Beyond the Use of Linear Approximations for Modelling Nash-Cournot Equilibria

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***Abstract*— A comparison of Nash-Cournot equilibria computed using DC and AC approximations of the transmission system is presented in this paper. The goal of doing this comparison is to identify the impact of reactive power and voltage-related issues on Nash equilibria. Following the same rationality of competition and the same solution scheme, a considerable mismatch in outcomes is found. This shows that the use of common linear approximations of the transmission system to model strategic behaviour in electricity markets may produce outcomes that are unlikely to be observed in real markets. Three- and 14-node systems are used to analyze different study cases.**

I. Introduction

Various simulation models have been proposed to characterize oligopolistic competition in electricity markets. The models vary depending on the rationality taken for competition, the degree of stylization of the transmission system, the market clearing scheme, and the incentives from other market activities, among others. As the transmission system has strong implications for market outcomes, the study of strategic behaviour and market power assessment must account for its impact. Early models of strategic behaviour either neglected the transmission system, or incorporated it using its simplest approximation as a transportation network model. Most other formulations use direct current (DC) linear approximations of the transmission system, and although this accounts for both current and voltage laws, reactive power and voltage constraints are neglected, even though they are inherent features of an electrical power system. A few models incorporate detailed representations of the transmission system by using an alternate current (AC) model.

The use of DC approximations for modelling strategic behaviour is ubiquitous in the technical literature. DC linear approximations are suitable to calculate active power flows and voltage angles, but they do not provide any insight regarding voltage magnitudes and reactive power flows. The aim in using an AC approach is to capture the reactive power and voltage issues that a DC model cannot. However, the AC model introduces significant technical complications, and this explains its limited use. The main assumption for the usual DC approximation is that sufficient reactive power compensation is available at all nodes to keep voltage levels constant at one per unit values. However, power systems operate within a voltage band. This provides some degree of flexibility in the operation of the system to accommodate insufficient local provision of reactive power. The study of reactive power in a market-oriented environment is a complex matter; even its pricing is not a resolved issue [1]–[3]. In the study of strategic behaviour in electricity markets, a common practice to account for the impact of the transmission system, i.e., for congestion, is to use fixed thermal limits in MW. However, voltage constraints are also a source of congestion. Therefore, reactive power and voltage constraints may have an impact on the strategic behaviour of market participants as well. Due to the complexities in implementing an AC-based model, this impact has been overlooked in the study of market power. In [4], a detailed AC model is used to represent the transmission system for a multi-leader single-follower Cournot game. It is shown how GenCos can manipulate not only active but also reactive generation to maximize their profits. In this paper, the work in [5] is used to compare the DC and AC approaches for the study of strategic behaviour with a Nash-Cournot rationality for competition.

In Section II, the formulation of the game for both the DC and AC approximations is presented. A comparison of both approaches with transmission line limits constraining the market is studied in Section III. In Section IV, the same comparison is carried out when voltage limits are binding. Conclusions are given in Section V.

II. AC and DC approximations

As well known in the power literature, a power system in steady state can accurately be modelled with an AC approximation which relate four system variables: i) Voltage angle, ii) Magnitude voltage angle, iii) active power injections, and iv) reactive power injections. Given the system parameters and the values of those four variables, any other variable of the system can be computed, such as power flows in the transmission lines.

In polar coordinates, the expression for the active power in transmission line ij is:

$$z_{ij}(V,\theta) = g_{ij}V_i^2 - g_{ij}V_iV_j\cos(\theta_i-\theta_j) \\ -b_{ij}V_iV_j\sin(\theta_i-\theta_j) \qquad (1)$$

where g_{ij} and b_{ij} are the series conductance and susceptance of the transmission line, respectively; while V_i and θ_i

are the Magnitude and angle for the voltage at node i.

