Geometry of Semidefinite Max-Cut Relaxations via Matrix Ranks

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June 14, 2001
(Revised October 25, 2001)

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Research Report CORR 2001-39

Key words: Semidefinite Programming, Discrete Optimization, Lagrangian Relaxation, Max-Cut Problem.

Abstract

Semidefinite programming (SDP) relaxations are proving to be a powerful tool for finding tight bounds for hard discrete optimization problems. This is especially true for one of the easier NP-hard problems, the Max-Cut problem (MC). The well-known SDP relaxation for Max-Cut, here denoted SDP1, can be derived by a first lifting into matrix space and has been shown to be excellent both in theory and in practice.

Recently the present authors have derived a new relaxation using a second lifting. This new relaxation, denoted SDP2, is strictly tighter than the relaxation obtained by adding all the triangle inequalities to the well-known relaxation. In this paper we present new results that further describe the remarkable tightness of this new relaxation. Let $\mathcal{F}_n$ denote the feasible set of SDP2 for the complete graph with n nodes, let F_n denote the appropriately defined projection of $\mathcal{F}_n$ into $\mathcal{S}^n$, the space of real symmetric $n \times n$ matrices, and let C_n denote the cut polytope in $\mathcal{S}^n$. Further let $Y \in \mathcal{F}_n$ be the matrix variable of the SDP2 relaxation and $X \in F_n$ be its projection. Then for the complete graph on 3 nodes,

*Research partially supported by a Ph.D. Research Scholarship from the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche (Fonds FCAR) of Québec. Email anjos@stanfordalumni.org

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$F_3 = C_3$ holds. We prove that the rank of the matrix variable $Y \in \mathcal{F}_3$ of SDP2 completely characterizes the dimension of the face of the cut polytope in which the corresponding matrix X lies. This shows explicitly the connection between the rank of the variable Y of the second lifting and the possible locations of the projected matrix X within C_3. The results we prove for $n = 3$ cast further light on how SDP2 captures all the structure of C_3, and furthermore they are stepping stones for studying the general case $n \geq 4$. For this case, we show that the characterization of the vertices of the cut polytope via $\operatorname{rank} Y = 1$ extends to all $n \geq 4$. More interestingly, we show that the characterization of the one-dimensional faces via $\operatorname{rank} Y = 2$ also holds for $n \geq 4$. Furthermore, we prove that if $\operatorname{rank} Y = 2$ for $n \geq 3$, then a simple algorithm exhibits the two rank-one matrices (corresponding to cuts) which are the vertices of the one-dimensional face of the cut polytope where X lies.

1 Introduction

The Max-Cut problem (MC) is a combinatorial optimization problem on undirected graphs with weights on the edges. Given such a graph, the problem consists in finding a partition of the set of vertices into two parts that maximizes the sum of the weights on the edges that have one end in each part of the partition. We consider the general case where the graph is complete and we require no restriction on the type of edge weights. So, in particular, negative or zero edge weights are permitted.

The MC problem has applications in circuit layout design and statistical physics, see e.g. [5, 29]. Moreover, since the celebrated work of Goemans and Williamson [15, 16], this problem has become the flagship problem when studying applications of semidefinite programming (SDP) to combinatorial optimization [14, 34, 12, 13, 22, 18, 21, 7, 8, 19, etc.]. Furthermore, the MC problem is closely related to the so-called cut polytope, an important structure in the area of integer programming. The book of Deza and Laurent [9] presents many theoretical results about the cut polytope, and some connections to general integer programming are elaborated in [20].

It is well-known that MC is an NP-complete problem [24] and that it remains NP-complete for some restricted versions, see e.g. [11]. Nonetheless, some special cases can be solved efficiently. If the graph is not contractible to K_5, the complete graph on five vertices, then Barahona [3] proved that the polyhedral relaxation obtained from the triangle inequalities yields exactly the optimal value of MC. (The triangle inequalities model the constraints that for any three mutually connected vertices in the graph, it is only possible to cut either zero or two of the edges joining them.)

Semidefinite programming is arising as a powerful tool in many areas of optimization, see for example the recent Handbook [35]. For the Max-Cut problem, one particular SDP relaxation, denoted SDP1 in this paper and formally defined in Section 2, is now quite well-known. The feasible set of the SDP1 relaxation is the set of correlation matrices, or elliptope, and it has been well studied in the literature, see e.g. [27, 28] as well as [9] and the references therein. Goemans and Williamson [15, 16, 14] used the SDP1 relaxation in their celebrated approximation algorithm for finding a cut whose weight is guaranteed to be within 14% of the weight of the maximum cut, provided that there are no negative weights. Recently, Feige and Schechtman [10] analyzed the SDP1 relaxation with all the triangle inequalities included and constructed examples for which it is within 11% of the weight of the maximum cut. Further approximation results for problems with general quadratic objective functions are presented in [31, 36, 32]. In particular, Nesterov [31, 32] uses the SDP1 bound to provide estimates of μ^* for arbitrary edge weights with constant relative accuracy. All the results in [1, 2] as well as those in this paper also hold for arbitrary edge weights. This allowance for negative weights is important in the application of Max-Cut to Ising spin glass problems, see e.g. [1, 5].

Several new SDP relaxations for MC have recently appeared in the literature, see [25, 33, 26, 1, 2]. Of the two strengthened SDP relaxations for Max-Cut introduced in [2], one was observed to be particularly tight. Indeed, in [2] the authors proved that $F_n \subseteq M_n$ for $n \geq 3$, where F_n denoted the feasible set of this new relaxation, and M_n denotes the metric polytope, the polyhedral relaxations defined by the triangle inequalities. Using well-known results of polyhedral theory, it immediately follows that $F_n = C_n$ for $n = 3, 4$, since the triangle inequalities are the only facets of C_3 and C_4 (see e.g. page 503 of [9]). Extensive computational results illustrating the strength of this new relaxation are presented in [1].

In this paper we present new results that provide further understanding of the tightness of this new SDP relaxation. We prove that given $Y \in \mathcal{F}_3$, where $\mathcal{F}_n$ denotes the feasible set of SDP2 for the complete graph with n nodes, the rank of Y characterizes the dimension of the face of C_3 where the projection X lies. (This projection is formally defined in Section 2.) This shows explicitly the connection between the rank of the variable Y of the second lifting and the possible locations of the projected matrix X within C_3. The results we prove for $n = 3$ cast further light on how SDP2 captures all the structure of C_3, and furthermore they are stepping stones for studying the general case $n \geq 4$. For this general case, we show that the characterization of the vertices of the cut polytope via $\operatorname{rank} Y = 1$ extends to all $n \geq 4$. More interestingly, we show that the characterization of the one-dimensional faces via $\operatorname{rank} Y = 2$ also holds for $n \geq 4$. In fact, we prove that if $\operatorname{rank} Y = 2$ for $n \geq 3$, then it is possible to extract the two rank-one matrices (corresponding to cuts) which are the vertices of the one-dimensional face of the cut polytope where X lies. Indeed, we present a simple algorithm that does precisely this.

This paper is structured as follows. In Section 2 we introduce some notation and present several formulations and relaxations of Max-Cut that will be used in this paper. These include the formal definitions of the elliptope and the cut and metric polytopes, as well as some vector and matrix formulations of Max-Cut. We also show some elementary properties and relationships of the ranks of the matrices in these relaxations which, to the best of our knowledge, are not explicitly stated in the literature. In Section 3 we prove the complete characterization of the cut polytope for $n = 3$ in terms of the rank of the matrix Y feasible for SDP2. Finally, in Section 4 we show that the characterization of the faces of dimensions 0 and 1 in terms of the rank of the variable Y holds for all $n \geq 4$.

2 Formulations, Relaxations, and Ranks

Following [30], we can formulate the MC problem as follows. Let the given graph G have vertex set $\{1, \dots, n\}$ and let it be described by its weighted adjacency matrix $A(G)$. We assume that the graph in question is complete (if not, missing edges can be given weight 0 to complete the graph) and that the edge set is not empty, so $A(G) \neq 0$. Let L denote the *Laplacian matrix* associated with the graph; hence $L := \mathrm{Diag}\,(A(G) \cdot e) - A(G)$, where the linear operator Diag returns a diagonal matrix with diagonal formed from the vector given as its argument, and e denotes the vector of all ones. Let the vector $v \in \{\pm 1\}^n$ represent any cut in the graph via the interpretation that the sets $\{i : v_i = +1\}$ and $\{i : v_i = -1\}$ form a partition of the vertex set of the graph. Then we can formulate MC as

$$\begin{array}{rll} \mu^* = & \max & \frac{1}{4} v^T L v \\ & \text{s.t.} & v_i^2 = 1, \quad i = 1, \dots, n, \end{array}$$

where μ^* denotes the optimal value of MC. Define $Q := \frac{1}{4} L$ and consider the change of variable $X := vv^T, v \in \{\pm 1\}^n$. Then $v^T Q v = \mathrm{trace}\, QX$, and an equivalent formulation for MC is

$$(\mathrm{MC1}) \qquad \begin{array}{rll} \mu^* = & \max & \mathrm{trace}\, QX \\ & \text{s.t.} & \mathrm{diag}\,(X) = e \\ & & \mathrm{rank}\,(X) = 1 \\ & & X \succeq 0, X \in \mathcal{S}^n\,, \end{array}$$

where diag denotes the linear operator that returns a vector with the diagonal elements of its matrix argument, $X \succeq 0$ denotes that X is positive semidefinite, and $\mathcal{S}^n$ denotes the space of $n \times n$ symmetric matrices. This space has dimension $t(n) := n(n+1)/2$ and inner product $\langle A, B \rangle := \mathrm{trace}\, AB$.

Denote the feasible set of MC1, less the rank constraint, by

$$\mathcal{E}_n := \{X \in \mathcal{S}^n : \mathrm{diag}\,(X) = e, X \succeq 0\}.$$

The set $\mathcal{E}_n$ is the elliptope studied in [27, 28] and it is a convex relaxation of the feasible set of problem MC1 since we dropped the rank constraint. With this notation, let us define the semidefinite programming problem SDP1:

$$(\mathrm{SDP1}) \qquad \begin{array}{rll} \nu_1^* = & \max & \mathrm{trace}\, QX \\ & \text{s.t.} & X \in \mathcal{E}_n. \end{array}$$

This SDP relaxation has been well studied in the literature, see [35] and the references therein. In particular, Goemans and Williamson [15, 16] provided estimates for the quality of the SDP1 bound for Max-Cut. They proved that the optimal value of this relaxation is at most 14% above the value of the maximum cut, provided there are no negative edge weights. More precisely,

$$\mu^* \geq \alpha_{SDP1}\, \nu_1^*,$$

where $\alpha_{SDP1} := \min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta}{1-\cos\theta} \approx 0.87856$, and since $\frac{1}{\alpha_{SDP1}} \leq 1.14$, we have that $\nu_1^* \leq 1.14\, \mu^*$. Furthermore, by randomly rounding a solution to the SDP relaxation, they obtain an α_{SDP1}-approximation algorithm, i.e. an algorithm that produces a cut with value at least α_{SDP1} times the optimal value. (Note that Håstad [17] proved that it is NP-hard to find a ρ-approximation algorithm for Max-Cut with ρ greater than $\frac{16}{17} \approx 0.9412$.)

Other convex relaxations of the feasible set of MC1 have been proposed in the literature. The smallest convex set containing all the matrices X which are feasible for MC1 is their convex hull, called the *cut polytope*:

$$C_n := \operatorname{conv}\{X : X = vv^T, v \in \{\pm 1\}^n\}.$$

Optimizing trace QX over C_n would yield exactly μ^*, but an efficient description of the cut polytope is not known. A well-known relaxation of the cut polytope is the *metric polytope* M_n, defined as the set of all matrices satisfying the triangle inequalities:

$$\begin{aligned} M_n := \{X \in \mathcal{S}^n \ :\ & \operatorname{diag}(X) = e, \text{ and} \\ & X_{ij} + X_{ik} + X_{jk} \geq -1, X_{ij} - X_{ik} - X_{jk} \geq -1, \\ & - X_{ij} + X_{ik} - X_{jk} \geq -1, -X_{ij} - X_{ik} + X_{jk} \geq -1, \\ & \forall\, 1 \leq i < j < k \leq n\}. \end{aligned}$$

The triangle inequalities model the following constraints: for any three mutually connected vertices in the graph, it is only possible to cut either zero or two of the edges joining them. In fact, the triangle inequalities are sufficient to describe the cut polytope for graphs with less than five vertices, i.e. $C_n = M_n$ for $n \leq 4$; however, $C_n \subsetneq M_n$ for $n \geq 5$, see for example [9]. Nonetheless, if the graph G is not contractible to K_5, the complete graph on 5 vertices, then Barahona [4] proved that the linear programming problem

$$\begin{array}{ll} \max & \operatorname{trace} QX \\ \text{s.t.} & X \in M_n \end{array}$$

has optimal value equal to μ^*.

