

Robust Interior Point Methods and FR for Key Rate Computation in Quantum Key Distribution

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Motivation/Outline

- find reliable, efficient numerical method for calculating **key rates** for quantum key distribution (QKD) protocols
- Currently: **ill-posed models**; i.e., we want to (minimize)

find **reliable provable lower bound**

for the convex relative entropy: $\text{trace } \rho \log \rho - \rho \log \sigma$

$\sigma, \rho \succeq 0$ (positive semidefinite **matrices**), even though **singular** (opt. currently on boundary of SDP cone)

regulariz. using FACIAL REDUCTION, FR;
on both constraints and nonlinear objective

- (I) theoretically proven upper and **lower** bounds with possible approximate FR; **high precision**
- (II) (Gauss-Newton) interior point approach on regularized problem; (originally **singularity degree ONE > 0**)
- avoid current perturbation approach to get $\rho \succ 0$ (pos. def.)

QKD Background (Details in References)

- Quantum key distribution, QKD: the art of distributing secret keys between two honest parties, traditionally known as Alice and Bob;
- secret key rate (number of bits of secret key obtained per exchange of quantum signal) calculation is at the core of a security proof for any QKD protocol;
- calculation is a convex minimization (lower bound) problem, s.t. constraints to detect presence of any third party (Eve eavesdropping); fundamentally: security comes from the Heisenberg uncertainty principle as eavesdropping means detectable disturbances so Alice and Bob can detect presence of Eve;
- even with a quantum computer, a secret key generated by QKD remains secure.

(asymptotic) Key Rate Calculation

Winick, Lütkenhaus, Coles [9]

$$\begin{aligned} p^* = \quad & \min_{\rho} \quad D(\mathcal{G}(\rho) \| \mathcal{Z}(\mathcal{G}(\rho))) \\ \text{s.t.} \quad & \Gamma(\rho) = \gamma, \quad (\text{trace } \rho = 1) \\ & \rho \succeq 0 \quad (\text{density matrices}) \end{aligned}$$

Here:

- $D(\delta \| \sigma) = f(\delta, \sigma) = \text{trace } \delta [\log \delta - \log \sigma]$ is the **quantum relative entropy**;
- $\Gamma : \mathbb{H}^n \rightarrow \mathbb{R}^m$ lin. transf., $\Gamma(\rho) = (\text{trace } \Gamma_i \rho) = (\langle \Gamma_i, \rho \rangle)$;
- $\mathbb{H}^n$ linear space Hermitian matrices over $\mathbb{R}$; $\gamma \in \mathbb{R}^m$
- $\mathcal{G}$ and $\mathcal{Z}$ are **linear, completely positive maps, CP**
(here, sums of products $Z_i \rho Z_i^*$)

CP $\mathcal{G}, \mathcal{Z}$; e.g., $\mathcal{G} : \mathbb{H}^n \rightarrow \mathbb{H}^k, k > n, \mathcal{G}(\mathbb{H}_+^n) \subseteq \mathbb{H}_+^k$

e.g. $\mathcal{G}(\rho) = \sum_{j=1}^t K_j \rho K_j^*, \quad \sum_{j=1}^t K_j^* K_j \preceq I.$

Definition ($\mathcal{G} : \mathbb{H}^n \rightarrow \mathbb{H}^k$ (Kraus repres.))

$$\mathcal{G}(\rho) := \sum_{j=1}^{\ell} K_j \rho K_j^*,$$

$K_j \in \mathbb{C}^{k \times n}$, $\sum_{j=1}^{\ell} K_j^* K_j \preceq I$; adjoint $\mathcal{G}^*(\delta) := \sum_{j=1}^{\ell} K_j^* \delta K_j$;

Generally $k = i * n > n, i = 2, 3, \dots$;

and so typically $\mathcal{G}(\rho)$ **rank deficient** $\forall \rho \succ 0$ (positive definite)

Definition (self-adjoint (projection) $\mathcal{Z} : \mathbb{H}^k \rightarrow \mathbb{H}^k$)

$$\mathcal{Z}(\delta) := \sum_{j=1}^N Z_j \delta Z_j,$$

$Z_j = Z_j^2 = Z_j^* \in \mathbb{H}_+^k$ and $\sum_{j=1}^N Z_j = I_k$

Properties of QKD Problem

Lemma

The linear map $\mathcal{Z}$ is an orthogonal projection on $\mathbb{H}^k$ and $\text{trace}(\delta) \leq 1, \delta \succeq 0$ implies:

$$\text{trace}(\delta \log \mathcal{Z}(\delta)) = \text{trace}(\mathcal{Z}(\delta) \log \mathcal{Z}(\delta))$$

