Foreword: for Special Issue on Semidefinite Programming

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Semidefinite programming (denoted SDP) is an extension of linear programming (LP), with vector variables replaced by matrix variables and nonnegativity elementwise replaced by positive semidefiniteness. Let us express a (primal) SDP as

$$\begin{array}{ll} \min & C \bullet X \\ \text{s.t.} & \mathcal{A}X = b \\ & X \succeq 0, \end{array}$$

where: C, X are in $\mathcal{S}^n$, the space of symmetric $n \times n$ matrices; the inner product $C \bullet X = \operatorname{trace} CX$; $\succeq$ denotes nonnegativity in the Löwner partial order, i.e. $A \succeq B$ if $A - B \succeq 0$, i.e. $A - B$ is positive semidefinite; and

$$\mathcal{A} : \mathcal{S}^n \to \Re^m$$

is a linear operator, i.e. $(\mathcal{A}X)_i = \operatorname{trace}(A_i X)$, for given $A_i \in \mathcal{S}^n,\ i = 1, \ldots m$.

The close similarity between SDP and LP is apparent, the essential difference being that $\mathcal{P}$, the cone of positive semidefinite matrices, replaces $\Re^n_+$, the nonnegative orthant. Unlike LP, SDP is a *nonlinear* convex programming problem, because the boundary of the cone $\mathcal{P}$ is nonlinear. Nonetheless, LP and SDP share two key aspects:

- **Duality theory**: much of the theory of duality extends directly from LP to SDP. Indeed, this was recognized as early as 1963 by Bellman and Fan [2]. They also observed that duality theorems require somewhat stronger assumptions for SDP than for LP: specifically, they used a Slater-type constraint qualification to guarantee a zero duality gap.

- **Interior-point methods**: SDP can be very effectively solved by generalizing interior-point methods developed originally for LP. The credit for this important discovery goes primarily to Nesterov and Nemirovski [9] (who gave a far-reaching theory for general convex programs) and Alizadeh [1] (who argued that many interior-point methods could be extended from LP to SDP in a simple, unified way). While the question of superiority of simplex versus interior-point methods for LP remains a controversial question, there is no contest for SDP, because the nonlinearity of the feasible set makes a simplex method impractical. The fact that SDP can be efficiently solved by interior-point methods, both in practice and, from a theoretical point of view, in polynomial time, has driven the recent surge of interest in the subject.

In this volume, a wide variety of aspects of SDP are studied. One way to organize these is as follows:

1. THEORETICAL

(a) A theorem of Goldman and Tucker shows that every (finite optimal valued) LP has an optimal primal-dual pair that satisfies strict complementarity. However, this result does not hold for SDP. Furthermore, the formulation of nondegeneracy conditions requires study of the geometry of the semidefinite cone $\mathcal{P}$. The boundary of $\mathcal{P}$, like that of $\Re^n_+$, is composed of smooth pieces, but the pieces are nonlinear. This topic is studied by Alizadeh, Haeberly and Overton in our first paper, "Complementarity and nondegeneracy in semidefinite programming."

(b) As already mentioned, strong duality may fail for SDP if a constraint qualification does not hold. It is possible to construct nonstandard dual programs which avoid constraint qualifications. The first such strong dual program whose size is polynomial in the data of the original problem is given by our second paper "An exact duality theory for semidefinite programming and its complexity implications", by Ramana.

2. ALGORITHMIC

(a) Our third paper, "A projective method for solving linear matrix inequalities", by Gahinet and Nemirovski, defines and analyzes a potential-reduction interior-point method for SDP. This method has been implemented in the Matlab LMI Toolbox, perhaps at present the most widely used software for SDP. The term LMI means linear matrix inequality and is commonly used in control applications (see below).

(b) Our fourth paper, "The long-step method of analytic centers for fractional problems", by Nemirovski, extends the method of analytic centers to solve a broad class of fractional programming problems, including linear fractional programming, SDP, and certain eigenvalue optimization problems for matrix pencils.