As the AC power flows expressions are involved and difficult to include in the modeling of imperfect competition, typical DC approximations are used. In order to linearize the power flow expressions, the following assumptions are used:

1. There is enough voltage support at all nodes such that the voltage profile can be kept at 1 p.u.; hence $V_1 = V_2 = ... = V_N = 1$ and expression (1) is

$$z_{ij}(\theta) = g_{ij} - g_{ij}\cos(\theta_i - \theta_j) - b_{ij}\sin(\theta_i - \theta_j) \tag{2}$$

2. The difference between angles is small in magnitude; hence $\cos(\theta_i - \theta_j) \approx \cos(0) = 1$ and $\sin(\theta_i - \theta_j) \approx \theta_i - \theta_j$. This yields the following power flow expression

$$z_{ij}(\theta) = -b_{ij}(\theta_i - \theta_j) \tag{3}$$

3. Series resistance is much smaller than the series reactance; thus $b_{ij} = -1/x_{ij}$, where x_{ij} is the series reactance of the line. Thus, the active power flow expression for the transmission line ij can be linearly defined in terms of only the voltage angle variables, i.e.,

$$z_{ij}(\theta) = \frac{1}{x_{ij}}(\theta_i - \theta_j) \tag{4}$$

As this expression resembles the standard DC resistor, this approximation is well known as a lossless DC linearization.

For the sake of simplicity in the introduction of the model, a generic variable u is used to denote all system variables such as the voltage-related and generation variables; magnitude and angle in the AC model, and only magnitude in the DC model. Hence, the power flow expression for a transmission line ij can simply be denoted as $z_{ij}(u)$.

Furthermore, the nodal balances for active power can be computed as

$$P_i = \sum_{i \in N_i} z_{ij}(u) \tag{5}$$

For the AC model, the above expression can be used to stand for both active and reactive power balances, while for the DC model, it only stands for the active power balance.

III. Mathematical Model at a Glance

Using a model for a pool market where an ISO clear the market and manages congestion, the objective of the ISO is stated as the minimization of the social cost given by the difference between the total generation cost incurred from all the competitive generators and the total demand benefit. Moreover, the ISO acts as a follower, taking as given the dominant (Cournot) suppliers' decisions in order to find an optimal dispatch. Therefore, the market problem can be cast mathematically as

$$\min \sum_{h,i} c_{h,i}(g_{h,i}) - \sum_i b_i(d_i) \tag{6}$$

s.t.

$$\sum_{\nu,\hbar} g^*_{\nu,\hbar,i} + \sum_h g_{h,i} - d_i - P_i(u) = 0, \quad : \lambda_i, \ \forall i, \tag{7}$$

$$|\hat{z}_{ij}(u)| \leq \overline{z}_{ij}, \quad : \overline{\mu}_{ij}, \quad \forall\, (i,j) \in \mathcal{L}, \tag{8}$$

$$\underline{e}(u) \leq e(u) \leq \overline{e}(u), \quad : \underline{\mu}^e_{h,i}, \overline{\mu}^e_{h,i}, \quad \forall\, i,\, h, \tag{9}$$

The objective function (6) stands for the social cost. Expression (7) is the power flow balances of the system (supply meets demand). On the left hand side of (7), the first sum stands for the contribution of dominant suppliers, while the second sum accounts for the competitive suppliers. The dominant suppliers' decision variables are denoted as $(\cdot)^*$ because they are exogenous to the ISO problem and thus their generation costs are not appended in the objective function as they are not going to be dispatched. However, they are the GenCo decision variables within each dominant GenCo problem. Expression (8) accounts for the transmission line limits in either direction. This expression can denote active or MVA limits in the Ac model, or just active in the DC model. Expression (9) is a generic constraint to stand for any other operational constraint such as voltage and reactive injection limits (AC model), and generation limits (AC and DC models).

Problem (6)-(9) is a classical OPF, except for the parametrization over the leaders' decision variables. On the right hand side of each constraint, its corresponding dual variable is placed. These dual variables provides the economical information for the market. The dual variables associated with the power balances set the locational marginal prices for trading power.