$\mathcal{G}, \mathcal{Z} \circ \mathcal{G}$ May not Preserve Positive Definiteness

$$\begin{aligned} p^* = \quad & \min_{\rho} \quad D(\mathcal{G}(\rho) \| \mathcal{Z}(\mathcal{G}(\rho))) \\ \text{s.t.} \quad & \Gamma(\rho) = \gamma, \\ & \rho \succeq 0 \quad (\text{density matrices}) \end{aligned}$$

Known Properties

$$\min_{\rho, \sigma, \delta} \text{trace}(\delta(\log \delta)) - \text{trace}(\sigma(\log \sigma))$$

quantum relative entropy D is finite under range condition;
jointly convex in both δ and σ

Ready for FR, Facial Reduction (Regularization)

$$\begin{aligned} p^* = & \min_{\rho, \sigma, \delta} \quad \text{trace}(\delta(\log \delta)) - \text{trace}(\sigma(\log \sigma)) \\ \text{s.t.} \quad & \Gamma(\rho) = \gamma \\ & \sigma = \mathcal{Z}(\delta) \\ & \delta = \mathcal{G}(\rho) \\ & \rho, \sigma, \delta \succeq 0. \end{aligned}$$

Our Goal: Final Asymptotic Key Rate

obtained by getting a **reliable lower bound** of this problem (and then removing the cost of error correction, a constant).

Slater Constraint Qualification, Strict Feasibility, Stability

Slater: $\exists \hat{p} \succ 0 : \Gamma \hat{p} = \gamma$

Strong Duality, Stability

Slater is sufficient for strong duality;
equivalent to numerical stability under RHS perturbations;
Slater fails in surprisingly many applications

Advantages of FR

FR can be used to obtain strict feasibility and regularize the problem, and often simultaneously simplify the problem.

current applications: e.g.

hard discrete opt.; (distance geometry); EDM and low rank matrix completion etc ... (recent survey Drusvyatskiy-W. [5])

Facial Reduction, FR, Preliminaries cont...

convex cone: $K : \lambda K \subseteq K, \forall \lambda \geq 0, K + K \subseteq K,$

dual cone: $S^* = \{\phi \in \mathbb{H} : \langle \phi, s \rangle \geq 0, \forall s \in S\}.$

convex cone F is a face of a convex cone $K, F \trianglelefteq K$, if

$$x, y \in K, x + y \in F \implies x, y \in F.$$

Faces of the positive semidefinite cone are characterized by the range or nullspace of any element in the relative interior:

Lemma

Let F a convex subset of $\mathbb{H}_+^n$ with $X \in \text{ri } F$ with orthogonal spectral decomposition $X = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & Q \end{bmatrix}^*$, with

$D \in \mathbb{H}_{++}^r$. Then TFAE:

(i) $F \trianglelefteq \mathbb{H}_+^n$;

(ii) $F = \{Y \in \mathbb{H}_+^n : \text{range}(Y) \subset \text{range}(X)\}$
 $= \{Y \in \mathbb{H}_+^n : \text{null}(Y) \supset \text{null}(X)\};$

(iii) $F = P\mathbb{H}_+^r P^*$; (iv) $F = \mathbb{H}_+^n \cap (QQ^*)^\perp$ (exposing QQ^*)

Facial Reduction via Theorem of Alternative

Lemma (theorem of the alternative)

For the feasible constraint system, exactly one of the following statements holds:

- 1 *there exists $\rho \succ 0$ such that $\Gamma(\rho) = \gamma$ (Slater);*
- 2 *there exists y (and exposing vector Z) such that*

$$0 \neq Z = \Gamma^*(y) \succeq 0, \quad \langle \gamma, y \rangle = 0.$$

The matrix $Z = \Gamma^*y$ above is an **exposing vector** for the feasible set.

Definition (minimal face)

K a closed convex cone; $S \subseteq K$ a convex set; then $\text{face}(S) \trianglelefteq K$ is the *minimal face*, the intersection of all faces of K that contain S .

