SEMIDEFINITE PROGRAMMING

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Abstract

Semidefinite programming is an extension of linear programming where (some of) the vector variables are replaced by matrix variables and (some of) the nonnegativity elementwise constraints are replaced by positive semidefiniteness constraints. These are convex problems which can be solved efficiently by interior-point methods. They arise in many applications, e.g. in combinatorial optimization, matrix completion problems, stability of differential systems, and, more generally, as the dual of Lagrangian relaxations of quadratic models of numerically hard problems.

1 Introduction

1.1 Background

Semidefinite programming (denoted SDP) is an extension of linear programming (LP), where (some of) the vector variables are replaced by matrix variables and (some of) the nonnegativity elementwise constraints are replaced by positive semidefiniteness constraints. (The semidefiniteness constraints are also referred to as linear matrix inequalities.) We can express a linear (primal) SDP as

$$(PSDP) \qquad \begin{array}{rl} \mu^* := \min & C \bullet X \\ \text{s.t.} & \mathcal{A}X = b \\ & X \succeq 0, \end{array}$$

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where: C, X are in $\mathcal{S}^n$, the space of symmetric, real, $n \times n$ matrices; the inner product $C \bullet X = \operatorname{trace} CX$; $\succeq$ denotes nonnegativity in the Löwner partial order, i.e. $A \succeq B$ if $A - B \succeq 0$, i.e. $A - B \in \mathcal{S}^n_+$, the cone of positive semidefinite matrices; and

$$\mathcal{A} : \mathcal{S}^n \rightarrow \Re^m$$

is a linear operator, i.e. $(\mathcal{A}X)_i = \operatorname{trace}(A_i X)$, for given $A_i \in \mathcal{S}^n$, $i = 1, \ldots m$. In fact, SDP is a special case of the cone programming problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x) \succeq_K 0, \end{array}$$

where f, g are appropriate functions, K is a convex cone, and $g(x) \succeq_K 0$ denotes the cone partial order, i.e. $g(x) \in K$. This is a very general mathematical program which can include standard equality and inequality constraints, since K can consist of the cross product of many cones, e.g. the nonnegative orthant, the semidefinite cone, and the origin $\{0\}$, $K = \Re^n_+ \otimes \mathcal{S}^n_+ \otimes \{0\}$.

Thus SDP includes LP as a special case; and so it has all the elegant duality and geometric results that LP does. In addition, it is a nonlinear problem and so has extra subtleties that create many interesting difficulties; the geometry of the cone $\mathcal{P}$ is particular elegant and well studied. Linear SDPs are convex programs that can be solved very efficiently using interior-point methods. And, SDP also has many very important applications. It arises as the dual of Lagrangian relaxations for quadratic models of numerically hard problems. Thus, we have an efficient tool for finding bounds and estimates for many hard problems in a variety of different areas, e.g. in Engineering, Combinatorics, Statistics.

1.1.1 SDP Relaxation for Max-Cut Problem

The success and popularity of SDP is exemplified by the results of the relaxation for the max-cut problem. Let $G = (V, E)$ be an undirected graph with edge set $V = \{v_i\}_{i=1}^n$ and weights w_{ij} on the edges $(v_i, v_j) \in E$. We want to find the index set $\mathcal{I} \subset \{1, 2, \ldots n\}$, to maximize the weight of the edges with one end point with index in $\mathcal{I}$ and the other in the complement. This is equivalent to

$$(MC) \qquad mc^* := \max \ \tfrac{1}{2} \textstyle\sum_{i<j} w_{ij}(1 - x_i x_j), \quad x \in \mathcal{F},$$

where $\mathcal{F} := \{\pm 1\}^n$, and $x_i = 1$ if $i \in \mathcal{I}$ and -1 otherwise. The objective function is a (homogeneous) quadratic form, $x^T Q x$. Therefore, we get the equivalent quadratic model

$$(QMC) \qquad \max \ x^T Q x, \quad x_i^2 = 1, \quad \forall i.$$

The LP relaxation is

$$(LPMC) \qquad \max \ x^T Q x, \quad -1 \leq x_i \leq 1, \quad \forall i,$$

which yields very little useful information, e.g. the maximum is $x = 0$ when $Q \preceq 0$ and is intractable otherwise.