In standard Nonlinear Programming notation, the OPF problem can be compactly stated as

$$\begin{aligned} \min \quad & b(\boldsymbol{w}_0,\, \boldsymbol{w}_1) && (10)\\ \text{s.t.} \quad & \boldsymbol{c}_{\mathcal{E}}(\boldsymbol{x}, \boldsymbol{w}_0,\, \boldsymbol{w}_1) = \boldsymbol{0}, \quad : \boldsymbol{\lambda}, && (11)\\ & \boldsymbol{c}_{\mathcal{I}}(\boldsymbol{w}_1) \geq \boldsymbol{0}, \quad : \boldsymbol{\mu}, && (12)\\ & \boldsymbol{w}_0 \geq \boldsymbol{0}, && (13) \end{aligned}$$

where equality and inequality constraints are comprised into $\boldsymbol{c}_{\mathcal{E}}(\boldsymbol{x}, \boldsymbol{w}_0,\, \boldsymbol{w}_1)$ and $\boldsymbol{c}_{\mathcal{I}}(\boldsymbol{w}_1)$ respectively, while their corresponding dual variables are comprised into vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. The vectors $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ contain all the system variables that are free and nonnegative, respectively. The vector $\boldsymbol{x}$ contains all the decisions variables of the Cournot GenCos.

This primal problem can be defined in terms of its first order optimality conditions as follows

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{h}_{\mathcal{E}}(\boldsymbol{x},\, \boldsymbol{y}_0,\, \boldsymbol{y}_1), && (14)\\ \boldsymbol{0} &= \boldsymbol{h}_{\mathcal{I}}(\boldsymbol{y}_0,\, \boldsymbol{y}_1) - \boldsymbol{s}, && (15)\\ \boldsymbol{0} &\leq \boldsymbol{s} \quad \perp \quad \boldsymbol{y}_0 \geq \boldsymbol{0}. && (16) \end{aligned}$$

where primal and dual variables are gathered in vectors $\boldsymbol{y}_0$ and $\boldsymbol{y}_1$ accordingly, and $\boldsymbol{s}$ is a slack variable. This is a system of stationary conditions used to mathematically characterize the ISO primal problem. As it contains complementarity conditions ($\perp$), usually it is refer as a Complementarity problem (CP). For details of its derivation please refer to [5].

A. Dominant Players

Within the game, each Cournot GenCo aims to maximize its own profit (revenues minus costs). This is stated in (17) below as a minimization of negative profit. The revenues are defined by the power it sells at nodal prices, $\lambda_i \sum_h g_{\nu,h,i}$, while the cost for providing such power is $c_{\nu,h,i}(g_{\nu,h,i})$. Thus, the problem of the Cournot Gencos can be cast as follows:

$$\left(\begin{array}{ll} \min & -\Pi_\nu = f_\nu(\boldsymbol{x}_\nu, \boldsymbol{y}_1) \\ \text{s.t.} & \boldsymbol{h}_\nu(\boldsymbol{x}_\nu) \geq \boldsymbol{0}, \\ & \boldsymbol{h}_{\mathcal{E}}(\boldsymbol{x}_\nu,\, \boldsymbol{y}_0,\, \boldsymbol{y}_1;\, \boldsymbol{x}_{-\nu}) = \boldsymbol{0}, \\ & \boldsymbol{h}_{\mathcal{I}}(\boldsymbol{y}_0,\, \boldsymbol{y}_1) - \boldsymbol{s} = \boldsymbol{0}, \\ & \boldsymbol{0} \leq \boldsymbol{y}_0 \quad \perp \quad \boldsymbol{s} \geq \boldsymbol{0}, \end{array} \right) \quad \forall \nu. \tag{17}$$

Note that the stationarity conditions of the ISO problem are appended as constraints within the Cournot problems, giving rise to a set of Mathematical Problems with Complementarity Constraints (MPCCs), well known as an Equilibrium Problem with Complementarity Constraints (EPCC). Due to the complementarity conditions EPCC are hard-to-solve problems. One of the main reasons is that they violate any classical constraint qualifications.