First (Partial) FR Step on S_R (Analytical/Accurate)

affine manifold constraint is divided into two sets

observable, reduced density operator constraint sets, $S_O \cap S_R$; with Kronecker product, $\otimes$

$$S_O = \left\{ \rho \succeq 0 : \langle P_s^A \otimes P_t^B, \rho \rangle = p_{st}, \forall s, t \right\},$$

$$\begin{aligned} S_R &= \{ \rho \succeq 0 : \text{trace}_B(\rho) = \rho_A \} \quad (\text{partial trace}) \\ &= \{ \rho \succeq 0 : \langle \Theta_j \otimes I_B, \rho \rangle = \theta_j, \forall j = 1, \dots, m_R \}, \end{aligned}$$

where

data $\theta_j = \langle \Theta_j, \rho_A \rangle$; and $\rho_A \in \mathbb{H}_+^{n_A}$ often singular
 $\{\Theta_j\}$ orthonormal basis system A.

Let $\text{range } P = \text{range } \rho_A \subsetneq \mathbb{H}^{n_A}$, $P^*P = I_r$, and let $V = P \otimes I_B$.

Then **FR:** $\rho \in S_R \implies \rho = VRV^*$, for some $R \in \mathbb{H}_+^{r \cdot n_B}$

Lemma (useful equivalent form for entropy function)

Let $Y = VRV^ \in \mathbb{H}_+$, $R \succ 0$ be the compact spectral decomposition of Y with $V^*V = I$. Then*

$$\text{trace}(Y \log Y) = \text{trace}(R \log R).$$

Proof.

We obtain a unitary matrix $U = \begin{bmatrix} V & P \end{bmatrix}$ by completing the basis. Then $Y = UDU^*$, where $D = \text{BlkDiag}(R, 0)$. We conclude, with $0 \cdot \log 0 = 0$, that

$$\text{trace } Y \log Y = \text{trace } D \log D = \text{trace } R \log R.$$


Exposing Vectors Analytically; Spectral Decomposition

We use the following simple result to obtain the exposing vectors of the minimal face in the problem analytically, i.e., we find the matrices V with orthonormal columns.

Lemma (analytic FR)

Let $\mathcal{C} \subseteq \mathbb{H}_+^n$ be a given closed convex set with nonempty interior. Let $Q_i \in \mathbb{H}^{k \times n}$, $i = 1, \dots, t$, be given matrices. Define the linear map $\mathcal{A} : \mathbb{H}^n \rightarrow \mathbb{H}^k$ and matrix V by

$$\mathcal{A}(X) = \sum_i^t Q_i X Q_i^*, \quad \text{range}(V) = \text{range} \left(\sum_{i=1}^t Q_i Q_i^* \right).$$

Then the minimal face,

$$\text{face}(\mathcal{A}(\mathcal{C})) = V \mathbb{H}_+^r V^*.$$

Obtain the Minimal Faces, $V\mathbb{H}_+ V^*$

$$\begin{aligned}\rho &= V_\rho R_\rho V_\rho^* \in \mathbb{H}_+^n, & R_\rho &\in \mathbb{H}_+^{n_\rho} \\ \delta &= V_\delta R_\delta V_\delta^* \in \mathbb{H}_+^k, & R_\delta &\in \mathbb{H}_+^{k_\delta} \\ \sigma &= V_\sigma R_\sigma V_\sigma^* \in \mathbb{H}_+^k, & R_\sigma &\in \mathbb{H}_+^{k_\sigma}\end{aligned}$$

Define the linear maps

$$\begin{aligned}\Gamma_V : \mathbb{H}_+^{n_\rho} &\rightarrow \mathbb{R}^m & \text{by} & \quad \Gamma_V(R_\rho) = \Gamma(V_\rho R_\rho V_\rho^*), \\ \mathcal{G}_V : \mathbb{H}_+^{n_\rho} &\rightarrow \mathbb{H}_+^k & \text{by} & \quad \mathcal{G}_V(R_\rho) = \mathcal{G}(V_\rho R_\rho V_\rho^*), \\ \mathcal{Z}_V : \mathbb{H}_+^{k_\delta} &\rightarrow \mathbb{H}_+^k & \text{by} & \quad \mathcal{Z}_V(R_\delta) = \mathcal{Z}(V_\delta R_\delta V_\delta^*).\end{aligned}$$

Substitute with Linear Mapping $\mathcal{V}_\delta(\cdot) := V_\delta \cdot V_\delta^*$