The SDP relaxation can be obtained easily from (QMC) by noting $x^T Qx = \operatorname{trace} x^T Qx = \operatorname{trace} Qxx^T$. We then substitute $X = xx^T$ and replace/relax the rank one condition on X to $X \succeq 0$. This yields

$$(SDPMC) \qquad pmc^* := \max \ \operatorname{trace} QX, \quad \operatorname{diag}(X) = e, \quad X \succeq 0,$$

where $\operatorname{diag}(X)$ denotes the vector formed from the diagonal of X and e is the vector of ones.

There are several other equivalent forms. Another approach is to use the Lagrangian relaxation of the quadratic model (QMC).

$$\begin{array}{rcl} mc^* \leq dmc^* : & = & \min\limits_{\lambda} \max\limits_{x} x^T Qx - \sum_{i=1}^n \lambda_i (x_i^2 - 1) \\ & = & \min\limits_{Q - \operatorname{Diag}(\lambda) \succeq 0} e^T \lambda, \end{array}$$

where the semidefinite constraint arises from assuming boundedness of the inner unconstrained quadratic maximization problem, which then has a maximum attained at $x = 0$. Since the Slater constraint qualification holds, $Q - \operatorname{Diag}(\lambda) \succ 0$, we conclude that strong duality holds with (SDPMC), i.e. $mc^* \leq dmc^* = pmc^*$ and (SDPMC) is attained.

An alternative approach to this relaxation uses the fact that, for any fixed normalized vector $u \in \Re^n$, we can reformulate (MC) as

$$(MC) \qquad mc^* = \max \ \tfrac{1}{2} \sum_{i<j} w_{ij}(1 - v_i^T v_j), \quad v_i \in \{-u, u\}, \quad \forall i. \tag{1}$$

This leads to the relaxation used in GOEMANS and WILLIAMSON [1995] which is again equivalent to the SDP relaxation. (Surprisingly, see [POLJAK and WOLKOWICZ, 1995, POLJAK et al., 1995], this relaxation is equivalent to several others such as the LP relaxation (box) with perturbations on the objective function.) GOEMANS and WILLIAMSON [1995] use (1) and a randomized algorithm which provides an approximate solution to (MC) whose objective value is within 1.14 of the optimal value. Previous results were within 1.5 of the optimal value. In practice, much better results (e.g. 1.06 on average in HELMBERG et al. [1996b]) are obtained empirically.

In summary, the SDP relaxation provides an efficient way of solving the Lagrangian relaxation which has been shown to be very strong both theoretically [GOEMANS and WILLIAMSON, 1995] and in practice [HELMBERG et al., 1996a, HELMBERG and RENDL, 1997, BENSON et al., 1997, BURER and MONTEIRO, 1999].

1.2 Historical notes

It is only recently that SDP has become a mathematical discipline in its own right. This is due to the many important applications that have arisen, as well as to the development of efficient primal-dual interior-point (denoted p-d i-p) algorithms that can solve SDPs.

Up until this time, LP models were, and still are, the principal tool used in approximations of numerically hard problems. DANTZIG [1991] discusses his well known tale about his confrontation with Hotelling at his first presentation of the simplex method for LP, i.e. after his presentation Hotelling stated: “But we all know that the world is nonlinear” and that Von Neumann replied for him: “The speaker titled his talk ‘linear programming’ and carefully stated his axioms. If you have an application that satisfies the axioms, well use it. If it does not, then don’t.” And, this provides a very good reason for the success of linear programming, i.e. there are many applications and the software that has been developed over the years provides us with very efficient solution techniques, e.g. NEMHAUSER et al. [1989]. The main algorithms were originally based on Dantzig’s simplex method. Then, begun by the work of KARMARKAR [1984], a revolution started in the way we solve LPs and led to the p-d i-p methods which has resulted in a much greater increase in size and speed, e.g. LUSTIG et al. [1994].