As derived in [5], the above EPCC can be efficiently treated by manipulating its strong stationarity conditions, resulting in the following standard NLP problem:

$$\min \sum_\nu \left\{ \boldsymbol{\phi}_\nu^T \boldsymbol{\tau}_\nu + \boldsymbol{\sigma}_\nu^T \boldsymbol{s} + \boldsymbol{\psi}_\nu^T \boldsymbol{y}_0 \right\} + \boldsymbol{y}_0^T \boldsymbol{s} \tag{18}$$

$$\text{s.t. } \boldsymbol{0} = \nabla_{\boldsymbol{x}_\nu} f - \nabla_{\boldsymbol{x}_\nu}^T \boldsymbol{h}_\nu \boldsymbol{\phi}_\nu - \nabla_{\boldsymbol{x}_\nu}^T \boldsymbol{h}_{\mathcal{E}_\nu} \overline{\boldsymbol{\vartheta}}_\nu, \quad \forall \nu, \tag{19}$$

$$\boldsymbol{0} = \nabla_{\boldsymbol{y}_0} f - \nabla_{\boldsymbol{y}_0}^T \boldsymbol{h}_{\mathcal{E}_\nu} \overline{\boldsymbol{\vartheta}}_\nu - \nabla_{\boldsymbol{y}_0}^T \boldsymbol{h}_{\mathcal{I}_\nu} \underline{\boldsymbol{\vartheta}}_\nu + \boldsymbol{s} \circ \boldsymbol{\xi}_\nu - \boldsymbol{\psi}_\nu, \quad \forall \nu, \tag{20}$$

$$\boldsymbol{0} = \nabla_{\boldsymbol{y}_1} f - \nabla_{\boldsymbol{y}_1}^T \boldsymbol{h}_{\mathcal{E}_\nu} \overline{\boldsymbol{\vartheta}}_\nu - \nabla_{\boldsymbol{y}_1}^T \boldsymbol{h}_{\mathcal{I}_\nu} \underline{\boldsymbol{\vartheta}}_\nu, \quad \forall \nu, \tag{21}$$

$$\boldsymbol{0} = \underline{\boldsymbol{\vartheta}}_\nu + \boldsymbol{y}_0 \circ \boldsymbol{\xi}_\nu - \boldsymbol{\sigma}_\nu, \quad \forall \nu, \tag{22}$$

$$\boldsymbol{0} = \boldsymbol{h}_\nu - \boldsymbol{\tau}_\nu, \quad \forall \nu, \tag{23}$$

$$\boldsymbol{0} = \boldsymbol{h}_{\mathcal{E}}, \tag{24}$$

$$\boldsymbol{0} = \boldsymbol{h}_{\mathcal{I}} - \boldsymbol{s}, \tag{25}$$

$$\boldsymbol{0} \leq \boldsymbol{\tau}_\nu,\, \boldsymbol{\phi}_\nu,\, \boldsymbol{\psi}_\nu,\, \boldsymbol{\sigma}_\nu,\, \boldsymbol{\xi}_\nu, \quad \forall \nu, \tag{26}$$

$$\boldsymbol{0} \leq \boldsymbol{y}_0,\, \boldsymbol{s}. \tag{27}$$

This formulation requires a set of second-level dual variables, such as $\boldsymbol{\tau}_\nu$ and $\boldsymbol{\psi}_\nu$ that are related to each single constraint of problem (17).

IV. Computational Results

For simplicity in the comparison, let us consider the three-node system shown in Fig. 1. The transmission network consists of three identical transmission lines; they are accordingly labeled in Fig. 1. The lines parameters are R_{ij}=0.005, X_{ij}=0.01 and b_{ij}^{sh}=0.4. They are in per unit (p.u.) with a base of 100 MVA; this network setting is taken from [1]. For the sake of illustration, only line 2 has a limit of 350 MVA.

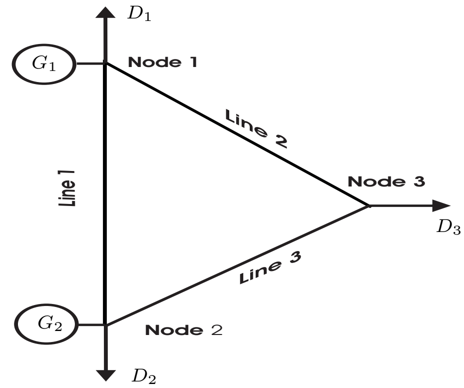

Fig. 1. Three-node power system.