More recently, the authors introduced the relaxation SDP2 [2]:

$$\text{(SDP2)} \qquad \begin{array}{rll} \nu_2^* = & \max & \operatorname{trace} Q\, \operatorname{sMat}(Y_{1:t(n),0}) \\ & \text{s.t.} & \operatorname{diag}(Y) = e \\ & & Y_{0,t(i)} = 1, i = 1, \ldots, n \\ & & Y_{0,T(i,j)} = Y_{T(i,k),T(k,j)}, \ \ \forall\, k = 1, \ldots, n, \forall\, 1 \leq i < j \leq n \\ & & Y \succeq 0, Y \in \mathcal{S}^{t(n)+1}, \end{array}$$

where the rows and columns of Y are indexed by $\{0, 1, \ldots, t(n)\}$, the index $T(i, j)$ is defined as

$$T(i,j) := \begin{cases} t(j-1)+i, \text{ if } i \leq j \\ t(i-1)+j, \text{ otherwise,} \end{cases}$$

and sMat is the linear operator that, given a vector $x \in \Re^{t(n)}$, returns a matrix $X \in \mathcal{S}^n$ obtained by filling in columnwise the upper triangular part of X with the $t(n)$ components of x and completing the strictly lower triangle by symmetry. Hence,

$$\mathrm{sMat}\begin{pmatrix} x_1 \\ \vdots \\ x_{t(n)} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_4 & \cdots & x_{t(n-1)+1} \\ x_2 & x_3 & x_5 & \cdots & x_{t(n-1)+2} \\ x_4 & x_5 & x_6 & \cdots & x_{t(n-1)+3} \\ & & \vdots & & \\ x_{t(n-1)+1} & x_{t(n-1)+2} & x_{t(n-1)+3} & \cdots & x_{t(n)} \end{pmatrix}.$$

(Recall that $t(i) = \frac{i(i+1)}{2}$.)

Let $\mathcal{F}_n$ denote the feasible set of SDP2. Using sMat , we define the projection of the feasible set of SDP2 onto $\mathcal{S}^n$ as

$$F_n := \{X \in \mathcal{S}^n \ : X = \mathrm{sMat}\,(Y_{1:t(n),0}), \ \ Y \in \mathcal{F}_n\}.$$

Since the feasible set of SDP2 is convex and compact, and since F_n is its image under a linear transformation, it follows that F_n is also convex and compact. It is easy to check that $C_n \subseteq F_n \subseteq \mathcal{E}_n$, and in fact $F_n \subseteq M_n$ also holds [2]. Since $C_n = M_n$ for $n = 3, 4$, it thus follows that $F_n = M_n$ for $n = 3, 4$. Furthermore, in [2] the authors proved that

Theorem 2.1 $C_n \subsetneq F_n \subsetneq \mathcal{E}_n \cap M_n$ *for* $n \geq 5$. ■

2.1 Ranks of Matrices in Low Dimensional Faces of C_n

In this section we present some elementary facts about the cut polytope C_n. First we recall from [9] the definition of k-neighbourliness: Given $k \geq 1$, a polytope is said to be k-neighbourly if, for every subset of at most k vertices of the polytope, the convex hull of this subset of vertices is a face of the polytope. This implies, in particular, that every subset of k vertices of the polytope is affinely independent. In the proof of Lemma 2.2, we will make use of the fact that the cut polytope is 2-neighbourly. The 2-neighbourliness property follows from the fact that the cut polytope is, in fact, 3-neighbourly [9, page 540], but was first proved independently in [6].

The results in Lemma 2.2 are elementary and reflect well-known results about the cut polytope. However, because we study the cut polytope in the matrix space $\mathcal{S}^n$ rather than in the vector space $\Re^{t(n-1)}$, these results are not explicitly stated in the literature to the best of our knowledge. Hence we present them for completeness.

Lemma 2.2 *Let $v_1, \ldots, v_{2^{n-1}}$ be elements of $\{\pm 1\}^n$ that represent the 2^{n-1} distinct cuts possible for the complete graph on n nodes, $n \geq 3$. By definition,*

$$C_n = conv\{v_i\, v_i^T, \quad i = 1, \ldots, 2^{n-1}\}.$$

Then

1. *X is a vertex of C_n if and only if $X = v_i\, v_i^T$ for some $i \in \{1, \ldots, 2^{n-1}\}$.*
2. *X is in the relative interior of a face of C_n of dimension 1 if and only if $X = \alpha\, v_i\, v_i^T + (1-\alpha)\, v_j\, v_j^T$ for some $\alpha \in (0,1)$ and $\{i, j\} \subseteq \{1, \ldots, 2^{n-1}\}$, $i \neq j$.*

Proof. To prove the first claim, note that if X is a vertex of C_n, then X must be an extreme point, and hence a generator of the convex hull, and thus $X = v_i\, v_i^T$ for some $i \in \{1, \ldots, 2^{n-1}\}$. The converse is also straightforward to prove, following for example the approach of the proof employed in [27] to prove Theorem 2.3 below.

To prove the second claim, first assume that X is in the relative interior of a face of dimension 1. Then by definition of such a face, X is the strict convex combination of two vertices, i.e. X is formed exactly as claimed. Conversely, if $X = \alpha\, v_i\, v_i^T + (1-\alpha)\, v_j\, v_j^T$, then the fact that the cut polytope is 2-neighbourly implies that the convex hull of the vertices $v_i\, v_i^T$ and $v_j\, v_j^T$ is a (one-dimensional) face of C_n, and hence X is in the relative interior of a face of dimension 1. ∎

2.2 Ranks of Matrices in $\mathcal{E}_n$

For the elliptope, only the vertices of C_n are completely characterized by the rank of X. Laurent and Poljak proved the following theorem (expressed here using our notation):

Theorem 2.3 *(Laurent and Poljak, Theorem 2.5 in [27])*
The elliptope $\mathcal{E}_n$ has precisely 2^{n-1} vertices, each of the form $v_i\, v_i^T$ for some $i \in \{1, \ldots, 2^{n-1}\}$.

By Lemma 2.2, the vertices of $\mathcal{E}_n$ are thus exactly the vertices of C_n. However, the correspondence between rank and geometric structure breaks down as soon as we consider matrices with rank equal to 2. Indeed, for $n = 3$ consider the matrix

$$\tilde{X} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

It can easily be checked that $\tilde{X} \in \mathcal{E}_3$ and has rank equal to two. However,

$$\tilde{X}_{12} + \tilde{X}_{13} + \tilde{X}_{23} = -\frac{3}{2} < -1$$

and so $\tilde{X}$ violates the triangle inequality $X_{12}+X_{13}+X_{23} \geq -1$. Hence $\tilde{X} \notin C_3$.

We now proceed to prove Theorem 3.14 which shows that, in contrast to the elliptope case, whenever $Y \in \mathcal{F}_3$ and $X = \mathrm{sMat}\,(Y_{1:6,0})$, the rank of Y completely characterizes the dimension of the face of C_3 in which X lies. In the next section, we prove some results about the ranks of the matrices in C_3 and apply them to prove the aforementioned characterization.

3 Ranks and Faces of C_3

For the rest of this section, we focus our attention on the polytope C_3 and the ranks of the matrices therein. First observe that there are four distinct cuts for the complete graph K_3, which we represent by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \text{and} \quad v_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

It is straightforward to check that every subset of three (or fewer) of these vectors is linearly independent. Recall that $X \in C_3$ if and only if

$$X = \sum_{i=1}^{4} \alpha_i \, v_i \, v_i^T, \quad \sum_{i=1}^{4} \alpha_i = 1, \quad \alpha_i \geq 0, i = 1, 2, 3, 4. \tag{3.1}$$

The affine independence of the set $\{v_i \, v_i^T \mid i = 1, 2, 3, 4\}$, which is also straightforward to check, implies that the convex combination in equation (3.1) is unique for each X.

Lemma 3.1 *$X \in C_3$ is positive definite if and only if the convex combination in (3.1) that equals X has at least three α_i positive.*

Proof. Every $X \in C_3$ is equal to a sum of matrices each with rank (at most) one, therefore

$$\mathrm{rank}\,(X) = \mathrm{rank}\, \left(\sum_{i=1}^{4} \alpha_i \, v_i \, v_i^T \right) \leq \sum_{i=1}^{4} \mathrm{rank}\, \left(\alpha_i \, v_i \, v_i^T \right).$$

If X is positive definite then $\mathrm{rank}\,(X) = 3$, therefore at least three of the matrices $\alpha_i \, v_i \, v_i^T$ must be non-zero. Hence at least three α_i are positive in any convex combination of vertices equal to X.

To prove sufficiency, suppose that the convex combination of vertices equal to X has at least three α_i positive. We prove that X is positive definite by contradiction.

Suppose there exists $w \in \Re^3$ such that $w \neq 0$ and $X\,w = 0$. Because every subset of three v_i's forms a basis for $\Re^3$, we may assume without loss of generality that $v_1^T w \neq 0$ and $v_2^T w \neq 0$ (that is, at least two v_i's are not orthogonal to w). Then

$$X\,w = \alpha_1 \, v_1 \, (v_1^T w) + \alpha_2 \, v_2 \, (v_2^T w) + \alpha_3 \, v_3 \, (v_3^T w) + \alpha_4 \, v_4 \, (v_4^T w) = 0 \tag{3.2}$$

therefore

$$\alpha_1\,(v_1^T\,w)\,v_1^T = -\left[\alpha_2\,(v_2^T\,w)\,v_2^T + \alpha_3\,(v_3^T\,w)\,v_3^T + \alpha_4\,(v_4^T\,w)\,v_4^T\right]$$

and, multiplying by w on the right, we obtain that

$$\alpha_1\,(v_1^T\,w)^2 = -\left[\alpha_2\,(v_2^T\,w)^2 + \alpha_3\,(v_3^T\,w)^2 + \alpha_4\,(v_4^T\,w)^2\right].$$

Now observe that each term $\alpha_i\,(v_i^T\,w)^2$ is non-negative. Hence both sides of the equation must equal zero. Since $v_1^T\,w \neq 0$, $\alpha_1 = 0$ must hold, and similarly $\alpha_2 = 0$ must also hold. This contradicts our assumption that the convex combination of vertices equal to X has at least three α_i positive. ■

Lemma 3.2 *$X \in C_3$ has rank equal to 1 if and only if $X = v_i\,v_i^T$ for some $i \in \{1,2,3,4\}$.*

Proof. Sufficiency is immediate. So suppose $X \in C_3$ has rank equal to 1. By Lemma 3.1, the linear combination of vertices equal to X has at most two α_i non-zero. To get a contradiction, suppose that some linear combination equal to X has two α_i non-zero. Further suppose $\alpha_3 = 0$ and $\alpha_4 = 0$ without loss of generality. Now proceed as follows:

1. Choose $w_1 \in \Re^3$ such that $v_2^T\,w_1 = 0$ and $v_1^T\,w_1 \neq 0$ (this is possible because v_1 and v_2 are linearly independent). Then

   $$X\,w_1 = \alpha_1\,(v_1^T\,w_1)\,v_1 + \alpha_2\,(v_2^T\,w_1)\,v_2 = \alpha_1\,(v_1^T\,w_1)\,v_1,$$

   where $\alpha_1\,(v_1^T\,w_1) \neq 0$. Hence v_1 is in the range of X.

2. Similarly, choose $w_2 \in \Re^3$ such that $v_1^T\,w_2 = 0$ and $v_2^T\,w_2 \neq 0$, and conclude that v_2 is in the range of X.

Since $\{v_1, v_2\}$ form a linearly independent set, the range of X has dimension at least 2. But the rank of X equals the dimension of its range, hence $\operatorname{rank} X \geq 2$, which is a contradiction. This concludes the proof. ■

Corollary 3.3 *$X \in C_3$ has rank equal to 2 if and only if there exists $\alpha \in (0,1)$ and $i \neq j$, $i,j \in \{1,2,3,4\}$ such that $X = \alpha\,v_i\,v_i^T + (1-\alpha)\,v_j\,v_j^T$.*

Proof. Again sufficiency is immediate by the linear independence of $\{v_i, v_j\}$ for $i \neq j$, $i,j \in \{1,2,3,4\}$. And, if $\operatorname{rank} X = 2$ then

- by Lemma 3.1, the convex combination equal to X has at most two α_i non-zero;

- by Lemma 3.2, the convex combination equal to X has at least two α_i non-zero.

Hence the result. ■

We summarize our results on the ranks of matrices $X \in C_3$ in the following theorem:

Theorem 3.4 *Let $X \in C_3$, i.e.*

$$X = \sum_{i=1}^{4} \alpha_i \, v_i \, v_i^T, \quad \sum_{i=1}^{4} \alpha_i = 1, \ \alpha_i \geq 0, i = 1, 2, 3, 4.$$

Then

- $\operatorname{rank} X = 1 \Leftrightarrow X$ *is a vertex of C_3;*
- $\operatorname{rank} X = 2 \Leftrightarrow X$ *is in the relative interior of a face of C_3 of dimension 1.*
- $\operatorname{rank} X = 3 \Leftrightarrow$ *the convex combination of vertices equal to X has at least three α_i positive.*

Proof. This result follows immediately from Lemmas 2.2, 3.1, 3.2, and Corollary 3.3. ■

We see that while the vertices and one-dimensional faces of C_3 are completely characterized by the rank of X, knowledge of the rank of X is insufficient to distinguish whether X lies in the relative interior of a facet (face of dimension 2) or in the interior of the cut polytope. We will prove that the rank of the variable Y in the second lifting achieves this purpose.

