AN INTERIOR-POINT METHOD FOR APPROXIMATE POSITIVE SEMIDEFINITE COMPLETIONS *

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October 20, 1997
University of Waterloo
CORR Report 95-11

Abstract

Given a nonnegative, symmetric matrix of weights, H, we study the problem of finding an Hermitian, positive semidefinite matrix which is closest to a given Hermitian matrix, A, with respect to the weighting H. This extends the notion of exact matrix completion problems in that, $H_{ij} = 0$ corresponds to the element A_{ij} being *unspecified* (free), while H_{ij} large in absolute value corresponds to the element A_{ij} being approximately *specified* (fixed).

We present optimality conditions, duality theory, and two primal-dual interior-point algorithms. Because of sparsity considerations, the dual-step-first algorithm is more efficient for a large number of free elements, while the primal-step-first algorithm is more efficient for a large number of fixed elements.

Included are numerical tests that illustrate the efficiency and robustness of the algorithms.

*This report (related software) is available by anonymous ftp at orion.uwaterloo.ca in the directory pub/henry/reports (henry/software/completions.d); or over WWW with URL ftp://orion.uwaterloo.ca/pub/henry/reports/ABSTRACTS.html (henry/software/readme.html).

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Key words: positive definite completions, best nonnegative approximation, semidefinite programming, primal-dual interior-point methods, complementarity problems.

AMS 1991 Subject Classification: Primary: 65K10; Secondary: 90C25, 15A45.

Contents

1 INTRODUCTION

Suppose we are given a partial Hermitian matrix A with certain elements fixed, for which all fully specified (fixed) principal submatrices are positive semidefinite. The positive semidefinite completion problem consists in finding numbers for the unspecified (free) elements in order to make A positive semidefinite. This problem has many applications and has been extensively studied, see [8, 2] and the references therein. Much is known in terms of existence of completions. In this paper, we generalize the completion problem in that we allow approximate completions. Using this approach, we introduce and test two primal-dual interior-point algorithms. These algorithms solve both approximate completion problems and exact completion problems and avoid the numerical difficulties that can arise from an exact completion approach. In addition, the algorithms can exploit sparsity of the problem, i.e. when only few elements are fixed or only few elements are free.

1.1 Background

Let $H = H^t \geq 0$ be a real, nonnegative (elementwise) symmetric matrix of weights with positive diagonal elements

$$H_{ii} > 0, \ \forall i,$$

and let $A = A^*$ be another given Hermitian matrix. (We often refer to A as the given partial Hermitian matrix, since some elements are specified as approximately fixed while others are free. However, for notational purposes, we assume that the free elements are originally set to 0 if they are not specified.) Let

$$||A||_F = \sqrt{\operatorname{trace} A^*A},$$

denote the *Frobenius norm* of A, and

$$f(P) := ||H \circ (A - P)||_F^2,$$

where $\circ$ denotes *Hadamard product.* We consider the weighted, best approximate completion problem

$$(AC) \qquad \begin{array}{rll} \mu^* := & \min & f(P) \\ & \text{subject to} & KP = b \\ & & P \succeq 0, \end{array}$$

where $K : \mathcal{H}_n \to \mathcal{C}^m$ is a linear operator from the Hermitian matrices to $\mathcal{C}^m$, and $P \succeq 0$ denotes *positive semidefiniteness*.

Note that $H_{ij} = 0$ means that P_{ij} is *free*, while $H_{ij} > 0$ puts a weight on forcing the components $P_{ij} \approx A_{ij}$, i.e. P_{ij} is approximately *fixed*. If the optimal value at an optimal $\bar{P}$ is $f(\bar{P}) = 0$, then we have an exact completion of A, where the positions corresponding to $H_{ij} = 0$ are free. The linear constraint $KP = b$ can force components of P to exactly equal the corresponding component of A, e.g. adding the constraint trace $E_{ij}P = A_{ij}$ is equivalent to fixing $P_{ij} = A_{ij}$, where

$$(E_{ij})_{kl} = \begin{cases} 1 & \text{if } i = k, j = l \text{ or } i = l, j = k \\ 0 & \text{otherwise} \end{cases}$$

i.e. the matrix E_{ij} is the zero matrix with a 1 in the i, j and j, i positions.

Suppose that the diagonal elements $a_{ii} > 0,\ \forall i$, i.e. $\operatorname{diag}(A) > 0$. Consider

$$\max \{\det(P) : P \succeq 0,\ \operatorname{diag}(P) = \operatorname{diag}(A)\}.$$

Then (Hadamard's inequality) the maximum occurs when P^{-1}, and so P, is diagonal. Thus, the positive definite completion matrix P, with maximum determinant, of a diagonal matrix is characterized by P^{-1} having zeros in the positions corresponding to the free components (the off diagonal components).

Now, suppose that $2k+1$ bands are specified in A, i.e. a_{ij} is fixed for $|i-j| \leq k$. And, suppose that any principal $k+1 \times k+1$ submatrix contained within the $2k+1$ bands is positive definite. Then:

Theorem 1.1 *([6])*

1. *If*
$$\mathcal{D}_k = \{B : B \succeq 0,\ b_{ij} = a_{ij}, \forall |i-j| \leq k\},$$
then $\mathcal{D}_k \neq \emptyset$.

