DUALITY FOR SEMIDEFINITE PROGRAMMING, *SDP Duality*

Basic Properties. Consider the primal **Semidefinite Program** (denoted *SDP*)

$$(P) \quad \begin{array}{rll} \mu^* := \text{minimize} & C \bullet X & \\ \text{subject to} & \mathbf{A}X & = \ b \\ & X & \succeq \ 0, \end{array}$$

where: $C \bullet X = \mathit{trace}\, CX$ denotes the inner product of the symmetric matrices C, X; $X \succeq Y$ denotes that the symmetric matrix $X - Y$ is positive semidefinite; and $\mathbf{A} : \mathbf{S}^n \longrightarrow \Re^m$ is a linear operator, on the space of symmetric matrices, with adjoint $\mathbf{A}^*$. Equivalently, the linear constraint can be written using symmetric matrices $A_i,\ i = 1, \cdots, m$, as

$$\mathbf{A}_i \bullet X = b_i, \text{ for all } i = 1, \cdots, m;$$

while the adjoint operation on $y \in \Re^m$ is

$$\mathbf{A}^* y := \sum_{i=1}^m y_i A_i.$$

The *Lagrangian* function is

$$\mathbf{L}(X, y) := C \bullet X + y^t (b - \mathbf{A}X).$$

The primal problem is equivalent to

$$\mu^* = \min_{X \succeq 0} \max_y \mathbf{L}(X, y) = C \bullet X + y^t (b - \mathbf{A}X).$$

The equivalence can be seen by using the *hidden* constraint in the outer minimization problem $b - \mathbf{A}X = 0$, i.e. if this constraint does not hold then the inner maximum value is $+\infty$.

By interchanging the max and the min and rewriting the order of terms in the Lagrangian, we get the dual problem and *weak duality*.

$$\mu^* \geq \nu^* := \max_y \min_{X \succeq 0} b^t y + (C - \mathbf{A}^* y) \bullet X.$$

Using the hidden constraint in the outer maximization problem $C - \mathbf{A}^* y \succeq 0$, this becomes equivalent to

$$(D) \quad \begin{array}{rll} \nu^* = \text{maximize} & b^t y & \\ \text{subject to} & \mathbf{A}^* y & \preceq \ C. \end{array}$$

The dual pair (P) and (D) look very much like a dual pair of linear programs (denoted LP) where the adjoint operator replaces the transpose and positive semidefiniteness of matrix variables replaces nonnegativity of vector variables. In fact, duality theory for SDP has a lot of similarities with that of LP: (i) *weak duality* $\mu^* \geq \nu^*$ follows from the interchange of max and min; from this we get that unboundedness of (P) (resp. (D)) implies infeasibility of (D) (resp. (P).

Weak duality illustrates one of the powerful uses of the dual program, i.e. it provides lower bounds on the optimal value of the primal program.

Other formulations of SDP provide similar duals. In fact, SDP is a special case of cone programming. Let K, L be two convex cones, i.e. K (and L) satisfy: the Minkowski sum $K + K \subset K$ and $\alpha K \subset K,\ \forall \alpha \in \Re$. Define the pimal cone program as

$$(PC) \quad \begin{array}{rl} \nu^* = \text{minimize} & < C, X > \\ \text{subject to} & \mathbf{A}X \succeq_K b \\ & X \succeq_L 0, \end{array}$$

where $X \succeq_L Y$ denotes $X - Y \in L$ (and similarly for K), and $< \cdot, \cdot >$ denotes the appropriate inner product. Then the above min-max argument yields the dual cone program

$$(DC) \quad \begin{array}{rl} \nu^* = \text{maximize} & < b, y > \\ \text{subject to} & \mathbf{A}^* y \preceq_{L^+} C \\ & y \succeq_{K^+} 0, \end{array}$$

where $\cdot^+$ denotes *polar cone.*

It is an interesting exercise to see that this elegant dual formulation works for linear programs that have mixtures of inequality and equality constraints with mixtures of free and nonnegative variables.