Equivalent Formulation

$$\begin{aligned} \min \quad & \text{trace}(R_\delta \log(R_\delta)) - \text{trace}(R_\sigma \log(R_\sigma)) \\ \text{subject to: } \quad & \Gamma_V(R_\rho) = \gamma \\ & \mathcal{V}_\sigma(R_\sigma) - \mathcal{Z}_V(R_\delta) = 0 \\ & \mathcal{V}_\delta(R_\delta) - \mathcal{G}_V(R_\rho) = 0 \\ & R_\rho, R_\sigma, R_\delta \succeq 0. \end{aligned}$$

After Rotation and Substitution; final model (QKD)

$$\begin{aligned} p^* = \min \quad & f(\rho) = \text{trace}(\hat{\mathcal{G}}(\rho)(\log \hat{\mathcal{G}}(\rho))) \\ & - \text{trace}(\hat{\mathcal{Z}}(\rho) \log \hat{\mathcal{Z}}(\rho)) \\ \text{subject to: } \quad & \Gamma_V(\rho) = \gamma_V \\ & \rho \in \mathbb{H}_+^{n_\rho}, \end{aligned}$$

Slater holds; smaller regularized problem;
positive definiteness preserved in obj. fn

Derivatives

Theorem (Derivatives of regularized objective)

Let $\rho \succ 0$. The gradient of f is

$$\nabla f(\rho) = \boxed{\hat{\mathcal{G}}^*(\log[\hat{\mathcal{G}}(\rho)]) + \hat{\mathcal{G}}^*(I)} - \boxed{\hat{\mathcal{Z}}^*(\log[\hat{\mathcal{Z}}(\rho)]) + \hat{\mathcal{Z}}^*(I)}.$$

The Hessian in direction $\Delta\rho$ is (1st order info)

$$\begin{aligned} \nabla^2 f(\rho)(\Delta\rho) = & \boxed{\hat{\mathcal{G}}^*(\log'[\hat{\mathcal{G}}(\rho)])(\hat{\mathcal{G}}(\Delta\rho))} - \\ & \boxed{\hat{\mathcal{Z}}^*(\log'[\hat{\mathcal{Z}}(\rho)])(\hat{\mathcal{Z}}(\Delta\rho))} \end{aligned}$$

Theorem (subdifferential)

Let $\{\rho_i\}_i \subseteq \mathbb{S}_{++}^{n_\rho}$ with $\rho_i \rightarrow \bar{\rho}$. If we have the convergence $\lim_i \nabla f(\rho_i) = \phi$, then

$$\phi \in \partial f(\bar{\rho}).$$

- **Duality**, and primal-dual **optimality conditions** with null-space representation
- Derive a **Gauss-Newton search direction** for the **nonlinear SDP** (with exact primal and dual feasibility possible)
- derive provable **lower** and upper **bounds**
- Empirics

Facially Reduced (Regularized) Nonlinear SDP

$$\begin{aligned} p^* = & \min && f(\rho) \quad (\text{regularized relative entropy}) \\ \text{subject to:} & && \Gamma_V(\rho) = \gamma_V \quad (\text{FR constraints}) \\ & && \rho \in \mathbb{H}_+^{n_\rho} \quad (\text{smaller SDP constr.}) \end{aligned}$$

Theorem (Basic Duality/Opt)

① Lagrangian $L(\rho, y) = f(\rho) + \langle y, \Gamma_V \rho - \gamma_V \rangle$, $y \in \mathbb{R}^{m_V}$.

② *Strong Duality*

$$\begin{aligned} p^* &= \max_y \min_{\rho \succeq 0} L(\rho, y) \\ &= d^* = \max_{Z \succeq 0, y} \left(\min_{\rho} (L(\rho, y) - \langle Z, \rho \rangle) \right) \end{aligned}$$

and d^* is attained for some $(y, Z) \in \mathbb{R}^{m_V} \times \mathbb{H}_+^{n_\rho}$.

③ p -d pair $(\rho, (y, Z))$, with $\partial f(\rho) \neq \emptyset$, is optimal iff

$$0 \in \partial f(\rho) + \Gamma_V^* y - Z \quad (\text{dual feasibility})$$

$$0 = \Gamma_V \rho - \gamma_V \quad (\text{linear primal feasibility})$$

$$0 = \langle \rho, Z \rangle \quad (\text{complementary slackness})$$

$$0 \preceq \rho, Z \quad (\text{SDP primal feasibility}).$$

Moreover, $\Gamma_V^* y \succeq 0$, $\langle y, \gamma_V \rangle < 0$, for some y , implies infeas.