2. $\max\{\det(B) : B \in \mathcal{D}_k\}$ *occurs at the unique matrix* $B \in \mathcal{D}_k$ *having the property that for all* $|i-j| \geq k+1$, *the* (i,j)*-entry of* B^{-1} *equals 0.*

□

Thus, we again have an optimum positive definite completion that is characterized by having an inverse with zeros in the positions corresponding to the free components.

Suppose that we are given the *partial Hermitian* matrix $A = A^*$, i.e. the components $A_{ij}, ij \in \mathcal{E}$ are fixed, while the others are free. This is denoted $A = A(\mathcal{G})$, for the undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. The partial Hermitian matrix A is called a $\mathcal{G}$-partial positive definite matrix if every specified principal minor is positive definite. The *positive semidefinite completion problem* consists in finding components $A_{ij}, ij \notin \mathcal{E}$, such that the resulting matrix $A \succeq 0$, is positive semidefinite. The graph $\mathcal{G}$ is *chordal* if there are no minimal cycles of length ≥ 4. A graph $\mathcal{G}$ is *completable* if and only if any $\mathcal{G}$-partial positive definite matrix has a positive definite completion.

Theorem 1.2 *([8]) $\mathcal{G}$ is completable if and only if $\mathcal{G}$ is chordal.*

□

Theorem 1.3 *([8]) Suppose that* $\operatorname{diag}(A)$ *is fixed and $A(\mathcal{G})$ has a positive completion. Then there is a unique such completion, B, with maximum determinant; and it is characterized by B^{-1} having 0 components in positions corresponding to all $kl \notin \mathcal{E}$.*

□

The unique completion with maximum determinant is again characterized by its inverse having zeros in the positions corresponding to free components. In this paper we see that this characterization follows through for AC with the inverse replaced by the "dual" matrix.

We assume that the diagonal of H is positive, for if $H_{kk} = 0$, then we can make P_{kk} as large as we like to maintain $P \succeq 0$. This allows us to set $P_{kj} = A_{kj}, \ \forall j$.

There are several special cases of AC:

1. Suppose that H has 0 valued elements corresponding to exactly the free elements of the partial matrix A, and, also, both K and b are 0. Then we found an exact completion, of the partial matrix A, if and only if the optimal value $\mu^* = 0$.

2. In the case that $A = 0$ and $H = E$, the matrix of ones, the problem AC is an exact completion problem or a best approximation problem with a semidefinite constraint. Exact completion problems with $P \succ 0$ and which maximize the determinant are studied in [8] and more recently in [2, 11, 12]. Existence and optimality conditions are presented. Numerical techniques are presented in [7]. Best nonnegative elementwise interpolation problems are studied in e.g. [18, 4, 3]. This includes optimality conditions as well as algorithms based on Newton's method.

3. Without the linear constraint, i.e. with $K = 0$ and $b = 0$, the problem AC is a quadratic program over a cone and can be rephrased as a complementarity problem over $\mathcal{P}$. Such problems have been studied in e.g. [14, 13]. Algorithms for general semidefinite programs with linear constraints have been recently studied in several papers, see e.g. [21, 5, 9]. These authors also have software packages, some of which are public domain, see e.g. the URL for semidefinite programming on the WWW ftp://orion.uwaterloo.ca/pub/henry/software/readme.html.

1.2 Main Results

In this paper we present the optimality conditions, a duality theory, and two primal-dual interior-point algorithms specialized for AC. One algorithm, the dual-step-first algorithm, is efficient for problems where H is sparse, since the dual variable Λ is 0 corresponding to 0 in H, i.e. corresponding to free elements of A. The other algorithm, the primal-step-first algorithm, is efficient when H has many relatively large (in absolute value) elements, i.e. when A, and so also the primal variable P, has few free elements.

We include numerical tests on randomly generated problems with various properties, i.e. $A \succeq 0, H \succeq 0$, and specified values for condition numbers of A, H, sparsity of H, and the number of sufficiently large elements of H. These tests show that the algorithms can efficiently solve such problems. The algorithms were tested on thousands of problems and never failed to find the optimal approximate completion. Thus, in the case that the matrix H contains zeros and an exact completion exists, the algorithm never failed to find an exact completion. In addition, since interior-point methods converge to the analytic center of the optimal face, the algorithm finds the completion of maximal rank. Degeneracy and ill-conditioning did not affect the algorithm, other than change the duality gap of the initial starting solution.

1.3 Outline

The paper is organized as follows. We complete this section with preliminary notation and results. In Section 2, we present the optimality characterizations and a dual program for AC. The primal-dual interior-point methods are presented in Section 3. The numerical tests are summarized in two tables in Section 4. We discuss the results in Section 5.

