and that $A=[a_{ij}]_G$ is a G-partial nonnegative matrix. Let $A_k=A+(1/k)I$, so that A_k is a G-partial positive matrix for each $k=1,2,\dots$. Let M_k be a positive completion of A_k. Since $L=V$, the sequence $\{M_k\}$ is bounded, and so has a convergent subsequence. The limit of this subsequence will then be a nonnegative completion of A.

Assume that G is nonnegative completable and that $A=[a_{ij}]_G$ is a G-partial positive matrix. Choose $\varepsilon>0$ such that $A-\varepsilon I$ is still a G-partial positive matrix. Let M be a nonnegative completion of $A-\varepsilon I$. Then $M+\varepsilon I$ is a positive completion of A. ■

In view of Proposition 2, we will henceforth only use the term "completable."

6. CHORDAL AND COMPLETABLE GRAPHS

In this section we characterize the set of completable graphs. A graph G is *chordal* if there are no minimal cycles of length $\geqslant 4$. An alternative characterization is that every cycle of length $\geqslant 4$ has a chord.

Chordal graphs have appeared in the literature in relation to the problem of perfect elimination orderings. An ordering α of G is a *perfect elimination ordering* if for every $v\in V$, the set

$$\{x\in V\mid\{v,x\}\in E,\ \alpha(v)<\alpha(x)\}$$

is a clique of G. This concept arises in connection with the problem of reordering the rows and columns of a (possibly sparse) matrix in such a way that no nonzero elements are created in the course of Gauss elimination. We quote from [3] the following important result.

THEOREM 6 [3]. *G has a perfect elimination ordering if and only if G is chordal.*

Notice that we may restate Theorem 1(i) as follows:

$$\textit{If } G \textit{ is a band graph, then } G \textit{ is completable.} \tag{6.1}$$

The main result in this section is the following.

THEOREM 7. *G is completable if and only if G is chordal.*

Theorem 7 is an extension of (6.1), since band graphs are chordal. We proceed with a sequence of lemmas preliminary to the proof of Theorem 7.

LEMMA 3. *G has no minimal cycles of length exactly* 4 *if and only if the following holds.*

For any pair of vertices u and v with $u \neq v$, $\{u, v\} \notin E$, the graph $G + \{u, v\}$ has a unique maximal clique which contains both u and v. (That is: if C and C' are both cliques in $G + \{u, v\}$ which contain u and v, then so is $C \cup C'$.) (6.2)

Proof of Lemma 3. Assume G has no minimal cycle of length 4, and that u, v, C, C' are as described in (6.2). We will show that $C \cup C'$ is a clique in $G + \{u, v\}$. Let $z \in C$, $z' \in C'$; we show that $\{z, z'\}$ is an edge of $G + \{u, v\}$. If $z = z'$ or $\{z, z'\} \cap \{u, v\} \neq \varnothing$ this is trivial, so we may assume u, z, v, z' are four distinct elements. Observe that (u, z, v, z') is a cycle of $G + \{u, v\}$, and so of G. Since G has no minimal cycles of length 4, either $\{u, v\}$ or $\{z, z'\}$ must be an edge of G. Since $\{u, v\}$ is not an edge of G by assumption, we have that $\{z, z'\}$ is an edge of G. But then, $\{z, z'\}$ is an edge of $G + \{u, v\}$, and $C \cup C'$ is a clique of $G + \{u, v\}$.

For the converse, assume that $\gamma = (x, u, y, v)$ is a minimal cycle of G, so that $\{x, y\}$ and $\{u, v\}$ are not edges of G. Then $C = \{x, u, v\}$ and $C' = \{y, u, v\}$ are cliques of $G + \{u, v\}$ which contain u and v, whereas $C \cup C'$ is not a clique of $G + \{u, v\}$, since $\{x, y\}$ is not an edge of G or $G + \{u, v\}$. Hence (6.2) fails. ■

We remark that a chordal graph must satisfy (6.2).

LEMMA 4. *Let $G = (V, E)$ be chordal. Then there exists a sequence of chordal graphs*

$$G_i = (V, E_i), \qquad i = 0, 1, \ldots, s,$$

such that $G_0 = G$, G_s is the complete graph, and G_i is obtained by adding an edge to G_{i-1} for all $i = 1, \ldots, s$.