Nullspace Representation/Residuals

Definition (nullspace representation)

$\hat{\rho} \in \mathbb{H}^{n_\rho}$ feasible point for $\Gamma_V(\cdot) = \gamma_V$.

$\mathcal{N}^* : \mathbb{R}^{n_\rho^2 - m_V} \rightarrow \mathbb{H}^{n_\rho}$ injective linear map in adjoint form so that we have the nullspace representation for the residual:

$$F_\mu^p = \Gamma_V \rho - \gamma_V \iff F_\mu^p = \mathcal{N}^*(v) + \hat{\rho} - \rho, \text{ for some } v.$$

Perturbed Optimality Conditions/Residuals

(i) dual feas.; (ii) primal feas.; (iii) perturbed compl. slack.

$$F_\mu(\rho, v, y, Z) = \begin{bmatrix} F_\mu^d \\ F_\mu^p \\ F_\mu^c \end{bmatrix} = \begin{bmatrix} \nabla_\rho f(\rho) + \Gamma_V^* y - Z \\ \mathcal{N}^* v + \hat{\rho} - \rho \\ Z\rho - \mu I \end{bmatrix} = 0, \quad \rho, Z \succ 0.$$

EXACT p-d feas if updated as: $\rho \leftarrow \Delta v$; $Z \leftarrow \Delta \rho, \Delta y$;
after a steplength = 1 is taken, exact p.f. is maintained.
exact dual feas. for Z is key for lower bound

Projected Gauss-Newton P-D I-P Method

Linearized System for GN Direction; OVERDETERMINED

$$F'_\mu d_{GN} = \begin{bmatrix} \nabla^2 f(\rho) \Delta \rho + \Gamma_v^* \Delta y - \Delta Z \\ \mathcal{N}^*(\Delta v) - \Delta \rho \\ Z \Delta \rho + \Delta Z \rho \end{bmatrix} \approx -F_\mu.$$

From First Block (for backsubstitution)

$$\begin{aligned} \Delta Z &= F_\mu^d + \nabla^2 f(\rho) \Delta \rho + \Gamma_v^* \Delta y \\ &= F_\mu^d + \nabla^2 f(\rho) (F_\mu^p + \mathcal{N}^*(\Delta v)) + \Gamma_v^* \Delta y. \end{aligned}$$

From Second Block (for backsubstitution)

$$\Delta \rho = F_\mu^p + \mathcal{N}^*(\Delta v).$$

Now substitute $\Delta Z, \Delta \rho$ into third block.

Projected GN direction

$$d_{GN} = (\Delta v \quad \Delta y) \quad (\text{backsubst. for } \Delta \rho, \Delta Z)$$

found from the least squares solution of (OVERDETERMINED)

$$\begin{aligned} [Z\mathcal{N}^*(\Delta v) + \nabla^2 f(\rho)\mathcal{N}^*(\Delta v)\rho] + [\Gamma_v^* \Delta y \rho] \\ = -F_\mu^c - ZF_\mu^p - (F_\mu^d + \nabla^2 f(\rho)F_\mu^p) \rho \end{aligned}$$

(Uses Hessian acting on a vector: $\nabla^2 f(\rho) : \mathbb{H} \rightarrow \mathbb{H}$)

Initialize: $\hat{\rho} \succ 0$, $\mu \in \mathbb{R}_{++}$, $\eta \in (0, 1)$

WHILE: stopping criteria is not met

solve for $(\Delta v, \Delta y)$

$$\Delta \rho = F_\mu^p + \mathcal{N}^*(\Delta v)$$

$$\Delta Z = F_\mu^d + \nabla^2 f(\rho)(F_\mu^p + \mathcal{N}^*(\Delta v)) + \Gamma_v^* \Delta y$$

choose steplength α

$$(\rho, y, Z) \leftarrow (\rho, y, Z) + \alpha(\Delta \rho, \Delta y, \Delta Z)$$

$$\mu = \text{trace}(\rho Z)/n; \mu \leftarrow \eta \mu$$

ENDWHILE

Sparse Nullspace Representation

We use a matrix representation M for Γ and a row and column permutation to get a well-conditioned near triangular basis matrix B . nullspace representation:

$$\hat{r} = \text{Hvec } \hat{\rho}; \Gamma_V \hat{\rho} = M \hat{r}(cp) = \gamma_V, M = \begin{bmatrix} B & E \end{bmatrix}, N^* = \begin{bmatrix} B^{-1} E \\ -I \end{bmatrix};$$