However, DANTZIG [1991] himself states: “In the final analysis, of course, Hotelling was right. The world is highly nonlinear. Fortunately systems of linear inequalities (as opposed to equalities) permit us to approximate most of the kinds of nonlinear relations encountered in practical planning.” Thus we can see the reason for the great success of LP: good approximations of many problems and efficient algorithms. However, what if we could efficiently solve quadratic approximations? That is where SDP has stepped in, i.e. it can efficiently solve approximations of quadratic models of many applied problems and, thus, obtain improved results compared to LP models.

As mentioned above, SDP is a special case of cone programming which is not a new area; nor is SDP itself. Much of the theory of duality extends directly from LP to SDP. Indeed, this was recognized for cone programs as early as 1963 by BELLMAN and FAN [1963]. Since that time many researchers in diverse areas from engineering, matrix theory, statistics, optimization have contributed to results on SDP.

2 THEORY

2.1 Derivation of SDP Relaxation for a General QQP

The SDP relaxation for general quadratically constrained quadratic programs, denoted QQP, can be derived from the Lagrangian relaxation. Define the quadratic function

$$q_i(y) = \frac{1}{2} y^t Q_i y + y^t b_i + c_i, \; y \in \Re^n$$

and the QQP

$$(\mathbf{QQP}) \qquad \begin{array}{rll} q^* := & \min & q_0(y) \\ & \text{s.t.} & q_i(y) \le 0, \quad i = 1, \dots m. \end{array} \tag{2}$$

Then the Lagrangian, with Lagrange multipliers x_i, is

$$L(y, x) \quad = \quad q_0(y) + \textstyle\sum_{i=1}^m x_i q_i(y), \tag{3}$$

or equivalently,

$$
\begin{array}{rcl}
L(y,x) & = & \frac{1}{2}y^t(Q_0+\sum_{i=1}^m x_iQ_i)y \\
& & +y^t(b_0+\sum_{i=1}^m x_ib_i) \\
& & +(c_0+\sum_{i=1}^m x_ic_i).
\end{array} \tag{4}
$$

Weak duality follows from the *hidden constraint*, i.e. the primal constraints arise from the maximization over $x \geq 0$ in

$$d^* := \max_{x\geq 0}\min_y L(y,x) \leq q^* = \min_y \max_{x\geq 0} L(y,x).$$

We now homogenize, i.e. we multiply the linear term by a new variable y_0 to get

$$y_0y^t(b_0+\sum_{i=1}^m x_ib_i), \quad y_0^2=1.$$

Therefore the dual value

$$
\begin{array}{rccc}
d^* & = & \max\limits_{x\geq 0}\min_y & L(y,x) \\
& = & \max\limits_{x\geq 0}\min\limits_{y_0^2=1} & \frac{1}{2}y^t(Q_0+\sum_{i=1}^m x_iQ_i)y+ty_0^2 \\
& & & +y_0y^t(b_0+\sum_{i=1}^m x_ib_i) \\
& & & +(c_0+\sum_{i=1}^m x_ic_i)-t \\
& = & \max\limits_{x\geq 0,t}\min\limits_{y} & \frac{1}{2}y^t(Q_0+\sum_{i=1}^m x_iQ_i)y+ty_0^2 \\
& & & +y_0y^t(b_0+\sum_{i=1}^m x_ib_i) \\
& & & +(c_0+\sum_{i=1}^m x_ic_i)-t.
\end{array}
$$

