Convex Conic Formulations of Robust Downlink Precoder Designs with Quality of Service Constraints

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Abstract

We consider the design of linear precoders (beamformers) for broadcast channels with Quality of Service (QoS) constraints for each user, in scenarios with uncertain channel state information (CSI) at the transmitter. We consider a deterministically-bounded model for the channel uncertainty of each user, and our goal is to design a robust precoder that minimizes the total transmission power required to satisfy the users' QoS constraints for all channels within a specified uncertainty region around the transmitter's estimate of each user's channel. Since this problem is not known to be computationally tractable, we will derive three conservative design approaches that yield convex and computationally-efficient restrictions of the original design problem. The three approaches yield semidefinite program (SDP) formulations that offer different trade-offs between the degree of conservatism and the size of the SDP. We will also show how these conservative approaches can be used to derive efficiently-solvable quasi-convex restrictions of some related design problems, including the robust counterpart to the problem of maximizing the minimum signal-to-interference-plus-noise-ratio (SINR) subject to a given power constraint. Our simulation results indicate that in the presence of uncertain CSI the proposed approaches can satisfy the users' QoS requirements for a significantly larger set of uncertainties than existing methods, and require less transmission power to do so.

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I. Introduction

Employing multiple antennas at the transmitter (base station) of a wireless downlink offers the potential for a substantial improvement in the quality of service (QoS) that the base station can offer to the assigned users. This potential can be realized by precoding the data symbols intended for each user in a manner that mitigates the multiuser interference at the (non-cooperating) receivers, and hence improves the fidelity of the received signals. The transmitter's ability to mitigate interference at the receivers is dependent on the availability of (accurate) channel state information (CSI) for all the users' channels. For scenarios in which one can assume perfect CSI is available at the transmitter, the problem of designing a linear precoder[1] that minimizes the transmitted power required to satisfy a set of QoS constraints specified by the users has been considered in [1], [2], [3], [4], [5], [6].

In practice, the CSI that is available at the transmitter is subject to uncertainties that arise from a variety of sources, such as estimation error, channel quantization and short channel coherence time, and downlink precoder design methods that assume perfect CSI are particularly sensitive to these uncertainties; e.g., [7]. This suggests that one ought to incorporate robustness to channel uncertainty into the formulation of the precoder design problem. One approach to doing so is to consider a bounded model for the errors in the transmitter's estimate of the (deterministic) autocorrelation matrices of the channel [3], [8]. This uncertainty model may be suitable for systems with uplink-downlink reciprocity in which the transmitter can estimate the users' channels. We will adopt an alternative approach in which we consider a bounded model for the error in the transmitter's estimate of the channels. This uncertainty model is particularly useful for systems in which users feed back quantized channel measurements to the transmitter, as knowledge of the quantization codebooks can be used to bound the quantization error. For this bounded channel uncertainty model, we consider the design of a linear precoder that minimizes the transmitted power required to ensure that each user's QoS requirement is satisfied for all channels within the specified uncertainty region. This problem is not known to be computationally tractable [9], and in order to obtain design methods that are known to be tractable we will obtain three conservative design approaches that yield convex and computationally-efficient restrictions of the original design problem.[2] The three approaches yield semidefinite program (SDP) formulations that offer different trade-offs between the degree of conservatism and the size of resulting SDP.

[1]Since we will focus on scenarios in which each user has a single antenna, linear precoding is analogous to downlink beamforming.

[2]Since these problems are restrictions of the original problem, the transmission power of the designed precoder is larger than (or equal to) the power that would be required if a tractable method for solving the original problem was available.

We will also show how these conservative design approaches can be used to obtain efficiently-solvable quasi-convex formulations of certain restrictions of related design problems. In particular, we consider the problem of determining the largest uncertainty region for which the QoS requirements can be satisfied for all channels within the region using finite transmission power. This problem is of considerable interest in scenarios in which the channel uncertainty is dominated by the quantization error incurred in a quantized feedback scheme. In that case, one might wish to choose the rate of the channel quantization scheme to be large enough (and the quantization cells small enough) for it to be possible to design a robust precoder with finite power. We provide quasi-convex formulations of conservative approaches to this problem that can be efficiently solved using a one-dimensional bisection search. We also consider the robust counterpart of the problem of maximizing the weakest user's signal-to-interference-plus-noise-ratio subject to a given power constraint on the transmitter (e.g., [5], [6]), and we provide quasi-convex formulations of conservative approaches to that design problem, too. Our numerical experiments will illustrate the impact that our proposed designs can have on a number of performance metrics. In particular, these experiments indicate that proposed approaches can satisfy the users' QoS requirements for a significantly larger set of uncertainties than existing methods, and require less transmission power to do so.