Strong Duality. However, unlike linear programming, *strong duality* for SDP needs a constraint qualification, e.g. strict primal feasibility (called Slater's condition),

$$\text{there exists } \hat{X} \succ 0, \text{ with } \mathbf{A}\hat{X} = b.$$

This constraint qualification implies strong duality holds, i.e. that $\mu^* = \nu^*$ and ν^* is attained. Conversely,

$$\mathbf{A}^* y \prec_{L^+} C$$

Semidefinite Program
Lagrangian

also implies that $\mu^* = \nu^*$ but with μ^* attained. If Slater's condition does not hold, then a duality gap $\mu^* > \nu^*$ can exist, and/or the dual (or primal) optimal value may not be attained, see e.g. [10].

Example

If the dual is

$$(D) \quad \begin{array}{rc} \nu^* = \sup & x_2 \\ \text{s. t.} & \begin{bmatrix} x_2 & 0 & 0 \\ 0 & x_1 & x_2 \\ 0 & x_2 & 0 \end{bmatrix} \preceq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

then the primal is

$$(P) \quad \begin{array}{rlc} \mu^* = & \inf & U_{11} \\ & \text{s. t.} & U_{22} = 0 \\ & & U_{11} + 2U_{23} = 1 \\ & & U \succeq 0 \end{array}$$

and we have the duality gap

$$\nu^* = 0 < \mu^* = 1.$$

In the case that strong duality holds, we get the following primal-dual characterization of optimality for the dual pair X, y, with $X \succeq 0$

$$\begin{array}{ll} \mathbf{A}X = b & \text{primal feasibility} \\ \mathbf{A}^*y \preceq C & \text{dual feasibility} \\ X(\mathbf{A}^*y - C) = 0 & \text{complementary slackness} \end{array}$$

These optimality conditions provide the basis for: (i) the primal simplex method (maintain primal feasibility and complementary slackness while striving for dual feasibility); (ii) the dual simplex method (maintain dual feasibility and complementary slackness while striving for primal feasibility); and (iii) the interior-point methods (maintain primal and dual feasibility while striving for complementary slackness. Unlike the LP case, there are currently no efficient algorithms for primal or dual simplex methods for SDP; however, interior-point methods have proven to be very successful. Thus we see the importance of duality for both theoretical and algorithmic purposes.

Strict Complementarity. Another example of the difference between LP and SDP arises in the complementary slackness conditions. If an optimal pair X, y exist, then in the LP case there also exists an optimal pair that satifies strict complementarity, i.e.

$$X + (C - \mathbf{A}^*y) \succ 0,$$

where in the LP case this is a sum of nonnegative diagonal matrices, see the Goldman-Tucker theorem [4, 5]. However, in the SDP case, there may not exist such a strict complementary optimal pair, though the existence is generic, see [8].

Closing the Duality Gap. Both the strict complementarity and strict feasiblity, or Slater's constraint qualification, are generic, see [8]. But there are classes of problems where strong duality fails, e.g. relaxations that arise from hard combinatorial problems, e.g. [13].

One can regularize semidefinite programs and guarantee that Slater's constraint qualification holds, e.g. [3, 12, 2]. This involves finding the minimal face of the semidefinite cone that contains the feasible set, i.e. the so-called *minimal cone.* A numerical procedure for regularization is presented in [3]. However, this process is not computationally tractable. An equivalent approach is the extended Lagrange-Slater dual program of Ramana [10, 9]. This provides a means of writing down a regularized program that is of polynomial size. Thus strong duality can be attained theoretically using the above techniques and exploiting the structure of specific problems. However, lack of regularity (Slater's condition) is an indication of an ill-posed problem. Thus, the question of whether regularization can be done computationally for general problems is still an open question, see e.g. [7].

Extensions. The SDP's that we consider have all contained linear objectives and constraints. There is no reason to restrict SDPs to this special case. Duality for general cone programs with possible nonlinear objectives and constraints is considered in [3, 2, 11]. Applications for quadratic objectives SDP appear in e.g. [6, 1].

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