We now have a new hidden constraint, i.e. a quadratic is unbounded below only if the Hessian is positive semidefinite, i.e. we get

$$B := \begin{pmatrix} 0 & b_0^t \\ b_0 & Q_0 \end{pmatrix},$$

$\mathcal{A} : \Re^{m+1} \to \mathcal{S}_{n+1}$

$$\mathcal{A}\begin{pmatrix} t \\ x \end{pmatrix} := -\begin{bmatrix} t & \sum_{i=1}^m x_ib_i^t \\ \sum_{i=1}^m x_ib_i & \sum_{i=1}^m x_iQ_i \end{bmatrix}$$

and the SDP constraint

$$B-\mathcal{A}\begin{pmatrix} t \\ x \end{pmatrix} \succeq 0.$$

The dual program is equivalent to the SDP (with $c_0=0$)

$$
\begin{array}{lccl}
 & d^* = & \sup & -t+\sum_{i=1}^m x_ic_i \\
(\mathbf{D}) & & \text{s.t.} & \mathcal{A}\begin{pmatrix} t \\ x \end{pmatrix} \preceq B \\
 & & & x\in\Re^m, t\in\Re.
\end{array}
$$

We can now form the Lagrangian dual again and get the dual of this SDP to get the SDP relaxation of the QQP. (More details on duality is given below in Section 2.3.)

$$
\begin{array}{llll}
 & d^* = & \inf & \operatorname{trace} BU \\
(\mathbf{DD}) & & \text{s.t.} & \mathcal{A}^*U = \begin{pmatrix} -1 \\ c \end{pmatrix} \\
 & & & U \succeq 0.
\end{array}
$$

(See SHOR [1987], BOYD et al. [1994a], POLJAK et al. [1995].)

Remark 1 *There is no hidden constraint on the Q_i if all q_i are convex. There is no need for SDP relaxations in the convex case and better algorithms exist.*

2.2 Geometry

Much of the elegant geometry of polyhedral sets developed for LP can be extended to SDP though the results are much more difficult and, if anything, more interesting. (See BOHNENBLUST [1948], TAUSSKY [1968], BARKER and CARLSON [1975] and more recently LEWIS [1998].) The semidefinite cone is *self-polar*

$$\mathcal{P} = \mathcal{P}^+ := \{Y : X \cdot Y \geq 0, \forall X \in \mathcal{P}\};$$

and it is *homogeneous*, i.e. for any $X, Y \in \operatorname{int}(\mathcal{P})$, there exists an invertible linear operator $\mathcal{A}$ that leaves $\mathcal{P}$ invariant and $\mathcal{A}(X) = Y$. The *faces* of $\mathcal{P}$ can be characterized using the null space, $\mathcal{N}$, (or range space, $\mathcal{R}$) of any matrix in the relative interior, i.e. if $\mathcal{F}$ is a face and X is in its relative interior then

$$\mathcal{F} = \{Y : \mathcal{N}(Y) \supset \mathcal{N}(X)\}.$$

The faces $\mathcal{P}$ are *exposed*, i.e. $\mathcal{F} = \mathcal{P} \cap \phi^\perp$, where ϕ is any matrix in the *conjugate face*, i.e. in $\mathcal{P} \cap \mathcal{F}^\perp$. (Here $\cdot\perp$ denote orthogonal complement.) also the faces are *projectionally exposed* [BORWEIN and WOLKOWICZ, 1981b], e.g. $\mathcal{F} = P\mathcal{P}P$, where P is a projection matrix and $\mathcal{R}(P) = \mathcal{R}(X)$. In addition, for any face $\mathcal{F}$, $\mathcal{P} + \mathcal{F}^\perp$ is always closed while $\mathcal{P} + \operatorname{span}(\mathcal{F})$ is never closed [RAMANA et al., 1997]. This latter condition can cause problems in optimality characterizations. Further results on the facial structure and multiplicity of eigenvalues in optimal faces can be found in e.g. PATAKI [1998, 1996].

One reason for the special nature of SDP is that the cone $\mathcal{P}$ is both homogeneous and self-dual. Such cones have been completely classified and include only the Lorentz cone (used for convex quadratic programming with convex quadratic constraints), the semidefinite cone $\mathcal{P}$ and its extensions (of which the most important is the cone of positive semidefinite complex Hermitian matrices), and products of these (including $\Re^n_+$, a product of n trivial semidefinite cones). For more details, see GÜLER and TUNCEL [1998].