3.1 Rank Connections Between $\mathcal{F}_3$ and C_3

In order to simplify the proofs, for the rest of this section and in Section 4 we will work with the principal submatrix of Y corresponding to the rows and columns $\{0, 2, 4, 5\}$. Note that the columns of Y that are not considered, namely columns 1,3, and 6, are all equal to column 0. This is because in general, whenever $Y \in \mathcal{F}_n$, the n columns indexed by $T(k,k)$, $k = 1, \ldots, n$, are all equal to column 0, since positive semidefiniteness of the 3×3 principal submatrix of Y corresponding to the rows and columns $\{0, T(k,k), T(i,j)\}$ gives

$$\begin{pmatrix} 1 & 1 & Y_{0,T(i,j)} \\ 1 & 1 & Y_{T(k,k),T(i,j)} \\ Y_{0,T(i,j)} & Y_{T(k,k),T(i,j)} & 1 \end{pmatrix} \succeq 0 \Rightarrow Y_{T(i,j),T(k,k)} = Y_{T(i,j),0}.$$

(The implication follows by symmetry and by Lemma 3.9 proved below.) Hence, the rank of $Y \in \mathcal{F}_n$ always equals the rank of the principal submatrix corresponding to the rows and columns $\{0\} \cup \{T(i,j) \mid 1 \leq i < j \leq n\}$, and all the rank results in this section and the next hold identically for the entire matrix Y.

Let us therefore set the notation (for the case $n = 3$):

$$X = \begin{pmatrix} 1 & X_{12} & X_{13} \\ X_{12} & 1 & X_{23} \\ X_{13} & X_{23} & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & X_{12} & X_{13} & X_{23} \\ X_{12} & 1 & X_{23} & X_{13} \\ X_{13} & X_{23} & 1 & X_{12} \\ X_{23} & X_{13} & X_{12} & 1 \end{pmatrix}.$$

We will make use of the following two well-known theorems from matrix theory (see for example [23]).

Theorem 3.5 *(Schur complement) Suppose that we partition A as*

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix},$$

where the principal submatrices A_1 and A_3 are square and A_1 is positive definite. Then $A \succeq 0$ if and only if $A_3 - A_2^T A_1^{-1} A_2 \succeq 0$.

Theorem 3.6 *(Interlacing eigenvalues theorem for bordered matrices)*
Let A be a given $n \times n$ Hermitian matrix, let y be a given n-vector, and let a be a given real number. Let $\hat{A}$ be the $(n+1) \times (n+1)$ Hermitian matrix obtained by bordering A with y and a as follows:

$$\hat{A} \equiv \begin{pmatrix} A & y \\ y^T & a \end{pmatrix}.$$

Let the eigenvalues of A and $\hat{A}$ be denoted by $\{\lambda_i(A)\}$ and $\{\lambda_i(\hat{A})\}$ respectively, and assume that they have been arranged in increasing order $\lambda_1(A) \leq \ldots \leq \lambda_n(A)$ and $\lambda_1(\hat{A}) \leq \ldots \leq \lambda_{n+1}(\hat{A})$. Then

$$\lambda_1(\hat{A}) \leq \lambda_1(A) \leq \lambda_2(\hat{A}) \leq \lambda_2(A) \leq \ldots \leq \lambda_{n-1}(A) \leq \lambda_n(\hat{A}) \leq \lambda_n(A) \leq \lambda_{n+1}(\hat{A}).$$

We will also make use of the following theorem from [2]:

Theorem 3.7 *Let X be an $n \times n$ symmetric matrix. Then*

$$X \succeq 0, X \in \{\pm 1\}^{n \times n} \quad \text{if and only if} \quad X = xx^T, \ \text{for some} \ \ x \in \{\pm 1\}^n.$$

■

The first lemma of this section establishes that the matrices Y with rank equal to 1 map to the vertices of C_3.

Lemma 3.8 *Let $Y \in \mathcal{F}_3$ and $X = \mathrm{sMat}\,(Y_{1:6,0})$. Then*

$$\operatorname{rank} Y = 1 \Leftrightarrow \operatorname{rank} X = 1.$$

Proof. First suppose $\operatorname{rank} Y = 1$. Then $\operatorname{rank} X \leq 1$, since X is a principal submatrix of Y. But $X \neq 0$, hence $\operatorname{rank} X = 1$.

Now suppose $\operatorname{rank} X = 1$. Since $X \succeq 0$, Theorem 3.7 implies that $X \in \{\pm 1\}^{3\times 3}$. Hence, $Y \in \{\pm 1\}^{4\times 4}$ and since $Y \in \mathcal{F}_3$ implies $Y \succeq 0$, Theorem 3.7 implies $\operatorname{rank} Y = 1$. ■

Having settled the vertex case, we proceed to prove some simple but very useful facts about the higher rank occurrences of Y. First we state and prove a simple technical lemma.

Lemma 3.9 *Suppose*

$$\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \succeq 0.$$

Then

1. *If* $a^2 = 1$ *then* $b = a\,c$*;*
2. *If* $b^2 = 1$ *then* $a = b\,c$*;*
3. *If* $c^2 = 1$ *then* $a = c\,b$*.*

Proof. We prove only the case $b^2 = 1$. The other cases are proved similarly, or may be reduced to this case by symmetrically permuting rows and columns of the matrix.

Positive semidefiniteness of the matrix implies that its determinant is non-negative, therefore

$$(1 - c^2) - a(a - bc) + b(ac - b) \geq 0$$

which implies

$$-(b^2\,c^2 + a^2 - 2\,a\,b\,c) \geq 0,$$

since $b^2 = 1$. Equivalently,

$$-(a - b\,c)^2 \geq 0$$

holds, and hence $a = b\,c$. ■

We note that all three implications of Lemma 3.9 may hold simultaneously. This is exemplified by the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}^T .$$

The next lemma establishes the possible ranks of Y whenever one or more of the elements of X have absolute value equal to 1.

Lemma 3.10 *Let* $Y \in \mathcal{F}_3$ *and* $X = \operatorname{sMat}(Y_{1:6,0})$*. Then the following hold:*

1. *If* $|X_{ij}| = 1$ *for some pair* $i, j, i \neq j$, *then* $\operatorname{rank} Y \leq 2$.

2. *If exactly one* X_{ij} *has absolute value equal to 1, then* $\operatorname{rank} Y = 2$.

3. *If two or three off-diagonal elements of* X *have absolute value equal to 1, then* $\operatorname{rank} Y = 1$.

Proof. The proof considers the possible cases with at least one element $X_{ij} \in \{\pm 1\}$. If all three of X_{12}, X_{13}, and X_{23} are equal to ± 1 then by Theorem 3.7, $\operatorname{rank} Y = 1$. If two of them are equal to ± 1, let us suppose without loss of generality that $|X_{12}| = 1$ and $|X_{13}| = 1$. If $X_{12} = 1$ then the leading 3×3 principal submatrix of Y is positive semidefinite, so applying Lemma 3.9,

$$\begin{pmatrix} 1 & 1 & X_{13} \\ 1 & 1 & X_{23} \\ X_{13} & X_{23} & 1 \end{pmatrix} \succeq 0 \Rightarrow X_{23} = X_{13},$$

and thus X_{23} also equals ± 1, therefore $\operatorname{rank} Y = 1$ again. Similarly, $X_{12} = -1 \Rightarrow X_{23} = -X_{13} \in \{\pm 1\}$ and $\operatorname{rank} Y = 1$.

Finally, if exactly one of them equals ± 1, suppose again without loss of generality that $|X_{12}| = 1$. If $X_{12} = 1$ then $X_{23} = X_{13}$ but neither equals ± 1 and

$$Y = \begin{pmatrix} 1 & 1 & X_{13} & X_{13} \\ 1 & 1 & X_{13} & X_{13} \\ X_{13} & X_{13} & 1 & 1 \\ X_{13} & X_{13} & 1 & 1 \end{pmatrix}$$

which has rank equal to two. Similarly, $X_{12} = -1 \Rightarrow X_{23} = -X_{13}$ and again $\operatorname{rank} Y = 2$. ■

Lemma 3.11, which we now prove, is perhaps a surprising result. It shows that if $X \in F_3$ has rank equal to two, then one of its elements has absolute value equal to one. Such a statement does not hold for the elliptope $\mathcal{E}_3$, as is demonstrated for example by the matrix $\tilde{X}$ presented in Section 2.2. Lemma 3.11 will also be very useful in the sequel.

Lemma 3.11 *If* $X \in F_3$ *and* $\operatorname{rank} X = 2$ *then* $|X_{ij}| = 1$ *for some* $i \neq j$.

Proof. If X has rank equal to 2 then X is singular. Clearly not all three off-diagonal entries of X can equal zero. Suppose two elements of X equal zero, and without loss of generality, that $X_{12} = 0$ and $X_{13} = 0$, so that

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & X_{23} \\ 0 & X_{23} & 1 \end{pmatrix}.$$

Since X is singular, $\det X = 0$ holds and so

$$1 - X_{23}^2 = 0,$$

which implies $X_{23} = \pm 1$.

Now suppose at most one element of X equals zero. Since X is singular, there is a linear dependency between its three columns and therefore there exist two scalars α and β, not both zero, such that

$$1 = \alpha\, X_{12} + \beta\, X_{13} \tag{3.3}$$

$$X_{12} = \alpha + \beta\, X_{23} \tag{3.4}$$

$$X_{13} = \alpha\, X_{23} + \beta \tag{3.5}$$

If $\alpha = 0$ then equation (3.5) implies $X_{13} = \beta$, therefore equation (3.3) implies $X_{13}^2 = 1$ and we are done. Similarly, if $\beta = 0$ then equation (3.4) implies $X_{12} = \alpha$, therefore equation (3.3) implies $X_{12}^2 = 1$ and we are again done.

Therefore we may suppose that $\alpha \neq 0$ and $\beta \neq 0$. Using equations (3.4) and (3.5) to substitute for X_{12} and X_{13} in equation (3.3) and solving for X_{23}, we obtain the equation

$$X_{23} = \frac{1 - \alpha^2 - \beta^2}{2\,\alpha\,\beta}. \tag{3.6}$$

Now, $\operatorname{rank} X = 2$ implies $\lambda_1(X) = 0$ and therefore, by the interlacing of eigenvalues, $\lambda_1(Y) = 0$. So $\det Y = 0$ and substituting for X_{12} and X_{13} in Y using (3.4) and (3.5), we have

$$\det \begin{pmatrix} 1 & \alpha + \beta\, X_{23} & \alpha\, X_{23} + \beta & X_{23} \\ \alpha + \beta\, X_{23} & 1 & X_{23} & \alpha\, X_{23} + \beta \\ \alpha\, X_{23} + \beta & X_{23} & 1 & \alpha + \beta\, X_{23} \\ X_{23} & \alpha\, X_{23} + \beta & \alpha + \beta\, X_{23} & 1 \end{pmatrix} = 0$$

which, upon expanding the determinant, is equivalent to

$$(1 - 2\,\alpha^2 - 2\,\beta^2 + \alpha^4 + \beta^4 - 2\,\alpha^2\,\beta^2)\,(1 - 2\,X_{23}^2 + X_{23}^4) = 0. \tag{3.7}$$

Equation (3.6) implies

$$\begin{aligned} X_{23}^2 &= \frac{(1 - \alpha^2 - \beta^2)^2}{4\,\alpha^2\,\beta^2} \\ &= \frac{1 - 2\,\alpha^2 - 2\,\beta^2 + \alpha^4 + \beta^4 + 2\,\alpha^2\,\beta^2}{4\,\alpha^2\,\beta^2} \end{aligned}$$

and therefore we have the equation

$$1 - X_{23}^2 = \frac{4\,\alpha^2\,\beta^2 - (1 - 2\,\alpha^2 - 2\,\beta^2 + \alpha^4 + \beta^4 + 2\,\alpha^2\,\beta^2)}{4\,\alpha^2\,\beta^2}$$

from which it follows that

$$-4\,\alpha^2\,\beta^2\,(1 - X_{23}^2) = 1 - 2\,\alpha^2 - 2\,\beta^2 + \alpha^4 + \beta^4 - 2\,\alpha^2\,\beta^2. \tag{3.8}$$

Substituting for the first term on the left-hand side of equation (3.7) using equation (3.8), we conclude that

$$(-4\,\alpha^2\,\beta^2\,(1-X_{23}^2))\,(1-2\,X_{23}^2+X_{23}^4)=0$$

or, equivalently,

$$-4\,\alpha^2\,\beta^2\,(1-X_{23}^2)^3=0.$$

Since $\alpha \neq 0$ and $\beta \neq 0$, we must have $1-X_{23}^2=0 \Leftrightarrow X_{23}=\pm 1$. This concludes the proof. ■

The next lemma shows that the rank correspondence between X and Y also holds for the rank-two case.