1.4 Preliminaries

We now present some of the matrix theoretic background. More details can be found in e.g. [10]. We work in the space of Hermitian matrices, $\mathcal{H}_n$, equipped with the trace inner product, $\langle X, Y \rangle = \operatorname{trace} X^*Y$, where X^* denotes complex conjugate. The cone of positive semidefinite Hermitian matrices is denoted $\mathcal{P}$. Each Hermitian matrix $A \in \mathcal{H}_n$ can be decomposed (Moreau decomposition [20]) into the sum of two orthogonal, positive and negative, semidefinite matrices

$$A = A_{\succ} - A_{\prec}, \text{ with } A_{\prec} \succeq 0, A_{\succ} \succeq 0, \; A_{\prec}A_{\succ} = 0.$$

We use $A \otimes B$ to denote the *Kronecker product* of A and B; $\operatorname{vec}(A)$ denotes the vector formed from the columns of the matrix A; while $\operatorname{Diag}(v)$ denotes the diagonal matrix formed from the vector v, and $\operatorname{diag}(A)$ denotes the vector formed from the diagonal of the matrix A.

1.5 Notation

$\mathcal{H}_n$ the space of Hermitian $n \times n$ matrices

$\mathcal{P}_n$ **or** $\mathcal{P}$ the cone of positive semidefinite matrices in $\mathcal{H}_n$

$Q \preceq R$ $R - Q$ is positive semidefinite

$A_{\succ}$ the positive semidefinite part of A

$A_{\prec}$ the negative semidefinite part of A

A^* the adjoint of the linear operator A

$A^{\dagger}$ the Moore-Penrose generalized inverse of the linear operator A

C^+ the polar cone of a set C, $C^+ = \{\phi : \langle \phi, k \rangle \geq 0, \; \forall k \in C\}$

$A \circ B$ the Hadamard product of A and B

$A^{(2)}$ the Hadamard product of A and A

$A \otimes B$ the Kronecker product of A and B

$\operatorname{vec}(A)$ the vector formed from the columns of the matrix A

$\operatorname{diag}(A)$ the vector formed from the diagonal of the matrix A

Diag (v)	the diagonal matrix formed from the vector v
e_i	the i-th unit vector
E_{ij}	the ij unit matrix, $E_{ij} = e_i e_j^t$
E	the matrix of ones
AC	the approximate completion problem
DAC	the dual of the approximate completion problem
SDP	a semidefinite program
$\|\|A\|\|_F$	the Frobenius norm of the matrix A
$f(P)$	the objective function of AC: $\|\|H \circ (A - P)\|\|_F^2$
$L(P, y, \Lambda)$	the Lagrangian of AC: $f(P) + \langle y, (b - KP)\rangle - \operatorname{trace} \Lambda P$
$B_\mu(P)$	the log-barrier function of AC: $f(P) - \mu \log\det P$
$\bar{P}$	an optimum semidefinite matrix of AC
$\bar{\Lambda}$	an optimum semidefinite matrix of DAC

2 DUALITY and OPTIMALITY

2.1 Unconstrained Case

We first consider the special case without the linear equality constraint, i.e. with only the semidefinite constraint. The statement that all feasible directions are not descent directions translates into the following characterization of an optimal solution $\bar{P}$ of AC:

$$\nabla f(\bar{P}) \in (\mathcal{P} - \bar{P})^+, \tag{2.1}$$

where $\mathcal{P}$ denotes the *cone of positive semidefinite matrices* and

$$S^+ = \{P : \operatorname{trace} QP \geq 0,\ \forall Q \in S\}$$

is the *polar cone* of the set S. This yields the following characterization of optimality.

Theorem 2.1 *The matrix $\bar{P} \succeq 0$ solves (AC) if and only if*

$$\operatorname{trace}\,(H^{(2)} \circ (\bar{P} - A))(P - \bar{P}) \geq 0, \ \forall P \succeq 0, \tag{2.2}$$

where $H^{(2)} = H \circ H$ is the Hadamard squared term.

Proof. Note that the gradient acting on the Hermitian matrix h, see e.g. [17], is

$$\begin{aligned} \langle \nabla f(P), h \rangle &= 2\operatorname{trace}\,((H \circ P)(H \circ h) - (H \circ A)(H \circ h)) \\ &= 2\operatorname{trace}\,(H \circ (P - A))(H \circ h) \\ &= 2\operatorname{trace}\,(H \circ h)(H \circ (P - A)) \\ &= 2\operatorname{trace}\,(h \circ H)(H \circ (P - A)) \\ &= 2\operatorname{trace}\,h(H^{(2)} \circ (P - A)). \end{aligned} \tag{2.3}$$

Therefore the gradient of the objective function is

$$\nabla f(P) = 2H^{(2)} \circ (P - A). \tag{2.4}$$

The result follows upon replacing h by the direction $P - \bar{P}$, for $P \succeq 0$, and applying the so-called Pshenichnyi condition (2.1).