Proof of Lemma 4. The lemma follows easily from Lemma 6 of [2] and induction. ∎

LEMMA 5. *Let $G' = (V', E')$ be a subgraph of $G = (V, E)$ induced by a subset $V' \subset V$. If G is completable, then so is G'.*

Proof of Lemma 5. Let $A' = [a'_{ij}]_{G'}$ be any G'-partial nonnegative matrix. Define a G-partial nonnegative matrix $A = [a_{ij}]_G$ via

$$a_{ij} = \begin{cases} a'_{ij} & \text{if } \{i, j\} \in E', \\ 0 & \text{if } \{i, j\} \in E \setminus E'. \end{cases}$$

Since G is completable, there is (by Proposition 1) a nonnegative completion M of A. The principal submatrix M' of M corresponding to rows and columns indexed by V' is then a nonnegative completion of A'. ∎

LEMMA 6. *There is a unique k-by-k positive semidefinite matrix $A = [a_{ij}]$ which satisfies*

$$a_{ij} = 1, \qquad \textit{all} \quad |i - j| \leqslant 1. \tag{6.3}$$

Namely, A is the matrix of all 1's.

Proof of Lemma 6. We need only show the uniqueness. For $k = 2$ there is nothing to show, and for $k = 3$ the validity is easy to check. Assume then that $k \geqslant 4$, and that k-by-k A is positive semidefinite satisfying (6.3). Let $1 \leqslant i < j \leqslant k$, $|i - j| \geqslant 2$. The 3-by-3 principal submatrix of A corresponding to rows and columns $i, i+1, j$ is positive semidefinite and satisfies (6.3). Hence $a_{ij} = 1$, since Lemma 7 holds when $k = 3$. ∎

Proof of Theorem 7 (only if). Let G be a completable graph. To obtain a contradiction, assume that G has a minimal cycle γ of length $\geqslant 4$. We may assume without loss of generality that $\gamma = (1, 2, \ldots, k)$. Let G' be the graph on

$\{1,\ldots,k\}$ induced by G. Consider the k-by-k G'-partial matrix defined by

$$a'_{ij}=1 \qquad \text{if} \quad |i-j|\leqslant 1,$$

$$a'_{1k}=-1\left(=a'_{k1}\right).$$

It is easy to see that A' is a G'-partial nonnegative matrix (one need only check principal minors of order 2). By Lemma 6, A' is not completable to a positive semidefinite matrix, and so G' is not completable. However, G' *is* completable by Lemma 5, and we have a contradiction. ■

Proof of Theorem 7 (if). Assume that G is chordal and not the complete graph on $\{1,\ldots,n\}$. Let

$$G=G_0,G_1,\ldots,G_s$$

be a sequence of chordal graphs satisfying the conditions of Lemma 4. Let A be any G-partial positive matrix. If we can show that there exists a G_1-partial positive matrix A_1 which extends A, then the existence of a positive completion of A will follow by induction.

Denote by $\{u,v\}$ the edge of G_1 that is not an edge of G. By Lemma 3 there is a unique maximal clique C of G_1 which contains both u and v. We may assume without loss of generality that $C=\{1,\ldots,p\}$ and that $u=1$, $v=p$. For any complex z, let $A_1(z)$ denote the G_1-partial matrix extending A which has z in position $(1,p)$ and $\bar{z}$ in position $(p,1)$. Let $M(z)$ be the leading principal p-by-p submatrix of $A_1(z)$. Since any clique containing $\{1,p\}$ is a subset of $\{1,\ldots,p\}$, $A_1(z)$ is a G_1-partial positive matrix iff $M(z)$ is a positive matrix. Hence we need only show that $M(z_0)$ is positive definite for some z_0.