Lemma 3.12 *Let $Y \in \mathcal{F}_3$ and $X = \mathrm{sMat}\,(Y_{1:6,0})$. Then*

$$\operatorname{rank} Y = 2 \Leftrightarrow \operatorname{rank} X = 2.$$

Proof. If $\operatorname{rank} Y = 2$ then Theorem 3.6 yields

$$\lambda_1(Y) \leq \lambda_1(X) \leq \lambda_2(Y) \leq \lambda_2(X) \leq \lambda_3(Y) \leq \lambda_3(X) \leq \lambda_4(Y),$$

and $\lambda_1(Y) \geq 0$ by positive semidefiniteness.

Hence,

$$\begin{aligned} \operatorname{rank} Y = 2 \quad &\Rightarrow \quad \lambda_2(Y) = 0 \\ &\Rightarrow \quad \lambda_1(X) = 0 \\ &\Rightarrow \quad \operatorname{rank} X \leq 2. \end{aligned}$$

But by Lemma 3.8, $\operatorname{rank} X = 1 \Leftrightarrow \operatorname{rank} Y = 1$. Hence $\operatorname{rank} X = 2$.

Conversely, suppose that $\operatorname{rank} X = 2$. By Lemma 3.11, $X_{ij} = \pm 1$ for some pair $i, j, i \neq j$, and by Lemma 3.10 this implies $\operatorname{rank} Y \leq 2$. Finally, by Lemma 3.8, the rank of Y cannot equal 1 if $\operatorname{rank} X = 2$. Hence $\operatorname{rank} Y = 2$.

■

The only remaining possible value for $\operatorname{rank} X$ is 3, i.e. X is full-rank and thus lies in the relative interior of $\mathcal{E}_3$. This corresponds to Y having rank either 3 or 4. The next lemma shows that the rank of Y being equal to 3 characterizes the cases where one of the triangle inequalities holds tightly, and thus X lies in the relative interior of a facet of C_3.

Lemma 3.13 *Let $Y \in \mathcal{F}_3$ and $X = \mathrm{sMat}\,(Y_{1:6,0})$. Then*
$\operatorname{rank} Y = 3 \quad \Leftrightarrow \quad \operatorname{rank} X = 3$ *and one triangle inequality holds with equality.*

Proof. First suppose that $\operatorname{rank} X = 3$ and that

$$\alpha\, X_{12} + \beta\, X_{13} + \gamma\, X_{23} + 1 = 0$$

holds for given $\alpha, \beta, \gamma \in \{\pm 1\}$ such that $\alpha\,\beta\,\gamma = 1$. Then

$$\begin{aligned} Y \begin{pmatrix} 1 \\ \alpha \\ \beta \\ \gamma \end{pmatrix} &= \begin{pmatrix} 1 + \alpha\, X_{12} + \beta\, X_{13} + \gamma\, X_{23} \\ X_{12} + \alpha + \beta\, X_{23} + \gamma\, X_{13} \\ X_{13} + \alpha\, X_{23} + \beta + \gamma\, X_{12} \\ X_{23} + \alpha\, X_{13} + \beta\, X_{12} + \gamma \end{pmatrix} \\ &= \begin{pmatrix} 1 + \alpha\, X_{12} + \beta\, X_{13} + \gamma\, X_{23} \\ \frac{1}{\alpha}(\alpha\, X_{12} + \alpha^2 + \alpha\,\beta\, X_{23} + \alpha\,\gamma\, X_{13}) \\ \frac{1}{\beta}(\beta\, X_{13} + \alpha\,\beta\, X_{23} + \beta^2 + \beta\,\gamma\, X_{12}) \\ \frac{1}{\gamma}(\gamma\, X_{23} + \alpha\,\gamma\, X_{13} + \beta\,\gamma\, X_{12} + \gamma^2) \end{pmatrix} \\ &= 0, \end{aligned}$$

since $\alpha^2 = 1$, $\beta^2 = 1$, $\gamma^2 = 1$, and $\alpha\,\beta = \frac{1}{\gamma} = \gamma$, $\alpha\,\gamma = \frac{1}{\beta} = \beta$, $\beta\,\gamma = \frac{1}{\alpha} = \alpha$.

Since

$$\begin{pmatrix} 1 \\ \alpha \\ \beta \\ \gamma \end{pmatrix} \neq 0,$$

it follows that Y is singular, and so $\operatorname{rank} Y \leq 3$. Since $\operatorname{rank} X = 3$, by Lemmas 3.8 and 3.12, we conclude that $\operatorname{rank} Y = 3$.

Conversely, suppose that $\operatorname{rank} Y = 3$. By Lemmas 3.8 and 3.12, it follows that $\operatorname{rank} X = 3$. It remains to prove that one triangle inequality holds with equality.

Since $\operatorname{rank} Y = 3$, performing Gaussian elimination on Y will yield a matrix with the elements of the last row all equal to zeros. We will perform Gaussian elimination on Y, but before proceeding, we first observe that $\operatorname{rank} X = 3$ implies X is positive definite and thus all its principal submatrices are nonsingular. Specifically considering the top left 2×2 principal submatrix implies that $1 - X_{12}^2 \neq 0$. Furthermore,

$$\det X = 1 - X_{12}^2 - X_{13}^2 - X_{23}^2 + 2\, X_{12}\, X_{13}\, X_{23} > 0$$

since X is positive definite. Because we also have

$$\begin{aligned} (1 - X_{13}^2) - \frac{(X_{23} - X_{12}\, X_{13})^2}{1 - X_{12}^2} = 0 \quad &\Leftrightarrow \quad 1 - X_{12}^2 - X_{13}^2 - X_{23}^2 + 2\, X_{12}\, X_{13}\, X_{23} = 0 \\ &\Leftrightarrow \quad \det X = 0, \end{aligned}$$

and the latter does not hold, we conclude that

$$(1 - X_{13}^2) - \frac{(X_{23} - X_{12}\, X_{13})^2}{1 - X_{12}^2} \neq 0.$$

We now perform Gaussian elimination on Y. Applying type-3 elementary row operations as defined in [23], which consist of adding a scalar multiple of one row to another row, we can reduce Y to the form

$$\begin{pmatrix} 1 & X_{12} & X_{13} & X_{23} \\ 0 & 1-X_{12}^2 & \Delta_{231} & \Delta_{132} \\ 0 & 0 & (1-X_{13}^2)-\frac{\Delta_{231}^2}{1-X_{12}^2} & \Delta_{123}-\frac{\Delta_{231}\,\Delta_{132}}{1-X_{12}^2} \\ 0 & 0 & 0 & \left((1-X_{23}^2)-\frac{\Delta_{132}^2}{1-X_{12}^2}\right)-\frac{\left(\Delta_{123}-\frac{\Delta_{132}\,\Delta_{231}}{1-X_{12}^2}\right)^2}{\left((1-X_{13}^2)-\frac{\Delta_{231}^2}{1-X_{12}^2}\right)} \end{pmatrix},$$

where, for convenience, we use the notation

$$\Delta_{ijk} := X_{ij} - X_{ik}\,X_{kj}.$$

So, $\operatorname{rank} Y = 3$ implies

$$\left((1-X_{23}^2)-\frac{\Delta_{132}^2}{1-X_{12}^2}\right)-\frac{\left(\Delta_{123}-\frac{\Delta_{132}\,\Delta_{231}}{1-X_{12}^2}\right)^2}{\left((1-X_{13}^2)-\frac{\Delta_{231}^2}{1-X_{12}^2}\right)}=0$$

$\Leftrightarrow$

$$\left((1-X_{23}^2)-\frac{\Delta_{132}^2}{1-X_{12}^2}\right)\left((1-X_{13}^2)-\frac{\Delta_{231}^2}{1-X_{12}^2}\right)=\left(\Delta_{123}-\frac{\Delta_{132}\,\Delta_{231}}{1-X_{12}^2}\right)^2$$

$\Leftrightarrow$

$$\frac{[(1-X_{23}^2)(1-X_{12}^2)-(X_{13}-X_{12}\,X_{23})^2][(1-X_{13}^2)(1-X_{12}^2)-(X_{23}-X_{12}\,X_{13})^2]}{(1-X_{12}^2)^2}$$

$$=\frac{[(1-X_{12}^2)(X_{12}-X_{13}\,X_{23})-(X_{13}-X_{12}\,X_{23})(X_{23}-X_{12}\,X_{13})]^2}{(1-X_{12}^2)^2}$$

$\Leftrightarrow$

$$(1-X_{12}^2-X_{13}^2-X_{23}^2+2\,X_{12}\,X_{13}\,X_{23})^2=(X_{12}-2\,X_{13}\,X_{23}-X_{12}^3+X_{12}\,X_{13}^2+X_{12}\,X_{23}^2)^2$$

and therefore *exactly* one of the following equalities holds:

$$(1-X_{12}^2-X_{13}^2-X_{23}^2+2\,X_{12}\,X_{13}\,X_{23})=(X_{12}-2\,X_{13}\,X_{23}-X_{12}^3+X_{12}\,X_{13}^2+X_{12}\,X_{23}^2) \tag{3.9}$$

or

$$(1-X_{12}^2-X_{13}^2-X_{23}^2+2\,X_{12}\,X_{13}\,X_{23})=-(X_{12}-2\,X_{13}\,X_{23}-X_{12}^3+X_{12}\,X_{13}^2+X_{12}\,X_{23}^2) \tag{3.10}$$

Both cannot hold because the left-hand side equals $\det X$ and is therefore non-zero.

Case 1: Equation (3.9) holds: In this case, we have that equation (3.9) holds if and only if

$$1-X_{12}^2-X_{12}+X_{12}^3-(X_{13}^2+X_{23}^2-2\,X_{13}\,X_{23})+X_{12}\,(2\,X_{13}\,X_{23}-X_{13}^2-X_{23}^2)=0$$

$$\Leftrightarrow$$

$$(1-X_{12}^2)(1-X_{12})-(X_{13}-X_{23})^2-X_{12}\,(X_{13}-X_{23})^2=0$$

$$\Leftrightarrow$$

$$(1-X_{12})^2(1+X_{12})-(X_{13}-X_{23})^2(1+X_{12})=0.$$

Since we know $X_{12}\neq -1$, the last equation is equivalent to

$$(1-X_{12})^2=(X_{13}-X_{23})^2.$$

Therefore we have two subcases: either

$$1-X_{12}=X_{13}-X_{23} \tag{3.11}$$

or

$$1-X_{12}=-(X_{13}-X_{23}). \tag{3.12}$$

If (3.11) holds, then the triangle inequality $-X_{12}-X_{13}+X_{23}\geq -1$ holds with equality. If (3.12) holds, then the triangle inequality $-X_{12}+X_{13}-X_{23}\geq -1$ holds with equality. Hence if equation (3.9) holds, we have indeed one triangle inequality that holds with equality.

Case 2: Equation (3.10) holds: In this case, we have that equation (3.10) holds if and only if

$$1-X_{12}^2+X_{12}-X_{12}^3-(X_{13}^2+X_{23}^2+2\,X_{13}\,X_{23})+X_{12}\,(2\,X_{13}\,X_{23}+X_{13}^2+X_{23}^2)=0$$

$$\Leftrightarrow$$

$$(1+X_{12}^2)(1-X_{12})-(1-X_{12})(X_{13}+X_{23})^2=0,$$

by an argument similar to the one in Case 1. This equation is equivalent to

$$(1+X_{12})^2=(X_{13}+X_{23})^2$$

therefore we again have two subcases: either

$$1+X_{12}=X_{13}+X_{23} \tag{3.13}$$

or

$$1+X_{12}=-(X_{13}+X_{23}). \tag{3.14}$$

If (3.13) holds, then the triangle inequality $X_{12}-X_{13}-X_{23}\geq -1$ holds with equality. If (3.14) holds, then the triangle inequality $X_{12}+X_{13}+X_{23}\geq -1$ holds with equality. So if equation (3.10) holds, we also have one triangle inequality that holds with equality.

This completes the proof. ■

Theorem 3.14 summarizes the case $n = 3$. It shows how the rank of Y characterizes the dimension of the face of C_3 where X lies.

Theorem 3.14 *Let $Y \in \mathcal{F}_3$ and $X = \mathrm{sMat}\,(Y_{1:6,0})$. Then $X \in C_3$ and*

- $\operatorname{rank} Y = 1$ *iff X is a vertex of C_3;*
- $\operatorname{rank} Y = 2$ *iff X lies in the relative interior of a face of C_3 of dimension 1;*
- $\operatorname{rank} Y = 3$ *iff X lies in the relative interior of a facet (face of dimension 2) of C_3;*
- $\operatorname{rank} Y = 4$ *iff X lies in the relative interior of C_3.*

Proof. Since $F_3 = C_3$, the first two statements follow by Theorem 3.4 and Lemmas 3.8 and 3.12. The third statement then follows from Lemma 3.13 and the fact that the triangle inequalities define all the facets of C_3. Finally, if no triangle inequality holds with equality, then this last fact also implies that X must lie in the interior of the cut polytope. ■

The following corollary is immediate.