$\square$

Corollary 2.1 *If $H = E$, the matrix of ones, then the (unique) optimal solution of AC is $\bar{P} = A_{\succ}$, where $A_{\succ}$ is the positive part of A.*

Proof. We substitute $\bar{P} = A_{\succ}$ in (2.2) and see that

$$\operatorname{trace}\,(H^{(2)} \circ (A_{\succ} - A))(P - A_{\succ}) = \operatorname{trace}\,(A_{\succ} - A)(P - A_{\succ}) = \operatorname{trace}\,A_{\prec}P \geq 0,$$

where the *negative part* of A is denoted $A_{\prec}$, and satisfies $A = A_{\succ} - A_{\prec}$, $A_{\prec} \succeq 0$, and $\operatorname{trace}\,A_{\prec}A_{\succ} = 0$. The objective function is strictly convex in this case, i.e. the Hessian of f acting on h is $\operatorname{trace}\,2hH^{(2)} \circ h = 2||h||_F^2$. Therefore the optimum is unique.

$\square$

Corollary 2.2 *If $H \succ 0$, then the (unique) optimal solution $\bar{P}$ of AC is the solution of*

$$H \circ \bar{P} = (H \circ A)_{\succ}.$$

Proof. From the conclusion and (2.2) we get

$$\begin{array}{rcl} \operatorname{trace}(H^{(2)} \circ (\bar{P} - A))(P - \bar{P}) & = & \operatorname{trace}(H \circ \bar{P} - H \circ A)(H \circ P - H \circ \bar{P}) \\ & = & \operatorname{trace}(H \circ A)_{\prec}(H \circ P) \\ & \geq & 0, \quad \forall P \succeq 0. \end{array} \tag{2.5}$$

This is sufficient for optimality of $\bar{P}$. As in the previous proof, the objective function is strictly convex and the optimum is unique.

□

2.2 General Constrained Case

Another approach involves the dual program of AC which we now derive. For $\Lambda \in \mathcal{H}_n$ and $y \in \mathcal{C}^m$, let

$$L(P, y, \Lambda) = f(P) + \langle y, b - KP\rangle - \operatorname{trace} \Lambda P \tag{2.6}$$

denote the *Lagrangian* of AC. It is easy to see that the primal program AC is equivalent to

$$\mu^* = \min_{P} \max_{\substack{y \\ \Lambda \succeq 0}} L(P, y, \Lambda). \tag{2.7}$$

We assume that the generalized Slater's constraint qualification,

$$\exists\ P \succ 0 \text{ with } KP = b\ ,$$

holds for AC.

Note that if the linear equality constraint does not exist, then the assumption $\operatorname{diag}(H) > 0$ implies that Slater's condition holds. The linear constraint can be written as $(\langle P, A_i\rangle) = (b_i)$, for appropriate symmetric matrices A_i. Then the adjoint operator $K^*y = \sum_i y_i A_i$.

Slater's condition implies that strong duality holds, i.e. this means

$$\mu^* = \max_{\substack{y \\ \Lambda \succeq 0}} \min_{P} L(P, y, \Lambda) \tag{2.8}$$

and μ^* is attained for some $\Lambda \succeq 0$, see e.g. [16]. The inner minimization of the convex, in P, Lagrangian is unconstrained and we can differentiate to get the equivalent program

$$\mu^* = \max_{\substack{\nabla f(P) - K^* y = \Lambda \\ \Lambda \succeq 0}} f(P) + \langle y, b - KP\rangle - \operatorname{trace} \Lambda P. \tag{2.9}$$

We can now state the dual problem.

$$
(DAC)\quad
\begin{array}{lll}
\mu^* := & \max & f(P) + \langle y, b - KP \rangle - \operatorname{trace} \Lambda P \\
& \text{subject to} & \nabla f(P) - K^* y - \Lambda = 0 \\
& & \Lambda \succeq 0.
\end{array}
\tag{2.10}
$$

The above pair of dual programs, AC and DAC, provide an optimality criteria in terms of feasibility and complementary slackness. This provides the basis for many algorithms including primal-dual interior-point algorithms. In particular, we see that the duality gap, in the case of primal and dual feasibility, is given by the complementary slackness condition:

$$
\operatorname{trace} P \left(2H^{(2)} \circ (P - A) - K^* y \right) = 0, \tag{2.11}
$$

or equivalently

$$
P \left(2H^{(2)} \circ (P - A) - K^* y \right) = 0.
$$

Theorem 2.2 *The matrix $\bar{P} \succeq 0$ and vector-matrix $\bar{y}, \bar{\Lambda} \succeq 0$ solve AC and DAC if and only if*

$$
\begin{array}{cc}
K\bar{P} = b & \textit{primal feasibility} \\
2H^{(2)} \circ (\bar{P} - A) - K^* \bar{y} - \bar{\Lambda} = 0 & \textit{dual feasibility} \\
\operatorname{trace} \bar{\Lambda} \bar{P} = 0 & \textit{complementary slackness}
\end{array}
$$

□

The above yields an equation for the solution of AC. This is a very simple optimality characterization and compares to the characterizations of best nonnegative interpolants in [18]. This illustrates the connection between complementarity and the positive and negative parts. This also extends the results in [23].