There are two GenCos, which are placed at nodes 1 and 2, and can provide active and reactive power. Their generation characteristics are given in Table I. Demands for active power are considered to be price responsive, while reactive demands are considered as inelastic. Voltage and demand requirements are listed in Table IV.

TABLE I
Generation data for two GenCos.

GenCo ν	β_v ($\$/MWh$)	γ_v ($\$/MW^2h$)	MW point	MVAR min	MVAR max
1	15	0.004	0	-600	600
			3000	-400	400
2	20	0.004	0	-600	600
			3000	-400	400

TABLE II
Node characteristics.

Node i	$\underline{V}_i$ (pu)	$\overline{V}_i$ (pu)	ρ_{oi} ($\$/MWh$)	δ_i ($\$/MW^2h$)	Q_{Li} (MVAR)
1	0.97	1.03	120	0.05	175
2	0.97	1.03	120	0.05	175
3	0.97	1.03	125	0.05	175

A. A detailed AC model

First, the outcome for Cournot competition is obtained using a detailed AC transmission system. The resulting outcome is given in Table III.

The upper voltage at node 1 and the limit of transmission line 1-3 are binding. However, their congestion multipliers are zero. This feature is well known for DC models, and as shown in [5], this feature also applies to voltage limits with the AC formulation. Therefore, if the room exists to manipulate the market, GenCos will exploit not only line limits but also voltage constraints. GenCos can manipulate the voltage limits by varying their nodal injection or reactive power at their corresponding node. The upper voltage limit is reached at node 1 because G1 is providing most of the reactive power, which tends to increase the voltage magnitude at its node. The lowest voltage is at node 3, where the demand makes the voltage drop as there is no local support of reactive power.

TABLE III

COMPARISON OF MARKET OUTCOMES WHEN HAVING LINE AND VOLTAGE CONSTRAINTS

Case	AC 1	DC
λ_1 ($\$/MWh$)	58.89	57.82
λ_1 ($\$/MWh$)	58.90	57.82
λ_1 ($\$/MWh$)	60.50	57.82
g_1 (MW)	945.18	1000.00
g_2 (MW)	933.30	915.18
q_1 (MVAR)	361.28	–
q_2 (MVAR)	62.57	–
d_1 (MW)	611.03	621.72
d_2 (MW)	610.94	621.72
d_3 (MW)	644.97	671.72
Π_1 ($\$/h$)	37,917.10	38,827.24
Π_2 ($\$/h$)	32,825.93	31,268.68
V_1 (pu)	1.03	–
V_2 (pu)	1.02	–
V_3 (pu)	1.00	–
z_{12} (MW)	99.87	28.27
z_{13} (MW)	328.18	350.00
z_{23} (MW)	322.31	321.72
Losses (MW)	11.53	–

Although the active power flow in line 1-3 is only 328.2 MW, there is also a reactive-power component using transmission capacity; both together amount 350 MVAR.

B. A DC-based model

Using the same setting for competition and the same transmission system, the linearized DC version is used instead. The resulting outcome is given in Table III as well.

TABLE IV

COMPARISON OF MARKET OUTCOMES FOR ONLY-VOLTAGE CONSTRAINTS

Case	AC 1	DC
λ_1 ($\$/MWh$)	59.40	57.36
λ_1 ($\$/MWh$)	59.86	57.36
λ_1 ($\$/MWh$)	61.18	57.36
g_1 (MW)	1075.31	1024.99
g_2 (MW)	785.07	904.02
q_1 (MVAR)	-116.09	–
q_2 (MVAR)	547.66	–
d_1 (MW)	605.90	626.33
d_2 (MW)	601.36	626.33
d_3 (MW)	638.17	676.33
Π_1 ($\$/h$)	43,128.98	39,222.57
Π_2 ($\$/h$)	28,830.66	30,510.90
V_1 (pu)	1.01	–
V_2 (pu)	1.03	–
V_3 (pu)	0.99	–
z_{12} (MW)	97.56	40.33
z_{13} (MW)	367.06	358.33
z_{23} (MW)	277.16	318.00
Losses (MW)	14.95	–

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