Optimal Diagonal Preconditioning, [4]

$d_i = \|F_\mu^{c'}(e_i)\|$, for unit vectors e_i ; column precondition using

$$F_\mu^{c'} \leftarrow F_\mu^{c'} \text{Diag}(d)^{-1}$$

MATLAB: $d_{GN} = ((F_\mu^{c'}/\text{Diag}(d)) \backslash RHS) ./ d$

Performed exceptionally well; problems are VERY ill-conditioned.

Evaluate f at Feasible ρ ; Iterative Refinement if Needed

if approximate linear feasibility $\Gamma_V \hat{\rho} \approx \gamma_V$, we apply iterative refinement by finding the projection Let $\hat{\rho} \succ 0$, $F_\mu^\rho = \Gamma_V \hat{\rho} - \gamma_V$. Then

$$\rho = \hat{\rho} - \Gamma_V^\dagger F_\mu^\rho = \operatorname{argmin}_\rho \left\{ \frac{1}{2} \|\rho - \hat{\rho}\|^2 : \Gamma_V \rho = \gamma_V \right\},$$

where we denote $\Gamma_V^\dagger$, generalized inverse. If $\rho \succeq 0$, then $\rho^* \leq f(\rho)$.

EXACT Primal Feasibility

In our tests we take a Newton step quite early, and maintain **exact primal feasibility** (no roundoff error buildup) for the further iterations.

Lower Bound from Weak Duality

$\rho \succ 0, Z \succeq 0$ in Interior Point Algorithm

The gradients exist at $\rho \succ 0$; we can verify dual feasibility.

Corollary (Lower Bound for FR problem from EXACT Dual Feas.)

$\hat{\rho}, \hat{y}$ primal-dual iterate; $\hat{\rho} \succ 0$. Set $\bar{Z} = \nabla f(\hat{\rho}) + \Gamma_V^* \hat{y}$.

If $\bar{Z} \succeq 0$, then lower bound is:

$$p^* \geq f(\hat{\rho}) + \langle \hat{y}, \Gamma_V \hat{\rho} - \gamma_V \rangle - \langle \hat{\rho}, \bar{Z} \rangle.$$

Proof.

Consider the dual problem

$d^* = \max_{y, Z \succeq 0} \min_{\rho \in \mathbb{H}^{n_\rho}} L(\rho, y) - \langle Z, \rho \rangle$. Dual feasibility implies:

$$\bar{Z} \succeq 0, \nabla f(\hat{\rho}) + \Gamma_V^* \hat{y} - \bar{Z} = 0 \implies \hat{\rho} \in \operatorname{argmin}_{\rho} L(\rho, \hat{y}) - \langle \bar{Z}, \rho \rangle.$$

Result follows from weak duality. □

Numerics; GN vs FW with/without FR

Problem Data		Gauss-Newton		FW (FR)		FW (no FR)		cvxquad (FR)	
protocol	size	gap	time	gap	time	gap	time	gap	time
ebBB84	(4,16)	6.0e-13	0.4	1.0e-04	86.9	1.2e-04	93.9	5.5e-01	194
ebBB84	(4,16)	1.2e-12	0.3	1.7e-04	96.8	1.3e-04	110.5	5.4e-01	1938
ebBB84	(4,16)	1.1e-12	0.2	1.6e-04	86.5	2.2e-04	112.0	5.7e-01	1979
ebBB84	(4,16)	4.2e-13	0.2	2.2e-04	88.6	2.2e-04	111.9	6.3e-01	523
pmBB84	(8,32)	5.5e-13	0.2	3.1e-05	1.3	6.5e-04	1.6	5.3e-01	158
pmBB84	(8,32)	6.1e-13	0.2	1.6e-04	1.1	3.8e-04	93.0	5.2e-01	207
pmBB84	(8,32)	6.3e-13	0.2	5.5e-05	1.1	3.0e-04	112.3	5.6e-01	299
pmBB84	(8,32)	1.3e-12	0.2	2.6e-04	1.1	1.3e-03	87.0	5.9e-01	188
mdiBB84	(48,96)	7.9e-13	0.9	9.6e-05	1.6	5.4e-04	120.9	1.8e-01	570
mdiBB84	(48,96)	1.4e-12	0.7	5.5e-05	101.2	6.6e-04	119.7	2.4e-01	585
mdiBB84	(48,96)	5.7e-13	0.9	1.5e-04	101.9	1.7e-03	439.3	3.1e-01	584
mdiBB84	(48,96)	9.2e-13	0.8	1.8e-04	100.0	2.2e-03	441.3	3.7e-01	558