Our notation is as follows: We will use boldface capital letters to denote matrices, boldface lower case letters to denote vectors and medium weight lower case letters to denote individual elements; $\mathbf{A}^T$ and $\mathbf{A}^H$ denote the transpose and the conjugate transpose of the matrix $\mathbf{A}$, respectively. The notation $\|\mathbf{x}\|$ denotes the Euclidean norm of vector $\mathbf{x}$, while $\|\mathbf{A}\|$ denotes the spectral norm (maximum singular value) of the matrix $\mathbf{A}$, and $\mathrm{E}\{\cdot\}$ denotes the expectation operator. The term $\mathrm{tr}(\mathbf{A})$ denotes the trace of matrix $\mathbf{A}$, $\mathbf{A} \otimes \mathbf{B}$ denotes the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, and for symmetric matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \geq \mathbf{B}$ denotes the fact that $\mathbf{A} - \mathbf{B}$ is positive semidefinite. The expression $\mathrm{Diag}(\mathbf{x})$ will denote the diagonal matrix whose diagonal elements are the elements of $\mathbf{x}$. We will denote the identity matrix by $\mathbf{I}$, and we will provide its dimension as a subscript where that makes the formulae significantly easier to parse.

II. System Model

We consider a broadcast scenario with N_t antennas at the transmitter which are used to send independent messages to K receivers, each of which has a single antenna. Let $\mathbf{s} \in \mathbb{C}^K$ be the vector of data symbols intended for each receiver. The transmitter generates a vector of transmitted signals, $\mathbf{x} \in \mathbb{C}^{N_t}$, by linearly precoding the vector $\mathbf{s}$

$$\mathbf{x} = \mathbf{P}\mathbf{s} = \sum_{j=1}^{K} \mathbf{p}_j s_j, \tag{1}$$

where $\mathbf{p}_j$ is the j^{th} column of the precoding matrix $\mathbf{P}$, and s_j is the j^{th} element of $\mathbf{s}$. Without loss of generality, we will assume that $\mathrm{E}\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}$, and hence, the total transmitted power is given by

$$\operatorname{tr}(\mathbf{P}^H\mathbf{P}) = \sum_{k=1}^{K} \|\mathbf{p}_k\|^2. \tag{2}$$

At the k^{th} receiver, the received signal y_k is given by

$$y_k = \mathbf{h}_k\mathbf{x} + n_k, \tag{3}$$

where $\mathbf{h}_k \in \mathbb{C}^{1\times N_t}$ is a row vector[3] representing the channel gains from the transmitting antennas to the k^{th} receiver, and n_k represents the zero-mean additive white noise at the k^{th} receiver, whose variance is $\sigma_{n_k}^2$. We will find it convenient to use the vector notation:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{4}$$

where $\mathbf{H}$ is the broadcast channel matrix whose k^{th} row is $\mathbf{h}_k$, and the noise vector $\mathbf{n}$ has covariance matrix $\mathrm{E}\{\mathbf{n}\mathbf{n}^H\} = \operatorname{Diag}(\sigma_{n_1}^2, \ldots, \sigma_{n_K}^2)$.

We consider broadcast scenarios in which each receiver has a quality of service requirement that is specified in terms of a lower bound on its signal-to-interference-plus-noise-ratio

$$\text{SINR}_k = \frac{|\mathbf{h}_k\mathbf{p}_k|^2}{\sum_{j=1,j\neq k}^{K} |\mathbf{h}_k\mathbf{p}_j|^2 + \sigma_{n_k}^2} \geq \gamma_k. \tag{5}$$

This SINR constraint represents a rather general constraint on the minimum quality of service received by the k^{th} user. Indeed, the SINR constraint can be translated into an equivalent constraint on the symbol error rate or the achievable data rate; e.g., [10].