Corollary 3.15 *Let $Y^* \in \mathcal{F}_3$ be optimal for SDP2. Then $\nu_3^* = \mu^*$, and if* $\operatorname{rank} Y = 3$*, then*

$$\nu_3^* < \nu_1^*.$$

Proof. The equality follows from $F_3 = C_3$, and the strict inequality from the fact that if $\operatorname{rank} Y = 3$, then $X = \mathrm{sMat}\,(Y_{1:6,0}) \in \operatorname{relint} \mathcal{E}_3$ and hence cannot be optimal for SDP1. ■

This concludes our study of the case $n = 3$. In the next section, we show that the characterization of the vertices of the cut polytope via $\operatorname{rank} Y = 1$ extends to all $n \geq 4$. More interestingly, we show that the characterization of the one-dimensional faces via $\operatorname{rank} Y = 2$ also holds for $n \geq 4$. Furthermore, we will prove that if $\operatorname{rank} Y = 2$ for $n \geq 3$, then it is possible to extract the two rank-one matrices (corresponding to cuts) which are the vertices of the one-dimensional face of the cut polytope where X lies. Indeed, we will present an algorithm that does precisely this.

4 Ranks and Low Dimensional Faces of C_n

Theorems 2.3 and 3.7 show that SDP1 characterizes the vertices of C_n as exactly those matrices in the elliptope that have rank equal to one. This implies that if X^* is optimal for SDP1 and $\operatorname{rank} X^* = 1$ then $\nu_1^* = \mu^*$, i.e. SDP1 yields the optimal value of Max-Cut. Furthermore, since X^* is rank-one, SDP1 also yields an optimal cut.

In this section we show that this characterization extends to SDP2, i.e. $Y \in \mathcal{F}_n$ is rank-one precisely when $X = \operatorname{sMat}(Y_{1:t(n),0})$ is a vertex of the cut polytope. Thus, if Y^* is optimal for SDP2 and $\operatorname{rank} Y^* = 1$ then $X^* = \operatorname{sMat}(Y^*_{1:t(n),0})$ is also rank-one, $\nu_3^* = \mu^*$ holds, and SDP2 yields an optimal cut.

We then proceed to prove a stronger property for SDP2, namely that $Y \in \mathcal{F}_n$ is rank-two if and only if $X = \operatorname{sMat}(Y_{1:t(n),0})$ is rank-two. In fact, this matrix X has a very special structure which leads to a simple algorithm that extracts from X two rank-one matrices (corresponding to cuts) which are the vertices of a one-dimensional face of the cut polytope. Thus we prove that X lies in the relative interior of a face of C_n of dimension 1. Hence, if Y^* is optimal for SDP2 and $\operatorname{rank} Y^* \leq 2$ then $\nu_3^* = \mu^*$ holds, and SDP2 yields both the optimal value and an optimal cut for Max-Cut (see Corollary 4.11).

For simplicity of the proofs, in this section we continue to work with the principal submatrix of Y corresponding to the rows and columns $\{0\} \cup \{T(i,j) \mid 1 \leq i < j \leq n\}$. Recall that because the rank of Y always equals the rank of this principal submatrix, all our results hold identically for the entire matrix Y.

The study of the geometry of F_n for $n \geq 4$ is complicated by the fact that given $X \in F_n$, we do not immediately have a full description of the corresponding variable Y feasible for SDP2. Indeed, as was observed in [2], given $X \in F_n$, we have only partial knowledge about Y. Hence we have to address two different, but related, issues:

1. Determine the possible value(s) that the unspecified elements of Y can take, for a given $X \in F_n$;

2. Hence determine the possible rank(s) that the matrix Y can have, given the rank of X.

We begin by settling the rank-one case. There are no surprises there, as the matrix Y is unique for every rank-one X and the characterization of Theorem 3.14 carries through. We then proceed in Section 4.2 to the rank-two case where we prove that, surprisingly, the characterization from Theorem 3.14 also holds.

4.1 Rank Characterization of Vertices

The next theorem gives the complete description of the rank-one case.

Theorem 4.1 *For $Y \in \mathcal{F}_n$ and $X = \operatorname{sMat}(Y_{1:t(n),0})$:*

1. *If* $\operatorname{rank} X = 1$ *then the unspecified elements of* Y *are uniquely determined by*
$$Y_{T(i,j),T(k,l)} = X_{ij}\, X_{kl}$$
for all 4-tuples (i,j,k,l) *of distinct elements from* $\{1,\ldots,n\}$*. Hence* Y *is uniquely determined by* X*.*

2. $\operatorname{rank} Y = 1 \quad \Leftrightarrow \quad \operatorname{rank} X = 1 \quad \Leftrightarrow \quad X$ *is a vertex of* C_n*.*

Proof. If $\operatorname{rank} X = 1$ then Theorem 3.7 implies $|X_{ij}| = 1$ for all i,j, since X is positive semidefinite. Given any 4-tuple (i,j,k,l) of distinct elements from $\{1,\ldots,n\}$, consider the 3×3 principal submatrix of Y corresponding to rows and columns $\{0, T(i,j), T(k,l)\}$:

$$\begin{pmatrix} 1 & X_{ij} & X_{kl} \\ X_{ij} & 1 & y \\ X_{kl} & y & 1 \end{pmatrix} \succeq 0,$$

where y denotes the element $Y_{T(i,j),T(k,l)}$. By Lemma 3.9, $|X_{ij}| = 1$ implies that $y = X_{ij}\, X_{kl}$. This proves the first claim.

The implication $\operatorname{rank} Y = 1 \Rightarrow \operatorname{rank} X = 1$ is clear, and the implication

$$X \text{ is a vertex of } C_n \Rightarrow \operatorname{rank} X = 1$$

follows by Lemma 2.2.

If $\operatorname{rank} X = 1$ then $\operatorname{diag} X = e$ implies $X_{ij} = v_i\, v_j$ for some $v \in \{\pm 1\}^n$ and thus, by Lemma 2.2, X is a vertex of C_n. Furthermore, let $x = \operatorname{svec}(v\, v^T) = \operatorname{svec}(X)$, so that $x_{T(i,j)} = v_i\, v_j$, and consider the matrix

$$\hat{Y} = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & x\, x^T \end{pmatrix}.$$

It is clear that $\operatorname{sMat}(\hat{Y}_{1:t(n),0})$ equals X. Furthermore,

$$\hat{Y}_{T(i,k),T(k,j)} = x_{T(i,k)}\, x_{T(k,j)} = v_i\, v_k\, v_k\, v_j = X_{ij} = Y_{T(i,k),T(k,j)}$$

and

$$\hat{Y}_{T(i,j),T(k,l)} = x_{T(i,j)}\, x_{T(k,l)} = v_i\, v_j\, v_k\, v_l = X_{ij}\, X_{kl} = Y_{T(i,j),T(k,l)}.$$

Hence, $\hat{Y} = Y$ and so $\operatorname{rank} X = 1 \Rightarrow \operatorname{rank} Y = 1$. This concludes the proof. ■

This settles the rank-one case. We proceed to the case where $\operatorname{rank} X = 2$.

4.2 Rank Characterization of One-Dimensional Faces

4.2.1 Implications of Rank-Two for $X \in F_n$

In this section we prove that if $X \in F_n$ and $\operatorname{rank} X = 2$, then X has a very particular structure. We first prove Lemma 4.2 which is a straightforward generalization of previous results.

Lemma 4.2 *If $X \in F_n$ and $\operatorname{rank} X = 2$, then for every triple (i,j,k) of distinct indices from $\{1,\ldots,n\}$, at least one of X_{ij}, X_{ik}, X_{jk} has absolute value equal to 1.*

Proof. Given i,j,k, consider the 3×3 principal submatrix of X indexed by rows and columns $\{0, T(i,j), T(i,k)\}$:

$$X_{i,j,k} = \begin{pmatrix} 1 & X_{ij} & X_{ik} \\ X_{ij} & 1 & X_{jk} \\ X_{ik} & X_{jk} & 1 \end{pmatrix}.$$

Since $\operatorname{rank} X = 2$, $\operatorname{rank} X_{i,j,k} \leq 2$, and clearly $X_{i,j,k} \succeq 0$.

We have two cases. If $\operatorname{rank} X_{i,j,k} = 1$, then Theorem 3.7 implies $|X_{ij}|$, $|X_{ik}|$, and $|X_{jk}|$ all equal 1, and we are done. If $\operatorname{rank} X_{i,j,k} = 2$, then considering the 4×4 submatrix of Y

$$\begin{pmatrix} 1 & X_{ij} & X_{ik} & X_{jk} \\ X_{ij} & 1 & X_{jk} & X_{ik} \\ X_{ik} & X_{jk} & 1 & X_{ij} \\ X_{jk} & X_{ik} & X_{ij} & 1 \end{pmatrix}$$

and applying Lemma 3.11, we conclude that one of X_{ij}, X_{ik}, X_{jk} has absolute value equal to 1. ■

We now apply Lemma 4.2 to prove a somewhat surprising result which is essential for proving the results in this section about characterizing the one-dimensional faces of C_n.

Lemma 4.3 *Suppose $X \in F_n$, $n \geq 4$ and $\operatorname{rank} X = 2$. Then there exists a scalar $\alpha \in [0,1)$ such that for all off-diagonal entries X_{ij} of X, either*

$$|X_{ij}| = 1 \quad \text{or} \quad |X_{ij}| = \alpha.$$

Furthermore, there is at least one occurrence of $\pm\alpha$ in the first row of X, i.e. there exists $j \in \{2,\ldots,n\}$ such that $|X_{1j}| = \alpha$.

Proof. The proof is by contradiction. Suppose there exist $\alpha \in (-1,1)$ and $\beta \in (-1,1)$ such that $|\alpha| \neq |\beta|$ and $X_{ij} = \alpha$, $X_{kl} = \beta$ for indices i,j,k,l from $\{1,\ldots,n\}$. We have two cases.

Case 1: $\{i,j\} \cap \{k,l\} \neq \emptyset$: In this case, suppose $j = l$ without loss of generality, so $X_{ij} = \alpha$ and $X_{jk} = \beta$. Consider the 3×3 principal submatrix of X:

$$X_{i,j,k} = \begin{pmatrix} 1 & X_{ij} & X_{ik} \\ X_{ij} & 1 & X_{jk} \\ X_{ik} & X_{jk} & 1 \end{pmatrix} \succeq 0.$$

Clearly $\operatorname{rank} X_{i,j,k} \leq 2$ and since $|X_{ij}| \neq 1$ and $|X_{jk}| \neq 1$, by Theorem 3.7 $\operatorname{rank} X_{i,j,k} \neq 1$. Hence $\operatorname{rank} X_{i,j,k} = 2$ and by Lemma 4.2, $|X_{ik}| = 1$ holds. Now, Lemma 3.9 implies $X_{ij} = X_{ik}\, X_{jk}$, so $\alpha = \pm\beta$ and $|\alpha| = |\beta|$, which is a contradiction. This completes Case 1.

Case 2: $\{i,j\} \cap \{k,l\} = \emptyset$: As in the first case, we consider the principal submatrix $X_{i,j,k}$ of X which, as in Case 1, has rank equal to 2 (since $|X_{ij}| \neq 1$). Therefore (by Lemma 4.2), either $|X_{ik}| = 1$ or $|X_{jk}| = 1$.

If $|X_{ik}| = 1$ then by Lemma 3.9, $X_{ij} = X_{ik}\, X_{jk}$, so $|X_{jk}| = |X_{ij}| = |\alpha|$. Now consider another 3×3 principal submatrix of X:

$$X_{j,k,l} = \begin{pmatrix} 1 & X_{jk} & X_{jl} \\ X_{jk} & 1 & X_{kl} \\ X_{jl} & X_{kl} & 1 \end{pmatrix} \succeq 0.$$

Since $|X_{kl}| \neq 1$, $\operatorname{rank} X_{j,k,l} = 2$. Hence one of its other two entries has absolute value equal to one. But $|X_{jk}| = |\alpha| \neq 1$, hence $|X_{jl}| = 1$. By Lemma 3.9, $X_{jk} = X_{jl}\, X_{kl}$ and therefore $|X_{jk}| = |X_{kl}|$, or equivalently, $|\alpha| = |\beta|$, which contradicts our assumption that $|\alpha| \neq |\beta|$.

Alternatively, if $|X_{jk}| = 1$ then since the principal submatrix $X_{i,j,k}$ has rank equal to 2, it follows that $X_{ij} = X_{jk}\, X_{ik}$, i.e. $|X_{ij}| = |X_{ik}| = |\alpha|$. Now consider the 3×3 principal submatrix of X:

$$X_{i,k,l} = \begin{pmatrix} 1 & X_{ik} & X_{il} \\ X_{ik} & 1 & X_{kl} \\ X_{il} & X_{kl} & 1 \end{pmatrix} \succeq 0.$$

Since $|X_{ik}| \neq 1$ and $|X_{kl}| = |\beta| \neq 1$, $|X_{il}| = 1$ must hold. But then Lemma 3.9 implies $|X_{ik}| = |X_{kl}|$, or equivalently, $|\alpha| = |\beta|$, again a contradiction. This completes Case 2, and the proof of the first claim.

