Problem Solution for Problem 19-5 in IMAGE, The Bulletin of the International Linear Algebra Society, Issue Number 19, 1997.

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1 Problem 19-5

Let A and B be real symmetric matrices.

1. If A and B are positive definite (non-negative definite), then it is not true that $AB + BA$ is positive definite (non-negative definite).

   Proof. The proof follows (quickly) from using the following matlab program (courtesy of Mike Overton, Courant Institute, NY).

   ```
   for try = 1:10,
      X = diag(rand(10,1));
      Z = rand(10,10); Z = Z'*Z;
      XZZX = X*Z + Z*X;
      [eig(X) eig(Z) eig(XZZX)]
      pause
   end;
   ```

   A specific example given in [2] is

   $$A = \begin{bmatrix} 8 & 1 \\ 1 & 0.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}.$$

   The eigenvalues of A are: 8.1310 and 0.3690. The eigenvalues of $AB + BA$ are: -1 and 33.

   □

2. If A and $AB + BA$ are both positive semidefinite, then it is not true that B is positive semidefinite: consider the diagonal matrices

   $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

   But for A and $P = AB + BA$ both positive definite, then it is true that B is positive definite.

   Proof. Let $B = UDU^t$ be the orthogonal diagonalization of B. Then A and $U^tPU = (U^tAU)D + D(U^tAU)$ are both positive definite, i.e. we could have assumed that $B = D$ was diagonal and so we write

   $$P = AD + DA.$$

But, then we can solve the consistent linear equation in the diagonal matrix D to get

$$D_{ii} = P_{ii}/A_{ii} > 0, \ \forall i.$$

$\square$

OBSERVATIONS An immediate corollary to the proof of part 2 is

Corollary 1.1 *If A and $AB + BA$ are nonnegative definite and the diagonal of A is positive, then B is nonnegative definite.*

In fact, these types of questions arise indirectly in certain algorithms for solving semidefinite programming (SDP) problems such as

$$\min\{\operatorname{trace} CX : \mathcal{A}X = b, \ X \succeq 0\},$$

where $X \succeq 0$ denotes positive semidefiniteness, C is a given symmetric matrix and $\mathcal{A}$ is a given linear operator. These algorithms start with two given $n \times n$ symmetric positive definite matrices $\bar{Z}, \bar{X} \succ 0$ and the scalar $\mu = \operatorname{trace} \bar{Z}\bar{X}/n$. They then take a Newton step for solving the nonlinear system of equations:

$$\begin{pmatrix} Z + C - \mathcal{A}^* y \\ b - \mathcal{A}(X) \\ ZX - \mu I \end{pmatrix} =: \begin{pmatrix} F_d \\ F_p \\ F_c \end{pmatrix} = 0. \tag{1.1}$$

The first two equations are linear while the nonlinearity arises from the third equation. This equation also results in the system being overdetermined, since it maps symmetric matrices to possibly nonsymmetric matrices. One approach is to symmetrize this equation, resulting in $ZX + XZ - 2\mu I = 0$. (More general symmetrizations are discussed in [3].) The linearization of the resulting square system has a unique solution (called the AHO direction [1]) if $ZX + XZ$ is positive semidefinite. (This is shown in [5] and [4].) However, as the above example shows, this may not hold.

However, the following is clear: if $ZX - \mu I = 0$ (i.e. Z, X lie on the so-called central path), then $ZX + XZ \succ 0$, since commutativity means common orthogonal diagonalization. Therefore, a natural question to ask is:

Problem 1.1 *Given $Z, X \succ 0$, $\mu > 0$, define a neighbourhood of the central path*

$$||ZX - \mu I|| < \epsilon \text{ or } ||Z^{\frac{1}{2}} X Z^{\frac{1}{2}} - \mu I|| < \epsilon$$

which guarantees that $ZX + XZ \succ 0$.

Such a result is discussed by e.g. Monteiro [2]. He shows that

$$||Z^{\frac{1}{2}} X Z^{\frac{1}{2}} - \mu I|| < \mu/2$$

implies that the linearization has a unique solution. In fact, this does not imply $XZ + ZX \succ 0$ as the above example illustrates with $\nu = 8$.

However, the triangle inequality (observation due to Mike Todd) does show that:

$$||ZX - \mu I|| < \mu \text{ implies } ZX + XZ \succ 0,$$

thus providing a neighbourhood of the central path.

References

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