2.3 Duality Theory and Optimality Conditions

Much of the theory of duality extends directly from LP to SDP. As mentioned above, this was recognized as early as 1963 by BELLMAN and FAN [1963]. They also observed that strong duality theorems require a Slater-type constraint qualification (strict feasibility). Note that we get weak duality from

$$\begin{aligned}\mu^* &= \min_{X\succeq 0}\max_{y} C\cdot X + y^T(b-\mathcal{A}X)\\ &\leq \max_{y}\min_{X\succeq 0} y^T b + (C-\mathcal{A}^*y)\cdot X\\ &= \nu^*,\end{aligned}$$

where $\mathcal{A}^*$ denotes the adjoint operator of $\mathcal{A}$, $\mathcal{A}^*y = \sum_i y_i\mathcal{A}_i$, and

$$(DSDP) \qquad \begin{array}{rl}\nu^* := \max & b^T y\\ \text{s.t.} & \mathcal{A}^*y \preceq C\end{array}$$

defines the dual program. If Slater's constraint qualification, a strictly feasible solution, exists for the primal (dual, respectively) then there is a zero duality gap, $\mu^* = \nu^*$, and the dual (primal, respectively) is attained.

If the Slater constraint qualification fails, then one can work with the minimal cones and obtain characterizations of optimality and strong duality theorems, see BORWEIN and WOLKOWICZ [1981a]. (For (PSDP) this is the minimal face of $\mathcal{P}$ containing the feasible set. For (DSDP) this is the minimal face of $\mathcal{P}$ which contains $C-\mathcal{A}^*\mathcal{F}$, where $\mathcal{F}$ denotes the feasible set.) One can also obtain a strong dual program whose size is polynomial in the data of the original problem [RAMANA, 1997]. The relation between these duals is given in RAMANA et al. [1997].

If a constraint qualification holds, then the duality theory gives rise to the characterization of optimality

$$\begin{array}{ccll}\mathcal{A}^*y + Z - C & = & 0 & \text{dual feasibility}\\ b-\mathcal{A}(X) & = & 0 & \text{primal feasibility}\\ ZX & = & 0 & \text{complementary slackness}\\ Z, X \succeq 0. & & & \end{array} \tag{5}$$

We call $X, (y, Z)$ a primal-dual optimal pair. Z is the (dual) slack variable.

First and second order optimality conditions for general cone programming are given in e.g. SHAPIRO [1997, 1998].

2.4 Degeneracy and Strict Complementarity

The notions of degeneracy and strict complementarity do not directly follow through from LP to SDP. Primal (dual) nondegeneracy implies a unique dual (primal) solution. Moreover, the nondegeneracy does not imply strict complementary slackness. A theorem of Goldman and Tucker shows that every (finite optimal valued) LP has an optimal primal-dual pair that satisfies strict complementarity (for SDP this translates to $Z + X \succ 0$). However, this result fail to hold for SDP. Furthermore, the formulation of nondegeneracy conditions requires study of the geometry of the semidefinite cone $\mathcal{P}$. The boundary of $\mathcal{P}$, like that of $\Re^n_+$, is composed of smooth pieces, but the pieces are nonlinear. Degeneracy and strict complementary slackness is studied in ALIZADEH et al. [1997], BONNANS and SHAPIRO [1998]; generic results are given in PATAKI and TUNCEL [1997].

2.5 Complexity

SDP are convex programs and fall into the class of problems that can be approximately solved in polynomial time. The complexity is based on self-concordant barriers for the cone $\mathcal{P}$. The smallest (best) barrier parameter is given in GÜLER and TUNCEL [1998].

The complexity of determining the (exact) feasibility of a system of linear matrix inequalities is discussed in PORKOLAB and KHACHIYAN [1997], RAMANA [1993].