Corollary 2.3 *The optimal solution of AC is the solution of the two equations*

$$
\bar{P} - (\bar{P} - 2H^{(2)} \circ (\bar{P} - A) + K^* y)_{\succ} = 0 \tag{2.12}
$$

and

$$
K \left(\bar{P} - 2H^{(2)} \circ (\bar{P} - A) + K^* y \right)_{\succ} = b. \tag{2.13}
$$

□

Corollary 2.4 *Suppose that $A = 0$ and $H = \frac{1}{\sqrt{2}}E$. Then the optimal solution of AC is:*

$$\bar{P} = (K^*y)_{\succ}, \tag{2.14}$$

where

$$K\left(K^*y\right)_{\succ} = b. \tag{2.15}$$

$\square$

However, we do not apply Newton's method to (2.12) but rather to a perturbed equation which allows us to stay interior to $\mathcal{P}$.

3 INTERIOR-POINT ALGORITHMS

We now present the interior-point algorithms for AC. We present a primal-step-first and a dual-step-first version. The difference in efficiency arises from the fact that the primal variable P does not change very much if few elements of A are free, while the dual variable Λ does not change very much if many elements of A are free.

Since we can increase the weights in H to try and force an exact completion, we restrict ourselves to the case without the linear equality constraint. This avoids the need for a constraint qualification and simplifies the algorithms.

3.1 Dual Infeasible Algorithm

We now derive a primal-dual interior-point method using the log-barrier approach, see e.g. [9]. This is an alternative way of deriving the optimality conditions in Theorem 2.2. The log-barrier problem for AC is

$$\min_{P \succ 0} B_\mu(P) := f(P) - \mu \log\det P,$$

where $\mu \downarrow 0$. For each $\mu > 0$ we take one Newton step toward minimizing the log-barrier function. Therefore, we take one Newton step for solving the stationarity condition

$$\nabla B_\mu(P) = 2H^{(2)} \circ (P - A) - \mu P^{-1} = 0. \tag{3.1}$$

This equation yields the perturbed complementary slackness equation:

$$2PH^{(2)} \circ (P - A) = \mu I, \tag{3.2}$$

and the estimate of the barrier parameter

$$\mu = \frac{2}{n} \operatorname{trace} PH^{(2)} \circ (P - A). \tag{3.3}$$

The Newton direction is dependent on which of the equations (3.1),(3.2) we choose to solve. The equation (3.2) is shown to perform better in the applications to solving max-cut problems in [9]. A discussion on various choices is given in [1]. (See also [15].) A general discussion of adaptive-step methods for linear programming is given in [19].

The Hessian acting on the matrix h is

$$\nabla^2 B_\mu(P)(h) = 2H^{(2)} \circ h - \mu P^{-1} h P^{-1}. \tag{3.4}$$

The equation for the Newton step h is

$$\nabla^2 B_\mu(P)(h) = 2H^{(2)} \circ h - \mu P^{-1} h P^{-1} = -\nabla f(P) + \mu P^{-1}. \tag{3.5}$$

We use the Kronecker product to replace this equation by an ordinary matrix-vector equation

$$\begin{aligned} &\left[2\operatorname{Diag}\left(\operatorname{vec}\left(H^{(2)}\right) - \mu(P^{-1} \otimes P^{-1})\right] \operatorname{vec}(h) \\ &\qquad = \operatorname{vec}\left(-2H^{(2)} \circ (P - A) + \mu P^{-1}\right). \end{aligned} \tag{3.6}$$

We now have a choice of several interior-point methods, e.g. an infeasible primal-dual method or a feasible primal-dual method. If we assume that the diagonal of H is positive, then we can always start with strict dual feasibility and so we can have a standard feasible primal-dual method. Otherwise, there can not be a strictly feasible dual solution. In this case we must use a different convergence criteria.

3.2 Primal-Dual Feasible Algorithms

We could modify the above procedure and guarantee dual feasibility under the assumption that $\operatorname{diag}(H) > 0$. For then, there exists an initial $P \succ 0$, such that $2H^{(2)} \circ (P - A) \succ 0$ as well. We can fix $P_{ij} = A_{ij}$ throughout the algorithm, for elements corresponding to $H_{ij} = \infty$. We then solve for the step h and backtrack to ensure both primal and dual strict feasibility. This yields the primal-step-first algorithm since we only solve for the step h for changes in the primal variable P. We do need to evaluate the dual variable in order to update the barrier parameter μ using the perturbed complementarity condition.

Alternatively, we can work with dual feasibility and perturbed complementary slackness. (We follow the approach in [9]. See also [19].) We replace equations (3.1) and (3.2) with the following two:

$$\begin{array}{rlc} 2H^{(2)} \circ (P - A) - \Lambda = 0 & \text{dual feasibility} \\ -P + \mu\Lambda^{-1} = 0 & \text{perturbed complementary slackness} \end{array} \tag{3.7}$$

Note that the dual feasibility equation adds restrictions to the diagonal of P in an interior-point approach. Since we want $\Lambda \succ 0$, we cannot assume that the diagonal of P is fixed equal to the diagonal of A; nor can we assume that the diagonal of H is 0. This means that we cannot fix the diagonal of P nor set it completely free. However, we can have arbitrarily large, or small, weights on the diagonal of H.