Table: Numerical Report: Gauss-Newton, Frank-Wolfe (FW), cvxquad

- GN performs significantly better for both accuracy and running time
- only three protocols (each with four different parameter settings);
that is all that cvxquad could handle;
- FW is significantly improved by using our new FR

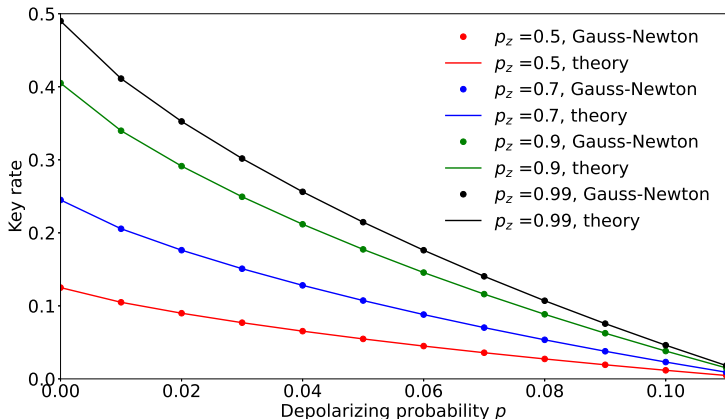
Larger Numerics; cvxquad Failed

Problem Data		Gauss-Newton		FW (FR)		FW (no FR)	
protocol	size	gap	time	gap	time	gap	time
TFQKD	(12,24)	5.9e-13	1.1	2.6e-09	1.9	1.6e-03	364.1
TFQKD	(12,24)	1.2e-12	0.8	3.8e-09	1.5	5.6e-04	369.1
TFQKD	(12,24)	3.2e-13	0.8	4.0e-09	1.3	1.7e-04	4.1
DMCV	(44,176)	2.7e-09	1326.1	2.4e-06	2808.4	3.4e-06	4933.9
DMCV	(44,176)	2.7e-09	1377.4	1.3e-06	974.2	2.5e-06	1281.2
DMCV	(48,192)	3.1e-09	1807.1	2.7e-06	3167.4	5.1e-06	5407.5
DMCV	(48,192)	3.2e-09	2110.6	2.6e-06	979.8	2.0e-06	1756.3
dprBB84	(12,48)	4.9e-13	1.3	3.8e-06	88.0	9.4e-05	123.0
dprBB84	(24,96)	1.0e-12	12.1	6.2e-06	15.9	3.6e-06	31.1
dprBB84	(36,144)	5.0e-13	69.3	6.5e-04	8.8	2.1e-02	30.1
dprBB84	(48,192)	1.1e-12	325.5	4.4e-05	17.1	9.8e-04	181.9

Table: Numerical Report: Gauss-Newton, Frank-Wolfe (FW)

- GN performs significantly better again
- FW does significantly better with our new FR again

GN is Exactly Analytical (Protocol mdiBB84)



The • are the lower bounds from GN; they coincide exactly with the analytical values on the curves.

This meets with the empirical evidence of gaps $\approx 10^{-12}$

- **regularized** the key rate calculation using **FACIAL REDUCTION** on both constraints and nonlinear (relative entropy) objective function over the Hermitians (complex);
- provided theoretically proven upper and **lower** bounds with **high precision**
- derived robust (Gauss-Newton) interior point approach on regularized problem
 - avoids current perturbation approach to get $\rho \succ 0$;
 - avoids roundoff error from backsubstitution steps;
 - attains exact primal feasibility during iterations
 - uses exact dual feasibility steps to improve on lower bounds

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Thanks for your attention!

Robust Interior Point Methods and FR for Key Rate Computation in Quantum Key Distribution

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Mon. April 5, 2021, 15:30 CEST
At: One World Optimization Seminar