3 ALGORITHMS

3.1 Interior-Point Algorithms

The SDP research is highlighted by the elegant primal-dual interior-point algorithms. The interior-point methods are extensions of polynomial time algorithms developed originally for LP. The seminal work of NESTEROV and NEMIROVSKI [1994] shows that $\log\det(X)$ is a self-concordant barrier function for SDP. Thus SDP can be solved in polynomial time using a sequence of barrier subproblems. ALIZADEH [1995] presented a transparent approach to extend potential function methods from LP to SDP. Simultaneously, strong numerical results for SDP and the special case of the min-max eigenvalue problem appeared in e.g. HELMBERG et al. [1996b] and JARRE [1993]. This started a flood of results in SDP from researchers in LP.

The most popular methods are the p-d i-p methods. These are based on applying Newton's method to solving the optimality conditions from the barrier problems, i.e. the perturbed optimality conditions

$$\begin{array}{rcl} Z + C - \mathcal{A}^* y & = & 0, \\ b - \mathcal{A}(X) & = & 0, \\ X - \mu Z^{-1} & = & 0. \end{array} \tag{6}$$

As in LP, the last equation can be written in many ways, e.g.

$$ZX = \mu I. \tag{7}$$

Different choices give rise to different search directions. However, unlike LP, choices like (7) result in (6) being an overdetermined system of nonlinear equations and so may not have a Newton direction. There are many ways to overcome this problem, e.g. by various symmetrization techniques, e.g. various symmetrization techniques [MONTEIRO and ZHANG, 1998, TODD, 1997] or solve the using Gauss-Newton approach [KRUK et al., 1998].

This is still an active area of research in particular for the large sparse problems.

3.2 Bundle Trust Algorithms

As seen above many SDP problems arise from eigenvalue problems. Conversely, many SDP's can be reformulated as eigenvalue optimization problems. For example, in HELMBERG and RENDL [1997] the dual SDP is transformed into an eigenvalue optimization problem by reformulating the SDP constraint as a nonnegativity constraint on the minimal eigenvalue of the slack matrix variable and lifting this constraint into the cost function by means of a Lagrange multiplier. The correct value of the Lagrange multiplier is known in advance if the primal feasible matrices have constant trace, which is the case for many combinatorial applications.

The result is the problem of maximizing the minimal eigenvalue of an affine matrix function with an additional linear objective term. This is a nonsmooth convex optimization problem. At each step of a descent method one needs a subgradient a function value. The subgradients are collected into a *bundle* to form a cutting plane model. This model is then used to get the next iterate. More details can be found in e.g. HELMBERG and RENDL [1997], OUSTRY [1998].

3.3 Large Sparse Case

The bundle trust method is motivated by the many large SDP problems with special structure that arise from e.g. combinatorial optimization. It is particularly well suited to exploit special structure. Other approaches for exploiting special structure in SDPs are given in BENSON et al. [1997], FUJISAWA et al. [1995], KOJIMA et al. [1997].

4 APPLICATIONS

What has made this area so interesting and exciting is that the theory and algorithms are motivated by many applications, e.g. in statistics, combinatorial problems, engineering (mentioned above), and matrix completion problems.

4.1 Combinatorial Optimization

Semidefinite programming plays an important role in combinatorial optimization where it is used to solve nonlinear convex relaxations of NP-hard problems. The interest in SDP techniques for combinatorial optimization was motivated by remarkable theoretical results that characterize the quality of the bounds obtained via semidefinite programming. we have already seen some details of the result of Goemans-Williamson on the complexity and strength of the SDP relaxations [GOEMANS and WILLIAMSON, 1995, 1994] for the Max-Cut problem. In general, one can model 0,1 discrete optimization problems using $x_i^2 - x_i = 0$ though this often leads to subproblems that are too hard to solve. One can then take the Lagrangian relaxation and its dual to get the tractable SDP relaxation. Thus SDP provides a means of efficiently solving relaxations of quadratic approximations, though the SDPs themselves are often very large problems. Early examples of applying SDP are: the stable set problem and the max-cut problem, see e.g. GRÖTSCHEL et al. [1988], LOVÁSZ and SCHRIJVER [1991]. Surveys in this area are e.g. RENDL [1999], GOEMANS [1997].