3. APPLICATIONS

(a) SDP has found many important applications in combinatorial optimization. Many models that approximate combinatorial problems use LP as a basic tool. Although quadratic approximations are better than linear ones, they often lead to subproblems that are too hard to solve. SDP provides a means of efficiently solving relaxations of quadratic approximations. This idea has been successfully applied to the stable set problem and the max-cut problem, see e.g. [5, 8]. Indeed, this volume is dedicated to Svata Poljak, one of the important early contributors to this subject. Our fifth paper, by Laurent, Poljak and Rendl, which was completed shortly before Poljak's untimely death, is "Connections between semidefinite relaxations of the max-cut and stable set problems." Besides exploring the links between these relaxations, it obtains tighter relaxations than previously available by introducing additional polyhedral constraints.

(b) In parallel with the applications to combinatorial optimization, many applications of SDP arose in engineering, especially in the field of robust control. For many such applications, see [3, 11]. As noted above, in this arena SDP is usually known as LMI: linear matrix inequality. An important but much harder problem in this field is the nonconvex bilinear matrix inequality. This is studied in our sixth paper, "A cone programming approach to the bilinear matrix inequality problem and its geometry," by Mesbahi and Papavassilopoulos.

(c) The trust region subproblem is a basic tool in unconstrained nonlinear programming, providing a local quadratic model of the objective function which is to be "trusted" only in a certain region. In this case, the SDP relaxation of the quadratic problem provides an exact solution. Our seventh paper, "A semidefinite framework for trust region subproblems with applications to large scale minimization", by Rendl and Wolkowicz, introduces an SDP framework that allows one to efficiently exploit sparsity when solving unconstrained nonlinear programming problems.

4. EXTENSIONS TO NONLINEAR PROBLEMS

   (a) So far, we have restricted the discussion to SDP with linear equality constraints, i.e. the only nonlinearity in the problem is the boundary of the cone $\mathcal{P}$. Equivalently, an affine matrix-valued function appears in the dual program, again with nonlinearity present only in the boundary of $\mathcal{P}$. However, one can consider SDP with nonlinear matrix-valued functions. First and second order local optimality conditions and sensitivity analysis for such nonlinear SDP's are studied in our eighth and last paper, "First and second order analysis of nonlinear semidefinite programs," by Shapiro.

A historical note: an influential volume in the early history of *Math Programming Study*, the predecessor of *Math Programming* Series B, was that on nondifferentiable optimization (Volume 3, 1975, M.L. Balinski and P. Wolfe, eds). That volume included one of the early papers on matrix eigenvalue optimization [4]. SDP is an eigenvalue optimization problem in the sense that the semidefinite constraint is equivalent to nonnegative bounds on eigenvalues. Solutions to SDP generally have multiple zero eigenvalues (just as solutions to LP have many nonbasic, i.e. zero, variables). However, it is not possible to formulate separate differentiable bounds on individual eigenvalues. For more on eigenvalue optimization, see [7].

Although the topic is not discussed in this volume, we mention that it has become appreciated recently that one reason for the special nature of SDP is that the cone $\mathcal{P}$ is both homogeneous and self-dual. Such cones have been completely classified and include only the Lorentz cone (used for convex quadratic programming with convex quadratic constraints), the semidefinite cone $\mathcal{P}$ and its extensions (of which the most important is the cone of positive semidefinite complex Hermitian matrices), and products of these (including $\Re^n_+$, a product of n trivial semidefinite cones). For more details, see [6].

The World Wide Web is a tremendous resource for information on SDP. This includes research papers, public domain software, and bibliography files. Several relevant web sites are listed below. These contain further pointers to other web sites with more information.

1. http://www.zib.de/helmberg/semidef.html
   contains a special home page for semidefinite programming organized by C. Helmberg, at Berlin Center for Scientific Computing, Konrad Zuse Zentrum fur Informationstechnik, Berlin.

2. ftp://orion.uwaterloo.ca/pub/henry/reports/psd.bib.gz
   contains a bib file with papers related to SDP.

3. http://www.mcs.anl.gov:80/home/otc/InteriorPoint
   is an archive of technical reports and papers on interior-point methods, maintained by S.J. Wright at Argonne National Laboratory. Many of the recent reports discuss primal-dual interior-point methods for SDP, a subject not discussed in this volume.

SDP is an exciting subject that remains very active, and will no doubt be so for some time to come. Here is an area where nonlinear convex programming has shown its strength, beauty, and versatility. One finds elegant theoretical results, provably efficient algorithms and a wide variety of applications.

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