We apply Newton's method to solve (3.7). We let h denote the step for P and l denote the step for Λ. Then linearization of the second equation yields

$$-(P + h) + \mu\Lambda^{-1} - \mu\Lambda^{-1}l\Lambda^{-1} = 0$$

or

$$P + h = \mu\Lambda^{-1} - \mu\Lambda^{-1}l\Lambda^{-1}. \tag{3.8}$$

We get

$$h = \mu\Lambda^{-1} - \mu\Lambda^{-1}l\Lambda^{-1} - P \tag{3.9}$$

and

$$l = \frac{1}{\mu}\{-\Lambda(P + h)\Lambda\} + \Lambda. \tag{3.10}$$

The linearization of the dual feasibility equation yields

$$2H^{(2)} \circ h - l = -(2H^{(2)} \circ (P - A) - \Lambda). \tag{3.11}$$

We assume that we start with an initial dual feasible solution. However, we include the feasibility equation on the right-hand-side of (3.11), because roundoff error can cause loss of feasibilty. (Since Newton directions maintain linear equations, we could theoretically substitute for h in this linearization with the right-hand side being 0.)

Dual-Step-First

We can eliminate the primal step h and solve for the dual step l.

$$\begin{array}{rcl} l & = & 2H^{(2)} \circ h + (2H^{(2)} \circ (P - A) - \Lambda) \\ & = & 2H^{(2)} \circ (\mu\Lambda^{-1} - \mu\Lambda^{-1}l\Lambda^{-1} - P) + (2H^{(2)} \circ (P - A) - \Lambda). \end{array} \tag{3.12}$$

Equivalently, we get the Newton equation

$$2H^{(2)} \circ (\mu\Lambda^{-1}l\Lambda^{-1}) + l = 2H^{(2)} \circ (\mu\Lambda^{-1} - A) - \Lambda. \tag{3.13}$$

We can now solve for l and substitute to recover h. We then backtrack and ensure both primal and dual positive definiteness.

The equation (3.13) is equivalent to

$$\begin{aligned}\left[2\mathrm{Diag}\,(\mathrm{vec}\,(H^{(2)})\mu(\Lambda^{-1} \otimes \Lambda^{-1}) + I\right] \mathrm{vec}\,(l) \\ = \mathrm{vec}\,\left(2H^{(2)} \circ (\mu\Lambda^{-1} - A) - \Lambda\right).\end{aligned} \tag{3.14}$$

We see that in this final form we can exploit the possible sparsity of H, i.e. $H_{ij} = 0 \Rightarrow l_{ij} = 0$. Therefore we can delete the corresponding rows and columns of the Kronecker product when solving the system of equations. Let F denote the $k \times 2$ matrix with row s denoting the s-th nonzero element of H, i.e. for $s = 1, \ldots, k$,

$$(F_{s1}, F_{s2})_{s=1,\ldots k} = \{ij : H_{ij} \neq 0\}.$$

Note that the (kl, ij) component of the Hadamard-Kronecker product $2H^{(2)} \circ (\mu\Lambda^{-1}l\Lambda^{-1})$ is found from

$$\begin{aligned} 2\mu\mathrm{trace}\, E_{kl}\left(H^{(2)} \circ \Lambda^{-1}E_{ij}\Lambda^{-1}\right) &= 2\mu\mathrm{trace}\, e_k e_l^t \left(H^{(2)} \circ \Lambda^{-1}e_i e_j^t\Lambda^{-1}\right) \\ &= 2\mu\mathrm{trace}\, e_l^t \left(H^{(2)} \circ \Lambda^{-1}_{:,i}\Lambda^{-1}_{j:}\right) e_k \\ &= 2\mu H^{(2)}_{lk}\Lambda^{-1}_{li}\Lambda^{-1}_{jk}. \end{aligned}$$

Suppose that we represent the Newton system as

$$L\mathrm{vec}\,(l) = r, \tag{3.15}$$

then for $s = kl, t = ij$, the st component of the matrix L is

$$L_{s,t} = \begin{cases} 2\mu H^{(2)}_{F_{s_2},F_{s_1}}\Lambda^{-1}_{F_{s_2},F_{t_1}}\Lambda^{-1}_{F_{t_2},F_{s_1}} & \text{if } s \neq t \\ 2\mu H^{(2)}_{F_{s_2},F_{s_1}}\Lambda^{-1}_{F_{s_2},F_{t_1}}\Lambda^{-1}_{F_{t_2},F_{s_1}} + 1 & \text{if } s = t \end{cases} \tag{3.16}$$

The $s = ij$-component of the right-hand-side of the system (3.15) is

$$r_s = 2H^{(2)}_s\left(\mu\Lambda^{-1}_s - A_s\right) - \Lambda_s.$$

This simplifies the construction of the linear system, especially in the large sparse case.