Finally, suppose that $|X_{1j}| = 1$ for all $j = 2, \ldots, n$. Then the 3×3 principal submatrix of X:

$$X_{1,i,j} = \begin{pmatrix} 1 & X_{1i} & X_{1j} \\ X_{1i} & 1 & X_{ij} \\ X_{1j} & X_{ij} & 1 \end{pmatrix} \succeq 0$$

implies, by Lemma 3.9, that $|X_{ij}| = 1$ for all i, j. But then, since $X \succeq 0$, Theorem 3.7 implies that $\operatorname{rank} X = 1$, a contradiction. Hence, we conclude that there exists $j \in \{2, \ldots, n\}$ such that $|X_{1j}| = \alpha$. ■

4.2.2 Algorithm for Constructing the Vertex Cuts

In this section we present an algorithm which, given $X \in F_n$ satisfying $\operatorname{rank} X = 2$, explicitly constructs two cut vectors v_1 and v_2 in $\{\pm 1\}^n$ and non-negative scalars α_1, α_2 such that

$$X = \alpha_1 \, v_1 \, v_1^T \; + \; \alpha_2 \, v_2 \, v_2^T \, , \quad \alpha_1 + \alpha_2 = 1.$$

By Lemma 2.2, this immediately implies that X is in the relative interior of a one-dimensional face of C_n.

We note that the case $\alpha = 0$, where α is the scalar defined in the statement of Lemma 4.3, will require special attention in the algorithm.

Algorithm 4.4 *(Construction of Extreme Points for $X \in F_n$, $\operatorname{rank} X = 2$)*

Input: *Matrix $X \in F_n$ satisfying $\operatorname{rank} X = 2$, and scalar $\alpha \in [0, 1)$ (as defined in the statement of Lemma 4.3).*

If $\alpha \neq 0$ **then**

1. *Let* $\alpha_1 = \frac{1+\alpha}{2}$, $\alpha_2 = \frac{1-\alpha}{2}$, $(v_1)_1 = 1$ *and* $(v_2)_1 = 1$.
2. *For* $j = 2, 3, \ldots, n$

   if $X_{1j} = 1$ **then** *let* $(v_1)_j = 1$ *and* $(v_2)_j = 1$;
   if $X_{1j} = -1$ **then** *let* $(v_1)_j = -1$ *and* $(v_2)_j = -1$;
   if $X_{1j} = \alpha$ **then** *let* $(v_1)_j = 1$ *and* $(v_2)_j = -1$;
   if $X_{1j} = -\alpha$ **then** *let* $(v_1)_j = -1$ *and* $(v_2)_j = 1$.

If $\alpha = 0$ **then** *by Lemma 4.3, and by symmetrically permuting rows and columns if necessary, we may assume without loss of generality that $X_{12} = \alpha = 0$. Note that, with this assumption, if $X_{1j} = 0$ then $|X_{2j}| = 1$ by Lemma 4.2.*

1. *Let* $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $(v_1)_1 = 1$, $(v_2)_1 = 1$, $(v_1)_2 = 1$ *and* $(v_2)_2 = -1$.
2. *For* $j = 3, 4, \ldots, n$

   if $X_{1j} = 1$ **then** *let* $(v_1)_j = 1$ *and* $(v_2)_j = 1$;
   if $X_{1j} = -1$ **then** *let* $(v_1)_j = -1$ *and* $(v_2)_j = -1$;
   if $X_{1j} = 0$ **then** *as observed above, we have two subcases:*
      if $X_{2j} = 1$ **then** *let* $(v_1)_j = 1$ *and* $(v_2)_j = -1$;
      if $X_{2j} = -1$ **then** *let* $(v_1)_j = -1$ *and* $(v_2)_j = 1$.

Output: v_1, v_2, α_1, *and* α_2.

We demonstrate the application of Algorithm 4.4 in the following two examples.

Example 4.5 *($\alpha \neq 0$) Consider*

$$X = \begin{pmatrix} 1 & \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} & 1 \\ 1 & \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} & 1 \end{pmatrix}.$$

It is straightforward to check that $\operatorname{rank} X = 2$*, and* $X \in F_4$ *since*

$$Y = \begin{pmatrix} 1 & 1 & \frac{1}{3} & 1 & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 \\ 1 & 1 & \frac{1}{3} & 1 & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} \\ 1 & 1 & \frac{1}{3} & 1 & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 \\ 1 & 1 & \frac{1}{3} & 1 & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} \\ 1 & 1 & \frac{1}{3} & 1 & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} \\ 1 & 1 & \frac{1}{3} & 1 & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} \\ 1 & 1 & \frac{1}{3} & 1 & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 & \frac{1}{3} & 1 \end{pmatrix} \in \mathcal{F}_4$$

and $\operatorname{sMat}(Y_{1:10,0}) = X$*. Note that Lemma 4.3 holds with* $\alpha = \frac{1}{3}$*.*

Let us apply Algorithm 4.4:

- $\alpha_1 = \frac{1+\alpha}{2} = \frac{2}{3}$, $\alpha_2 = \frac{1-\alpha}{2} = \frac{1}{3}$, $(v_1)_1 = 1$ *and* $(v_2)_1 = 1$*;*
- $X_{12} = \frac{1}{3} = \alpha$*, so* $(v_1)_2 = 1$ *and* $(v_2)_2 = -1$*;*
- $X_{13} = 1$*, so* $(v_1)_3 = 1$ *and* $(v_2)_3 = 1$*;*
- $X_{14} = \frac{1}{3} = \alpha$*, so* $(v_1)_4 = 1$ *and* $(v_2)_4 = -1$*.*

Hence the algorithm yields $\alpha_1 = \frac{2}{3}$*,* $\alpha_2 = \frac{1}{3}$*,* $v_1 = (1,1,1,1)^T$ *and* $v_2 = (1,-1,1,-1)^T$*. It is easy to verify that* $\alpha_1 v_1 v_1^T + \alpha_2 v_2 v_2^T = X$*.*

Example 4.6 *($\alpha = 0$) Consider*

$$X = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Again $\operatorname{rank} X = 2$, *and* $X \in F_4$ *because*

$$Y = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ -1 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{F}_4$$

and $\operatorname{sMat}(Y_{1:10,0}) = X$.

Note $X_{12} = 0$, *so* $X_{12} = \alpha$ *holds. Applying Algorithm 4.4:*

- $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $(v_1)_1 = 1$, $(v_2)_1 = 1$, $(v_1)_2 = 1$, *and* $(v_2)_2 = -1$;
- $X_{13} = 0$, *so we check* X_{23} *(which must equal* ± 1*):* $X_{23} = -1$, *so* $(v_1)_3 = -1$ *and* $(v_2)_3 = 1$;
- $X_{14} = -1$, *so* $(v_1)_4 = -1$ *and* $(v_2)_4 = -1$.

Hence the algorithm yields $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $v_1 = (1, 1, -1, -1)^T$ *and* $v_2 = (1, -1, 1, -1)^T$. *It is again easy to verify that* $\alpha_1 v_1 v_1^T + \alpha_2 v_2 v_2^T = X$.

We now prove the correctness of Algorithm 4.4. It is straightforward to check that $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, and $\alpha_1 - \alpha_2 = \alpha$. If we define

$$X^{\mathrm{alg}} := \alpha_1 v_1 v_1^T + \alpha_2 v_2 v_2^T,$$

we have the following result:

Lemma 4.7 *Let* $X \in F_n$, $\operatorname{rank} X = 2$, *and suppose that* v_1, v_2, α_1, *and* α_2 *are computed by Algorithm 4.4. Then* $X = X^{\mathrm{alg}}$.

Proof. The proof has two cases which parallel the two cases in Algorithm 4.4. However, first observe that for both cases,

$$X_{ii}^{\mathrm{alg}} = \alpha_1 (v_1)_i^2 + \alpha_2 (v_2)_i^2 = \alpha_1 \cdot 1 + \alpha_2 \cdot 1 = 1,$$

therefore $\operatorname{diag}(X^{\mathrm{alg}}) = e = \operatorname{diag}(X)$ holds for both cases. Now consider each case in turn.

Case 1: $\alpha \neq 0$: First check that the first rows of X and X^{alg} are equal.

For $j = 2, 3, \ldots, n$

$$\begin{array}{rcl} X^{\text{alg}}_{1j} & = & \alpha_1 (v_1)_1 (v_1)_j + \alpha_2 (v_2)_1 (v_2)_j \\ & = & \begin{cases} \alpha_1 \cdot (1) + \alpha_2 \cdot (1) = 1, & \text{if } X_{1j} = 1 \\ \alpha_1 \cdot (-1) + \alpha_2 \cdot (-1) = -1, & \text{if } X_{1j} = -1 \\ \alpha_1 \cdot (1) + \alpha_2 \cdot (-1) = \alpha_1 - \alpha_2 = \alpha, & \text{if } X_{1j} = \alpha \\ \alpha_1 \cdot (-1) + \alpha_2 \cdot (1) = \alpha_2 - \alpha_1 = -\alpha, & \text{if } X_{1j} = -\alpha \end{cases} \end{array}$$

therefore their first rows are equal.

Now for each pair of indices i, j from $\{2, \ldots, n\}$, $i \neq j$, Lemma 4.2 implies that at least one of X_{1i}, X_{1j}, X_{ij} has absolute value equal to 1. We consider three cases:

If $|X_{1i}| = 1$ **then** by Lemma 3.9, $X_{ij} = X_{1i} X_{1j}$. Let $X_{1i} = \sigma, \sigma^2 = 1$, hence the algorithm sets $(v_1)_i = \sigma$ and $(v_2)_i = \sigma$. We now consider two subcases:

If $X_{1j} = \delta, \delta^2 = 1$ **then** the algorithm sets $(v_1)_j = \delta$ and $(v_2)_j = \delta$ and therefore

$$\begin{array}{rcl} X^{\text{alg}}_{ij} & = & \alpha_1 \sigma \delta + \alpha_2 \sigma \delta \\ & = & \sigma \delta \\ & = & X_{1i} X_{1j} \\ & = & X_{ij}. \end{array}$$

If $X_{1j} = \delta \alpha, \delta^2 = 1$ **then** the algorithm sets $(v_1)_j = \delta$ and $(v_2)_j = -\delta$ and therefore

$$\begin{array}{rcl} X^{\text{alg}}_{ij} & = & \alpha_1 \sigma \delta + \alpha_2 \sigma (-\delta) \\ & = & \delta (\alpha_1 - \alpha_2) \sigma \\ & = & (\delta \alpha) \sigma \\ & = & X_{1j} X_{1i} \\ & = & X_{ij}. \end{array}$$

Hence, if $|X_{1i}| = 1$, we are done.

If $|X_{1j}| = 1$ **then** again by Lemma 3.9, $X_{ij} = X_{1i} X_{1j}$. Applying the same argument as for the previous case, we are done.

Otherwise, we have $X_{1i} = \sigma \alpha, \sigma^2 = 1$, $X_{1j} = \delta \alpha, \delta^2 = 1$, and $|X_{ij}| = 1$. In this case, the algorithm sets

$$(v_1)_i = \sigma, (v_2)_i = -\sigma, (v_1)_j = \delta, \text{ and } (v_2)_j = -\delta.$$

Furthermore, Lemma 3.9 implies $X_{1i} = X_{ij} X_{1j}$, which can be rewritten as $\sigma \alpha = X_{ij} \delta \alpha$, and since $\alpha \neq 0$, $X_{ij} = \sigma \delta$ (because $\delta^2 = 1$). Now,

$$\begin{array}{rcl} X^{\text{alg}}_{ij} & = & \alpha_1 \sigma \delta + \alpha_2 (-\sigma)(-\delta) \\ & = & (\alpha_1 + \alpha_2) \sigma \delta \\ & = & \sigma \delta \\ & = & X_{ij}. \end{array}$$

Therefore, if $\alpha \neq 0$, $X^{\text{alg}} = X$.

<u>**Case 2:** $\alpha = 0$**:**</u> As in the description of Algorithm 4.4, we suppose without loss of generality that $X_{12} = 0$. Now,

$$X_{12}^{\text{alg}} = \frac{1}{2}(v_1)_1(v_1)_2 + \frac{1}{2}(v_2)_1(v_2)_2 = \frac{1}{2} - \frac{1}{2} = 0 = X_{12},$$

as desired.