The graph on C induced by G has exactly the set of edges

$$\{\{i,j\}:|i-j|\leqslant p-2,\ 1\leqslant i\leqslant j\leqslant p\},$$

and so is a band graph. Therefore by (6.1) there exists z_0 as required. ■

7. THE COMPLETION PROCESS

Let $G=(V,E)$ be a chordal graph, and let $A=[a_{ij}]_G$ be a G-partial positive (or nonnegative) matrix. By an *empty position* we mean an edge $\{i,j\}$ not in E. In the proof of Theorem 7 it was seen that A can be completed to a positive definite (or semidefinite) matrix by filling up the empty positions of A sequentially, according to the sequence of chordal graphs described in Lemma 4. Recall that this results in a sequence $A = A_0, A_1, \dots, A_s$ where A_i is a G_i-partial positive (or nonnegative) matrix, and A_s is the desired completion.

In this section we consider the problem of constructing a sequence $G = G_0, G_1, \dots, G_s$ of chordal graphs which satisfies the conditions of Lemma 4. The answer to this question is obtainable by using the very efficient method in [2] for finding perfect elimination orderings of G.

Assume that $\alpha: V \to \{1, \dots, n\}$ is a (fixed) perfect elimination ordering of G. We will let v_k be the vertex $\alpha^{-1}(k)$, $k = 1, \dots, n$. Let s be the number of empty positions of A as in Lemma 4. (We remark that A has $2s$ undefined entries, but only s empty positions, since we assume A is Hermitian.) Define inductively the following sequence of graphs, $G_0 = (V, E_0), \dots,\ G_s = (V, E_s)$:

$$G_0 = G, \tag{7.1}$$

and if

$$k_i = \max\{ k \,|\, \{ v_k, v_m \} \notin E_i \text{ for some } m \}, \tag{7.2}$$

$$r_i = \max\{ r \,|\, \{ v_r, v_{k_i} \} \notin E_i \}; \tag{7.3}$$

we then let

$$G_{i+1} = G_i + \{ v_{k_i}, v_{r_i} \}, \qquad i = 0, \dots, s. \tag{7.4}$$

Notice that

$$r_i < k_i, \qquad i = 0, \dots, s-1,$$

and

$$k_0 \leqslant k_1 \leqslant \cdots \leqslant k_{s-1}.$$

In light of Theorem 6, our next result ensures that the sequence $G_0, G_1, \dots, G_s$ satisfies the conditions of Lemma 4.

THEOREM 8. *If α is a perfect elimination ordering for $G = G_0$, then α is a perfect elimination ordering for all the graphs $G_1, \dots, G_s$ defined inductively by* (7.1)–(7.4). (*Hence G_i is chordal for $i = 1, \dots, s$.*)

Proof of Theorem 8. We have to prove that for any vertex v_k, the set

$$S_i(v_k) = \{ v_m \mid \{ v_m, v_k \} \in G_i \text{ and } m > k \} \tag{7.5}$$

is a clique of G_i for $i = 0, \ldots, s-1$. By induction on i, assume $S_i(v_k)$ is a clique of G_i. If $v_k \neq v_{r_i}$, then

$$S_{i+1}(v_k) = S_i(v_k),$$

and so $S_{i+1}(v_k)$ is a clique of G_i and G_{i+1}. If on the other hand $v_k = v_{r_i}$, then

$$S_{i+1}(v_{r_i}) = S_i(v_{r_i}) \cup \{ v_{k_i} \}.$$

Now, from (7.2)–(7.4) v_{k_i} is linked in G_i to every vertex m with $m \geqslant r_i$, and so v_{k_i} is linked in G_i to every vertex of $S_i(v_{r_i})$. Hence $S_{i+1}(v_k) = S_{i+1}(v_{r_i}) = S_i(v_{r_i}) \cup \{ v_{k_i} \}$ is a clique. ∎

REFERENCES

1 H. Dym and I. Gohberg, Extensions of band matrices with band inverses, *Linear Algebra Appl.* 36:1–24 (1981).
2 G. S. Leuker, D. J. Rose, and R. E. Tarjan, Algorithmic aspects of vertex eliminations on graphs, *SIAM J. Comput.* 5(2):266–283 (1976).
3 D. J. Rose, Triangulated graphs and the elimination process, *J. Math. Anal. Appl.* 32:597–609 (1970).
4 D. J. Rose and R. E. Tarjan, Algorithmic aspects of vertex eliminations on directed graphs, *SIAM J. Appl. Math.* 34:176–197 (1978).

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