Primal-Step-First

Alternatively, if many elements of H are sufficiently large, i.e. if we fix (or specify) elements of A, then it is more efficient to eliminate l and solve for h first. By equating l in (3.10) and (3.11), and moving h to one side, we get the system

$$2H^{(2)} \circ h + \frac{1}{\mu}\Lambda h \Lambda = \Lambda - \frac{1}{\mu}\Lambda P \Lambda - \left(2H^{(2)} \circ (P - A) - \Lambda\right) \tag{3.17}$$

or, equivalently,

$$\begin{aligned} &\left[2\mathrm{Diag}\,(\mathrm{vec}\,(H^{(2)}) + \tfrac{1}{\mu}(\Lambda \otimes \Lambda)\right] \mathrm{vec}\,(h) \\ &\quad = \mathrm{vec}\,\left(\Lambda - \tfrac{1}{\mu}\Lambda P \Lambda - (2H^{(2)} \circ (P - A) - \Lambda)\right). \end{aligned} \tag{3.18}$$

We solve this system for h and then substitute to get l using (3.10).

Note that if the ij component of P is specified or fixed, i.e. if the corresponding element of H is sufficiently large, then the ij component of the primal step $h_{ij} = 0$. Therefore, if there are many elements of H which are sufficiently large, then the size of the system (3.18) is very small, i.e. it is the same order as the number of elements corresponding to relatively small values in H. Let F denote the $k \times 2$ matrix with row s denoting the s-th free element of P, i.e. for $s = 1, \ldots, k$,

$$(F_{s1}, F_{s2})_{s=1,\ldots k} = \{ij : P_{ij} \text{ is free}\}.$$

Note that the (kl, ij) component of the Kronecker product $\Lambda \otimes \Lambda$ is found from

$$\mathrm{trace}\, E_{kl}\Lambda E_{ij}\Lambda = \mathrm{trace}\, e_k e_l^t \Lambda e_i e_j^t \Lambda = \Lambda_{li}\Lambda_{jk}.$$

Also, the kl component of $\Lambda P \Lambda$ is

$$\mathrm{trace}\, E_{kl}\Lambda P \Lambda = \mathrm{trace}\, e_k e_l^t \Lambda P \Lambda = \Lambda_{l:} P \Lambda_{:k}.$$

If we represent the Newton system as

$$L\mathrm{vec}\,(h) = r, \tag{3.19}$$

then for $s = kl, t = ij$, the st component of the matrix L is

$$L_{s,t} = \begin{cases} \Lambda_{F_{s_1},F_{t_1}}\Lambda_{F_{t_2},F_{s_2}} & \text{if } s \neq t \\ 2\mu H^{(2)}_{F_{s_1},F_{s_2}} + \Lambda_{F_{s_1},F_{s_1}}\Lambda_{F_{s_2},F_{s_2}} & \text{if } s = t \end{cases} \tag{3.20}$$

If we denote the dual feasibility residual as

$$r_D := 2H^{(2)} \circ (P - A) - \Lambda,$$

then the $s = ij$-component of the right-hand-side of the system (3.19) is

$$r_s = \Lambda_s - \frac{1}{\mu} \Lambda_{j:} P \Lambda_{:i} - (r_D)_s.$$

4 NUMERICAL TESTS

The feasible interior-point algorithm was tested on randomly generated problems. We used MATLAB as the programming language. (The files are available by anonymous ftp, see the address and URL on the front page.) The tests were done on a SUN SPARC 1 with 16megs RAM and with SPECint 18.2 and SPECfp 17.9.

The major part of the work of one iteration is the construction and solution of the linear system to be solved for the Newton direction. Solving the linear system was much less expensive since it depended directly on the number of nonzeros and not on the dimension of the original problem.

For the dual-step-first algorithm, we need to form and solve the system in (3.14). This requires finding the inverse of the dual variable, Λ^{-1}, $O(n^3)$ operations. Then we need to form the Kronecker product $\Lambda^{-1} \otimes \Lambda^{-1}$, $O(\bar{n}^4)$ operations ($\bar{n}$ denotes the number of nonzero elements of H), and solve the resulting system, which involves $O(\bar{n}^6)$ operations. Note that the complexity of both these latter operations depends completely on the sparsity of H, i.e. on the number of nonzeros in H. The algorithm needs an initial positive definite matrix $P \succ 0$, for which the corresponding dual variable is positive definite as well, i.e. $2H^{(2)} \circ (P - A) = \Lambda \succ 0$. We obtain this by setting $P = A + tI$, where t is chosen using the smallest eigenvalue of A; which is, in turn, found using a Lanczos algorithm.

Similarly, for the primal-step-first algorithm, we use the system in (3.18). The complexity is as above except that it depends on the number of zeros, or "small" elements of H, i.e. elements which are not "sufficiently large". The initial point is found using a similar heuristic as in the dual-step-first algorithm.

In both algorithms, we have to be careful with the diagonal of P. We cannot have 0, or sufficiently large, elements on the diagonal of H. This can cause problems in finding an initial positive definite dual variable.

We first present the results for the dual-step-first algorithm. The matrix of weights H was chosen sparse, nonnegative, with positive diagonal, i.e. .1 was added to the diagonal to ensure that it is positive.