Next we consider the remaining non-diagonal entries of the first two rows of X^{alg}. For each $j = 3, 4, \dots, n$, there are two cases:

either $|X_{1j}| = 1$**,** so that $X_{1j} = \sigma, \sigma^2 = 1$ and the algorithm sets $(v_1)_j = \sigma$ and $(v_2)_j = \sigma$. Then Lemma 3.9 implies $X_{2j} = X_{1j} X_{12}$, and so $X_{2j} = 0$. Now,

$$\begin{array}{rcl} X_{1j}^{\text{alg}} & = & \frac{1}{2}(v_1)_j + \frac{1}{2}(v_2)_j \\ & = & \sigma \quad = \quad X_{1j}, \end{array}$$

and

$$\begin{array}{rcl} X_{2j}^{\text{alg}} & = & \frac{1}{2}(v_1)_2(v_1)_j + \frac{1}{2}(v_2)_2(v_2)_j \\ & = & \frac{1}{2}\sigma - \frac{1}{2}\sigma \\ & = & 0 \quad = \quad X_{2j}. \end{array}$$

or $X_{1j} = 0$**,** in which case Lemma 4.2 implies $|X_{2j}| = 1$ so $X_{2j} = \sigma, \sigma^2 = 1$, and the algorithm sets $(v_1)_j = \sigma$, $(v_2)_j = -\sigma$. Thus,

$$\begin{array}{rcl} X_{1j}^{\text{alg}} & = & \frac{1}{2}(v_1)_j + \frac{1}{2}(v_2)_j \\ & = & \frac{1}{2}\sigma + \frac{1}{2}(-\sigma) \\ & = & 0 \quad = \quad X_{1j} \end{array}$$

and

$$\begin{array}{rcl} X_{2j}^{\text{alg}} = & = & \frac{1}{2}(v_1)_2(v_1)_j + \frac{1}{2}(v_2)_2(v_2)_j \\ & = & \frac{1}{2}\sigma + \frac{1}{2}(-1)(-\sigma) \\ & = & \sigma \quad = \quad X_{2j} \end{array}$$

This proves that the first two rows of X^{alg} and X are identical.

Now consider any pair of indices i, j from $\{3, \dots, n\}$, $i \neq j$ and the 4×4 principal submatrix of X:

$$\begin{pmatrix} 1 & X_{12} & X_{1i} & X_{1j} \\ X_{12} & 1 & X_{2i} & X_{2j} \\ X_{1i} & X_{2i} & 1 & X_{ij} \\ X_{1j} & X_{2j} & X_{ij} & 1 \end{pmatrix} \succeq 0.$$

Since $X_{12} = 0$, the top left 2×2 principal submatrix is invertible, so applying the Schur complement (Theorem 3.5), the above principal submatrix is positive semidefinite if and only if

$$\begin{pmatrix} 1 & X_{ij} \\ X_{ij} & 1 \end{pmatrix} - \begin{pmatrix} X_{1i} & X_{2i} \\ X_{1j} & X_{2j} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} X_{1i} & X_{1j} \\ X_{2i} & X_{2j} \end{pmatrix} \succeq 0$$

$$\Leftrightarrow \begin{pmatrix} 1-(X_{1i}^2+X_{2i}^2) & X_{ij}-(X_{1i}\,X_{1j}+X_{2i}\,X_{2j}) \\ X_{ij}-(X_{1i}\,X_{1j}+X_{2i}\,X_{2j}) & 1-(X_{1j}^2+X_{2j}^2) \end{pmatrix} \succeq 0. \tag{4.15}$$

The fact that $X_{12}=0$ also implies:

- by Lemma 4.2, that
  - one of X_{1i}, X_{2i} has absolute value equal to 1, and
  - one of X_{1j}, X_{2j} has absolute value equal to 1;
- by Lemma 3.9, that the other element of each of the above two pairs must equal 0.

Hence,

$$X_{1i}^2+X_{2i}^2=1 \quad \text{and} \quad X_{1j}^2+X_{2j}^2=1,$$

and therefore the matrix inequality (4.15) implies that

$$X_{ij}=X_{1i}\,X_{1j}+X_{2i}\,X_{2j}.$$

We can now proceed to checking that X^{alg} indeed equals the right-hand side of this equation. There are four subcases:

Case i: $|X_{1i}|=1$ and $|X_{1j}|=1$: Let $X_{1i}=\sigma, \sigma^2=1$ and $X_{1j}=\delta, \delta^2=1$, and we also know $X_{2i}=0$ and $X_{2j}=0$. The algorithm sets $(v_1)_i=\sigma$, $(v_2)_i=\sigma$, $(v_1)_j=\delta$ and $(v_2)_j=\delta$. Therefore,

$$\begin{aligned} X_{ij}^{\text{alg}} &= \tfrac{1}{2}\,(v_1)_i\,(v_1)_j+\tfrac{1}{2}\,(v_2)_i\,(v_2)_j \\ &= \sigma\,\delta \\ &= X_{1i}\,X_{1j} \\ &= X_{1i}\,X_{1j}+X_{2i}\,X_{2j}, \quad \text{since } X_{2i}\,X_{2j}=0 \\ &= X_{ij}. \end{aligned}$$

Case ii: $|X_{1i}|=1$ and $|X_{2j}|=1$: Let $X_{1i}=\sigma, \sigma^2=1$ and $X_{2j}=\delta, \delta^2=1$, and also $X_{2i}=0$ and $X_{1j}=0$. The algorithm sets $(v_1)_i=\sigma$, $(v_2)_i=\sigma$, $(v_1)_j=\delta$ and $(v_2)_j=-\delta$. Therefore,

$$\begin{aligned} X_{ij}^{\text{alg}} &= \tfrac{1}{2}\,(v_1)_i\,(v_1)_j+\tfrac{1}{2}\,(v_2)_i\,(v_2)_j \\ &= \tfrac{1}{2}\,\sigma\,\delta+\tfrac{1}{2}\,\sigma\,(-\delta) \\ &= 0 \\ &= X_{1i}\,X_{1j}+X_{2i}\,X_{2j}, \quad \text{since } X_{1j}=0 \text{ and } X_{2i}=0 \\ &= X_{ij}. \end{aligned}$$

Case iii: $|X_{2i}|=1$ and $|X_{1j}|=1$: Let $X_{2i}=\sigma, \sigma^2=1$ and $X_{1j}=\delta, \delta^2=1$, and also $X_{2j}=0$ and $X_{1i}=0$. The algorithm sets $(v_1)_i=\sigma$, $(v_2)_i=-\sigma$, $(v_1)_j=\delta$ and $(v_2)_j=\delta$, and it is straightforward to check that $X_{ij}^{\text{alg}}=X_{ij}$ by applying the same kind of argument as in the previous cases.

Case iv: $|X_{2i}| = 1$ and $|X_{2j}| = 1$: Let $X_{2i} = \sigma, \sigma^2 = 1$ and $X_{2j} = \delta, \delta^2 = 1$, and also $X_{1i} = 0$ and $X_{1j} = 0$. The algorithm sets $(v_1)_i = \sigma$, $(v_2)_i = -\sigma$, $(v_1)_j = \delta$ and $(v_2)_j = -\delta$, and again it is straightforward to check that $X^{\text{alg}}_{ij} = X_{ij}$.

Therefore, if $\alpha = 0$, $X^{\text{alg}} = X$ also holds. This completes the proof. ∎

Corollary 4.8 *For $n \geq 3$,*

$$X \in F_n \text{ and } \operatorname{rank} X = 2 \qquad \textit{if and only if}$$
$$X \textit{ is in the relative interior of a face of } C_n \textit{ of dimension 1.}$$

Proof. If $X \in F_n$ and $\operatorname{rank} X = 2$, then Lemma 4.7 and Lemma 2.2 together imply that X is in the relative interior of a face of dimension 1.

Conversely, suppose that X is in the relative interior of a face of dimension 1. Then by Lemma 2.2 again, there exist vectors v_1 and v_2 in $\{\pm 1\}^n$, $v_1 \neq v_2$, and a scalar $\alpha \in (0,1)$ such that $X = \alpha\, v_1 v_1^T + (1-\alpha)\, v_2 v_2^T$.

By convexity, $X \in F_n$, and clearly $\operatorname{rank} X \leq 2$. To complete the proof, we suppose $\operatorname{rank} X = 1$ and obtain a contradiction.

Clearly $X \succeq 0$, therefore Theorem 3.7 implies $|X_{ij}| = 1$ for all i, j. We may also assume without loss of generality that $(v_1)_1 = 1$ and $(v_2)_1 = 1$. Since $v_1 \neq v_2$, there is an index $j \in \{2, \ldots, n\}$ such that $(v_1)_j = -(v_2)_j$. For such a choice of j, we have

$$\begin{aligned} X_{1j} &= \alpha\,(v_1)_j + (1-\alpha)\,(v_2)_j \\ &= \alpha\,(v_1)_j + (1-\alpha)\,(-(v_1)_j) \\ &= (2\,\alpha - 1)\,(v_1)_j. \end{aligned}$$

But since $\alpha \in (0,1)$, it follows that $|2\,\alpha - 1| < 1$, and therefore

$$|X_{1j}| = |(2\,\alpha - 1)\,(v_1)_j| = |2\,\alpha - 1| < 1,$$

a contradiction. ∎

4.2.3 Uniqueness of Y and Rank Characterization

This section proves the last implication required to obtain Theorem 4.10. We show that if $X \in F_n$, i.e. if there exists $Y \in \mathcal{F}_n$ such that $X = \operatorname{sMat}(Y_{1:t(n),0})$, and $\operatorname{rank} X = 2$, then $\operatorname{rank} Y = 2$ holds.

Suppose $X \in F_n$ and $\operatorname{rank} X = 2$. Applying Algorithm 4.4, we find v_1, v_2, α_1, and α_2 such that $X = \alpha_1\, v_1 v_1^T + \alpha_2\, v_2 v_2^T$. Define $x_1 := \operatorname{svec}(v_1 v_1^T)$ and $x_2 := \operatorname{svec}(v_2 v_2^T)$, and consider

$$\begin{aligned} Y &:= \alpha_1 \begin{pmatrix} 1 \\ x_1 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \end{pmatrix}^T + \alpha_2 \begin{pmatrix} 1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 \\ x_2 \end{pmatrix}^T \\ &= \begin{pmatrix} \alpha_1 + \alpha_2 & (\alpha_1 x_1 + \alpha_2 x_2)^T \\ \alpha_1 x_1 + \alpha_2 x_2 & \alpha_1 x_1 x_1^T + \alpha_2 x_2 x_2^T \end{pmatrix}. \end{aligned}$$

Clearly $\operatorname{rank} Y \leq 2$. Also, $Y \in \mathcal{F}_n$ (by convexity) and furthermore

$$\operatorname{sMat}(Y_{1:t(n),0}) = \operatorname{sMat}(\alpha_1 x_1 + \alpha_2 x_2) = X.$$

Hence, $\operatorname{rank} Y \neq 1$ by Theorem 4.1 (since $\operatorname{rank} X = 2$). Thus $\operatorname{rank} Y = 2$ and it suffices to prove that this is the only $Y \in \mathcal{F}_n$ such that $X = \operatorname{sMat}(Y_{1:t(n),0})$. This is the result of the next lemma.

Lemma 4.9 *Suppose $X \in F_n$ and $\operatorname{rank} X = 2$. Then the matrix $Y \in \mathcal{F}_n$ such that $X = \operatorname{sMat}(Y_{1:t(n),0})$ is unique. Specifically, let α be the scalar defined in the statement of Lemma 4.3; then for each 4-tuple (i,j,k,l) of distinct elements from $\{1,\ldots,n\}$, $Y_{T(i,j),T(k,l)}$ equals exactly one of the values $\{1,-1,\alpha,-\alpha\}$.*

Proof. Given X, the restriction that $Y \in \mathcal{F}_n$ fixes all the elements of Y except those of the form $Y_{T(i,j),T(k,l)}$, with i,j,k,l all distinct. We prove that all these elements are also uniquely determined in terms of the elements of X. This proves the uniqueness of Y.

We consider two cases:

Case 1: at least one of X_{ij},X_{kl} equals ± 1: If so, then the positive semidefiniteness of the 3×3 principal submatrix of Y corresponding to rows and columns $\{0, T(i,j), T(k,l)\}$:

$$\begin{pmatrix} 1 & X_{ij} & X_{kl} \\ X_{ij} & 1 & y \\ X_{kl} & y & 1 \end{pmatrix} \succeq 0,$$

where y denotes $Y_{T(i,j),T(k,l)}$, implies (by Lemma 3.9) that $y = X_{ij}\, X_{kl} \in \{\pm 1, \pm\alpha\}$, since at least one of X_{ij},X_{kl} equals ± 1.