4.2 Eigenvalue Problems

SDP is an eigenvalue optimization problem in the sense that the semidefinite constraint is equivalent to nonnegative bounds on eigenvalues. Solutions to SDP generally have multiple zero eigenvalues (just as solutions to LP have many nonbasic, i.e. zero, variables). However, it is not possible to formulate separate differentiable bounds on individual eigenvalues.

Min-max eigenvalue problems can be formulated as an SDP, i.e. one can maximize a variable λ subject to $A - \lambda I \preceq 0$. An early example of applications of min-max eigenvalues is CULLUM et al. [1975]. The converse is not true in general, but is true when the trace is a constant function on the feasible set, see e.g. HELMBERG and RENDL [1997]. This can be used to exploit structure and solve large scale problems.

For more on eigenvalue optimization, see LEWIS and OVERTON [1996].

4.3 Engineering

Many early applications of SDP appeared in the Engineering literature, e.g. [BOYD et al., 1993, YAKUBOVICH, 1965b,a]. This was particularly evident in control theory, where it has emerged as an important practical tool for the analysis of nonlinear and time-varying systems, and for controller synthesis. This is of great interest to computer-aided control system design. SDPs also arise in other fields of engineering and mathematics, such as structural optimization, circuit design, signal processing, and statistics, see e.g. [BOYD et al., 1993, VANDENBERGHE and BALAKRISHNAN, 1997, VANDENBERGHE and BOYD, 1994, FLETCHER, 1981, 1985, PUKELSHEIM, 1993, SAIGAL et al., 2000]. For other applications include solving Ricatti equations, see e.g. [BOYD et al., 1994b, VANDENBERGHE and BOYD, 1996]. (In this arena SDP is usually known as LMI: Linear Matrix Inequalities.) A typical example of an SDP application in Systems and Control is the numerical search for Lyapunov inequalities and also for Lyapunov functions for stability.

4.4 Matrix Completion Problems

The matrix completion problem consists in completing the missing elements of a matrix subject to certain conditions being met, e.g. positive definiteness. This is an extensively studied area in its right, see e.g. [GRONE et al., 1984, LAURENT, 1998, JOHNSON, 1990, BAKONYI and JOHNSON, 1995].

4.5 Nonlinear Programming

nonlinear programming TRS and SQQP The trust region subproblem is a basic tool in unconstrained nonlinear programming, providing a local quadratic model of the objective function which is to be "trusted" only in a certain region. In this case, the SDP relaxation of the quadratic problem provides an exact applications to large scale minimization", [RENDL and WOLKOWICZ, 1997]. In addition, this provides an SDP framework that allows one to efficiently exploit sparsity when solving unconstrained nonlinear programming problems. In the constrained case one obtains a quadratically constrained quadratic program as the local model. Again, an approximation of this model can be solved efficiently using SDP, [KRUK and WOLKOWICZ, 1998].

5 CONCLUSION

The World Wide Web is a tremendous resource for information on SDP. This includes research papers, public domain software, and bibliography files. Several relevant web sites are listed below. These contain further pointers to other web sites with more information.

1. http://www.zib.de/helmberg/semidef.html
   contains a special home page for semidefinite programming organized by C. Helmberg, at Berlin Center for Scientific Computing, Konrad Zuse Zentrum fur Informationstechnik, Berlin.

2. ftp://orion.uwaterloo.ca/pub/henry/reports/psd.bib.gz
   contains a bib file with papers related to SDP.

3. http://www.mcs.anl.gov:80/home/otc/InteriorPoint
   is an archive of technical reports and papers on interior-point methods, maintained by S.J. Wright at Argonne National Laboratory. Many of the recent reports discuss primal-dual interior-point methods for SDP, a subject not discussed in this volume.

SDP is an exciting subject that remains very active, and will no doubt be so for some time to come. Here is an area where nonlinear convex programming has shown its strength, beauty, and versatility. One finds elegant theoretical results, provably efficient algorithms and a wide variety of applications.

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