Different tolerances and dimensions were tested with no appreciable difference in the number of iterations. Typically, the duality gap is halved at each iteration. The algorithm stops with a guarantee that the duality gap meets the desired tolerance. Since the objective function is always nonnegative, this is equivalent to

$$\min \{f(P), \operatorname{trace} \Lambda P\} \leq \text{tolerance}.$$

The details for several of the tests follow in Tables 1 and 2. (Each test appears on one line and includes 20 test problems.)

dim	toler	H dens. / infty	$A \succeq 0$	cond(A)	$H \succ 0$	min/max	iters
60	10^{-6}	.01 / .001	yes	79.738	no	15/23	16.8
65	10^{-6}	.015 / .001	yes	49.8611	yes	18/24	21.25
83	10^{-6}	.007 / .001	no	235.1547	no	24/29	25.45
85	10^{-5}	.008 / .001	yes	94.6955	no	11/17	13.05
85	10^{-6}	.0075 / .001	no	299.8517	no	23/27	25.25
87	10^{-6}	.006 / .001	yes	74.163	yes	14/19	16.85
89	10^{-6}	.006 / .001	no	179.33	no	23/28	15.2
110	10^{-6}	.007 / .001	yes	172.255	yes	15/20	17.8
155	10^{-6}	.01 / 0	yes	643.9619	yes	14/18	15.3

Table 1: data for dual-step-first (20 problems per test): dimension; tolerance for duality gap; density of nonzeros in H/ density of infinite values in H; positive semidefiniteness of A; positive definiteness of H; min and max number of iterations; average number of iterations.

5 CONCLUSION

We have presented optimality conditions and an interior-point algorithm for the approximate completion problem AC. We have shown that the interior-point approach is very efficient and robust. We have tested the algorithm successfully on randomly generated problems. The algorithm never failed to find the optimal completion. In fact, the algorithm was very robust; the

dim	toler	H dens. / infty	$A \succeq 0$	cond(A)	$H \succ 0$	min/max	iters
15	10^{-5}	.0751 / .02	yes	222.52	no	8/17	10.25
15	10^{-6}	.1 / .95	yes	19.6102	no	10/23	15.55
15	10^{-6}	.01 / .95	yes	21.0473	no	10/20	13.2
19	10^{-6}	.005 / .1	yes	14.811	no	10/18	12.95
21	10^{-6}	.005 / .1	yes	20.3077	no	8/24	15
38	10^{-6}	1 / .99	yes	49.6068	yes	14/24	16.3
45	10^{-6}	1 / .99	yes	46.8396	yes	15/22	17.1
55	10^{-6}	1 / .99	yes	37.1883	yes	15/30	17.45
85	10^{-5}	.0219 / .02	yes	1374.54	no	16/23	18.95
95	10^{-5}	.0206 / .02	yes	2.6983	no	8/14	11.05
95	10^{-6}	1 / .999	yes	196.0130	yes	14/18	16.8
145	10^{-6}	.01 / .997	yes	658.5103	yes	13/17	14.95

Table 2: data for primal-step-first (20 problems per test): dimension; tolerance for duality gap; density of nonzeros in H/ density of infinite values in H; positive semidefiniteness of A; positive definiteness of H; min and max number of iterations; average number of iterations.

number of iterations consistently fell within a very small range. Degeneracy and ill-conditioning did not affect the algorithm. In particular, there were no difficulties for problems where only a singular semidefinite completion existed. If a positive definite completion existed, then one was always found, even if there were multiple optimal solutions for AC. In addition, the algorithm exploited sparsity and so was successful on relatively large problems.

We have not compared our algorithm directly with others. Other algorithms that solve similar problems have appeared in the literature. In particular, AC is a semidefinite programming problem and so can be expressed as a problem with a lower bound on the smallest eigenvalue. Algorithms for eigenvalue constrained problems have been solved by active set type methods, see e.g. [22]. However, recent testing has shown that our approach using primal-dual interior-point methods is superior, see e.g. [9]. Also, we did not include CPU times. The tests were done on a SPARC 1 using MATLAB. The major time was taken in forming the operator. MATLAB is notoriously slow when using loops. The CPU time can be reduced significantly if the operation is vectorized or the code is rewritten in C (for which MATLAB now has the capability). As an illustration of the poor CPU time due to the poor performance of MATLAB while looping, the last problem in Table 1, of dimension 155, took approximately 90 seconds per iteration to form the operator but only 6 seconds to solve the resulting system for the Newton direction. (This test was done on a SPARC 20.)

In addition, we do not present any convergence proofs. These proofs can be obtained by guaranteeing sufficient decrease at each step. This can be done using a potential function or via an Armijo-Goldstein-Wolfe type step, see e.g. [9].

There are still many unanswered questions. It is not clear that we have used the best primal-dual interior-point algorithm. We did not use a predictor-corrector approach or test out different forms of the complementarity condition, e.g. (3.2). It is not clear whether these changes, which do improve performance for SDP problems ([1],[15]), would improve performance here, since they would have to exploit sparsity to the same extent.

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