Case 2: both X_{ij},X_{kl} equal $\pm\alpha$: In this case, we write $X_{ij} = \sigma\,\alpha, \sigma^2 = 1$, $X_{kl} = \delta\,\alpha, \delta^2 = 1$, and consider the 3×3 principal submatrix of X:

$$X_{i,j,k} = \begin{pmatrix} 1 & X_{ij} & X_{ik} \\ X_{ij} & 1 & X_{jk} \\ X_{ik} & X_{jk} & 1 \end{pmatrix} \succeq 0.$$

By Lemma 4.2, either $|X_{ik}| = 1$ or $|X_{jk}| = 1$. Without loss of generality (symmetrically permuting rows and columns if necessary) suppose $|X_{ik}| = 1$. Then let $X_{ik} = \beta, \beta^2 = 1$, and observe that by Lemma 3.9, $X_{jk} = X_{ik}\, X_{ij} = \beta\,\sigma\,\alpha$. Now consider the 4×4 principal submatrix of Y indexed by the rows and columns $\{0, T(i,j), T(k,l), T(j,k)\}$:

$$\begin{pmatrix} 1 & \sigma\,\alpha & \delta\,\alpha & \beta\,\sigma\,\alpha \\ \sigma\,\alpha & 1 & y & \beta \\ \delta\,\alpha & y & 1 & X_{jl} \\ \beta\,\sigma\,\alpha & \beta & X_{jl} & 1 \end{pmatrix} \succeq 0.$$

Applying the Schur complement, positive semidefiniteness of this submatrix is equivalent to

$$\begin{pmatrix} 1 & y & \beta \\ y & 1 & X_{jl} \\ \beta & X_{jl} & 1 \end{pmatrix} - \begin{pmatrix} \sigma\,\alpha \\ \delta\,\alpha \\ \beta\,\sigma\,\alpha \end{pmatrix} \begin{pmatrix} \sigma\,\alpha \\ \delta\,\alpha \\ \beta\,\sigma\,\alpha \end{pmatrix}^T \succeq 0$$

$$\Leftrightarrow \begin{pmatrix} 1-\alpha^2 & y-\sigma\,\delta\,\alpha^2 & \beta\,(1-\alpha^2) \\ y-\sigma\,\delta\,\alpha^2 & 1-\alpha^2 & X_{jl}-\beta\,\sigma\,\delta\,\alpha^2 \\ \beta\,(1-\alpha^2) & X_{jl}-\beta\,\sigma\,\delta\,\alpha^2 & 1-\alpha^2 \end{pmatrix} \succeq 0.$$

Since $|\alpha| < 1$, $1-\alpha^2 > 0$ and so, scaling both rows and columns by $\frac{1}{\sqrt{1-\alpha^2}}$ (this is a congruence and therefore it does not affect positive semidefiniteness),

$$\Leftrightarrow \begin{pmatrix} 1 & \frac{y-\sigma\,\delta\,\alpha^2}{1-\alpha^2} & \beta \\ \frac{y-\sigma\,\delta\,\alpha^2}{1-\alpha^2} & 1 & \frac{X_{jl}-\beta\,\sigma\,\delta\,\alpha^2}{1-\alpha^2} \\ \beta & \frac{X_{jl}-\beta\,\sigma\,\delta\,\alpha^2}{1-\alpha^2} & 1 \end{pmatrix} \succeq 0.$$

By Lemma 3.9, it follows that

$$\begin{aligned} & \tfrac{y-\sigma\,\delta\,\alpha^2}{1-\alpha^2} = \beta\,\tfrac{X_{jl}-\beta\,\sigma\,\delta\,\alpha^2}{1-\alpha^2} \\ \Rightarrow\quad & y-\sigma\,\delta\,\alpha^2 = \beta\,X_{jl} - \beta^2\,\sigma\,\delta\,\alpha^2 \\ \Rightarrow\quad & y = \beta\,X_{jl} \in \{\pm 1, \pm\alpha\}. \end{aligned}$$

This completes the proof of the uniqueness of Y. ■

Theorem 4.10 *For $Y \in \mathcal{F}_n$ and $X = \mathrm{sMat}\,(Y_{1:t(n),0})$,*

$$\operatorname{rank} Y = 2 \quad\Leftrightarrow\quad \operatorname{rank} X = 2 \quad\Leftrightarrow\quad X \textit{ is in the relative interior of a face of } C_n \textit{ of dimension 1.}$$

Proof. Clearly $\operatorname{rank} Y = 2 \Rightarrow \operatorname{rank} X \leq 2$, and Theorem 4.1 implies $\operatorname{rank} X \geq 2$, hence $\operatorname{rank} X = 2$. We have just shown that if $\operatorname{rank} X = 2$, then Y is unique, and the construction of Y at the beginning of this section shows that $\operatorname{rank} Y = 2$. Finally, the second equivalence is exactly the statement of Corollary 4.8. ■

Corollary 4.11 *Let $Y^* \in \mathcal{F}_n$ be optimal for SDP2. If $\operatorname{rank} Y^* \leq 2$ then*

$$\nu_3^* = \mu^*,$$

and furthermore Y^* yields an optimal cut for Max-Cut.

Proof. Theorems 4.1 and 4.10 imply that if $\operatorname{rank} Y^* \leq 2$ then the matrix $X^* = \operatorname{sMat}(Y^*_{1:t(n),0})$ lies on the boundary of C_n, and hence $\nu_3^* = \mu^*$. If $\operatorname{rank} Y^* = 1$ then $\operatorname{rank} X^* = 1$ so it is straightforward to extract the optimal cut from X^*. Finally, if $\operatorname{rank} Y^* = 2$ then $\operatorname{rank} X^* = 2$ and Algorithm 4.4 yields two optimal cuts. ■

4.3 Counter-Example for the Rank-Three Case

We conclude our study of the strengthened SDP2 relaxation by showing that the equivalence of the ranks between Y and $X = \operatorname{sMat}(Y_{1:t(n),0})$ cannot be extended beyond rank-two.

Of course, it follows from Theorems 4.1 and 4.10 that $\operatorname{rank} Y = 3 \Rightarrow \operatorname{rank} X = 3$ for $n \geq 3$. However, the converse fails for $n = 3$, and hence for all $n \geq 3$. Indeed, an easy counterexample is provided by the 3×3 identity matrix since

$$Y = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

is feasible for SDP2 and has rank equal to 4, whereas $X = \operatorname{sMat}(Y_{1:6,0}) = I$ has rank equal to 3.

5 Conclusion

In this paper, we proved that the relaxation SDP2 derived in [1, 2] completely characterizes the faces of dimension 0 and 1 of the cut polytope in $\mathcal{S}^n$ (upon projecting its first column into $\mathcal{S}^n$ via the sMat operator) via the rank of its matrix variable Y. In doing so, we also proved that for the case $n = 3$, i.e. for the complete graph on 3 nodes, the rank of Y completely characterizes the dimension of the face of C_3 where $X = \operatorname{sMat}(Y_{1:6,0})$ lies. These results give further theoretical evidence of the remarkable tightness of this new SDP relaxation.

References

[1] M.F. ANJOS. *New Convex Relaxations for the Maximum Cut and VLSI Layout Problems.* PhD thesis, University of Waterloo, 2001.

[2] M.F. ANJOS and H. WOLKOWICZ. A strengthened SDP relaxation via a second lifting for the Max-Cut problem. *Discrete Applied Mathematics*, to appear.

[3] F. BARAHONA. The max-cut problem on graphs not contractible to K_5. *Oper. Res. Lett.*, 2(3):107–111, 1983.

[4] F. BARAHONA. On cuts and matchings in planar graphs. *Math. Programming*, 60(1, Ser. A):53–68, 1993.

[5] F. BARAHONA, M. GRÖTSCHEL, M. JÜNGER, and G. REINELT. An application of combinatorial optimization to statistical physics and circuit layout design. *Oper. Res.*, 36:493–513, 1988.

[6] F. BARAHONA and A.R. MAHJOUB. On the cut polytope. *Math. Programming*, 36(2):157–173, 1986.

[7] S. J. BENSON, Y. YE, and X. ZHANG. Solving large-scale sparse semidefinite programs for combinatorial optimization. *SIAM J. Optim.*, 10(2):443–461 (electronic), 2000.

[8] S. BURER and R.D.C. MONTEIRO. An efficient algorithm for solving the MAXCUT SDP relaxation. Technical report, Georgia Tech, Atlanta, GA, 1999.

[9] M.M. DEZA and M. LAURENT. *Geometry of cuts and metrics.* Springer-Verlag, Berlin, 1997.

[10] U. FEIGE and G. SCHECHTMAN. On the optimality of the random hyperplane rounding technique for MAX CUT. Technical report, Weizmann Institute, Rehovot, Israel, 2000.

[11] M.R. GAREY and D.S. JOHNSON. *Computers and Intractability: A guide to the theory of NP–completeness.* Freeman, San Francisco, 1979.

[12] M.X. GOEMANS. Semidefinite programming in combinatorial optimization. *Math. Programming*, 79:143–162, 1997.

[13] M.X. GOEMANS. Semidefinite programming and combinatorial optimization. *Documenta Mathematica*, Extra Volume ICM 1998:657–666, 1998. Invited talk at the International Congress of Mathematicians, Berlin, 1998.

[14] M.X. GOEMANS and F. RENDL. Combinatorial optimization. In H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors, *HANDBOOK OF SEMIDEFINITE PROGRAMMING: Theory, Algorithms, and Applications.* Kluwer Academic Publishers, Boston, MA, 2000.

[15] M.X. GOEMANS and D.P. WILLIAMSON. .878-approximation algorithms for MAX CUT and MAX 2SAT. In *ACM Symposium on Theory of Computing (STOC)*, 1994.

[16] M.X. GOEMANS and D.P. WILLIAMSON. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. Assoc. Comput. Mach.*, 42(6):1115–1145, 1995.

[17] J. HASTAD. Some optimal inapproximability results. In *Proc. of the 29th ACM Symp. on Theory Comput.*, 1997.

[18] C. HELMBERG. *An interior-point method for semidefinite programming and max-cut bounds.* PhD thesis, Graz University of Technology, Austria, 1994.

[19] C. HELMBERG and F. OUSTRY. Bundle methods to minimize the maximum eigenvalue function. In H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors, *HANDBOOK OF SEMIDEFINITE PROGRAMMING: Theory, Algorithms, and Applications.* Kluwer Academic Publishers, Boston, MA, 2000.

[20] C. HELMBERG, S. POLJAK, F. RENDL, and H. WOLKOWICZ. Combining semidefinite and polyhedral relaxations for integer programs. In *Integer Programming and Combinatorial Optimization (Copenhagen, 1995)*, pages 124–134. Springer, Berlin, 1995.

[21] C. HELMBERG and F. RENDL. A spectral bundle method for semidefinite programming. *SIAM J. Optim.*, 10(3):673 – 696, 2000.

[22] C. HELMBERG, F. RENDL, R. J. VANDERBEI, and H. WOLKOWICZ. An interior-point method for semidefinite programming. *SIAM J. Optim.*, 6(2):342–361, 1996.

[23] R.A. HORN and C.R. JOHNSON. *Matrix analysis.* Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[24] R. M. KARP. Reducibility among combinatorial problems. In R. E. Miller and J.W. Thatcher, editors, *Complexity of Computer Computation*, pages 85–103. Plenum Press, New York, 1972.

[25] J.B. LASSERRE. Optimality conditions and lmi relaxations for 0-1 programs. Laas research report, LAAS-CNRS, Toulouse, France, 2000.

[26] M. LAURENT. Tighter linear and semidefinite relaxations for max-cut based on the Lovász-Schrijver lift-and-project procedure. *Unpublished manuscript*, 2000.

[27] M. LAURENT and S. POLJAK. On a positive semidefinite relaxation of the cut polytope. *Linear Algebra Appl.*, 223/224:439–461, 1995.

[28] M. LAURENT and S. POLJAK. On the facial structure of the correlation matrices. *SIAM J. Matrix Anal. Appl.*, 17(3):530–547, 1996.

[29] T. LENGAUER. *Combinatorial algorithms for integrated circuit layout.* John Wiley & Sons Ltd., Chichester, 1990. With a foreword by Bryan Preas.

[30] B. MOHAR and S. POLJAK. Eigenvalues in combinatorial optimization. In *Combinatorial Graph-Theoretical Problems in Linear Algebra*, IMA Vol. 50. Springer-Verlag, 1993.

[31] Y.E. NESTEROV. Quality of semidefinite relaxation for nonconvex quadratic optimization. Technical report, CORE, Universite Catholique de Louvain, Belgium, 1997.

[32] Y.E. NESTEROV, H. WOLKOWICZ, and Y. YE. Semidefinite programming relaxations of nonconvex quadratic optimization. In H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors, *HANDBOOK OF SEMIDEFINITE PROGRAMMING: Theory, Algorithms, and Applications*, pages 361–420. Kluwer Academic Publishers, Boston, MA, 2000.

[33] P. PARRILO. *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization.* PhD thesis, Caltech, Pasadena, California, 2000.

[34] F. RENDL. Semidefinite programming and combinatorial optimization. *Appl. Numer. Math.*, 29:255–281, 1999.

[35] H. WOLKOWICZ, R. SAIGAL, and L. VANDENBERGHE, editors. *HANDBOOK OF SEMIDEFINITE PROGRAMMING: Theory, Algorithms, and Applications*. Kluwer Academic Publishers, Boston, MA, 2000. xxvi+654 pages.

[36] Y. YE. Approximating quadratic programming with bound and quadratic constraints. *Math. Programming*, 84:219–226, 1999.