A. Precoding with QoS Constraints: Perfect CSI Case

Given perfect CSI at the transmitter, the design of a precoder that minimizes the total transmitted power required to satisfy the users' QoS constraints can be stated as:

$$\min_{\mathbf{P}} \quad \sum_{k=1}^{K} \|\mathbf{p}_k\|^2 \tag{6a}$$

$$\text{subject to} \quad \frac{|\mathbf{h}_k\mathbf{p}_k|^2}{\sum_{j=1,j\neq k}^{K} |\mathbf{h}_k\mathbf{p}_j|^2 + \sigma_{n_k}^2} \geq \gamma_k. \tag{6b}$$

[3]Although treating $\mathbf{h}_k$ as a row vector is a mild abuse of notation, it is standard practice.

This is a convex optimization problem in the precoding matrix $\mathbf{P}$, and can be efficiently solved [1]–[6]. Indeed, if we make the following definitions,

$$\underline{\mathbf{h}}_k = \begin{bmatrix} \operatorname{Re}\{\mathbf{h}_k\} & \operatorname{Im}\{\mathbf{h}_k\} \end{bmatrix}, \tag{7}$$

$$\underline{\mathbf{P}} = \begin{bmatrix} \operatorname{Re}\{\mathbf{P}\} & \operatorname{Im}\{\mathbf{P}\} \\ -\operatorname{Im}\{\mathbf{P}\} & \operatorname{Re}\{\mathbf{P}\} \end{bmatrix}, \tag{8}$$

$$\underline{\mathbf{p}}_k = \begin{bmatrix} \operatorname{Re}\{\mathbf{p}_k\} \\ -\operatorname{Im}\{\mathbf{p}_k\} \end{bmatrix}, \tag{9}$$

we can formulate (6) as the following second order cone program (SOCP) with real variables [6]:

$$\min_{\underline{\mathbf{P}},\, t} \quad t \tag{10a}$$

$$\text{subject to} \quad \big\|\operatorname{vec}\big([\underline{\mathbf{p}}_1, \ldots, \underline{\mathbf{p}}_K]\big)\big\| \le t, \tag{10b}$$

$$\big\|[\underline{\mathbf{h}}_k\underline{\mathbf{P}}, \quad \sigma_{n_k}]\big\| \le \beta_k \underline{\mathbf{h}}_k \underline{\mathbf{p}}_k, \quad 1 \le k \le K, \tag{10c}$$

where $\beta_k = \sqrt{1 + 1/\gamma_k}$.

The primary goal of this paper is to obtain robust counterparts to (10) that mitigate the impact of imperfect CSI. Before we derive those counterparts, we would like to point out that when $\mathbf{H}$ has full row rank (which requires that $K \le N_t$), the perfect CSI problem (with finite SINR requirements) is always feasible. (The robust counterparts will not share this property.) Indeed, one feasible solution is to chose $\mathbf{P}$ to be the product of the right inverse of $\mathbf{H}$ and a diagonal power loading matrix with sufficiently large loadings. In practice, however, one may wish to constrain the transmission power in various ways, such as constraining the average power transmitted by each individual antenna (e.g., [11]), $\operatorname{E}\{|x_n|^2\} \le P_n$, $1 \le n \le N_t$. Another useful power constraint arises from the imposition of a spatially-shaped bound (e.g., [12], [13]) on the transmitted power, $\operatorname{E}\{\mathbf{x}^H\mathbf{Q}(\theta)\mathbf{x}\} \le P_{\text{shape}}(\theta)$ for all $\theta \in \Theta$, where $\mathbf{Q}(\theta) = \mathbf{v}(\theta)\mathbf{v}^H(\theta)$, with $\mathbf{v}(\theta)$ being the "steering vector" (e.g., [14]) of the transmitter's antenna array in the direction θ, $P_{\text{shape}}(\theta)$ is the maximum allowable power in the direction of θ, and Θ is the set of angles of interest. The later case is of particular interest in cellular systems in which interference to neighboring cells needs to be controlled; e.g., [15], [16]. Although we will focus on robust versions of the formulation in (10) in the presence of channel uncertainty, in the Appendix we will show how these two types of power constraints can be easily incorporated into our robust formulations.

B. Channel Uncertainty Model

We will model the channel uncertainty using a deterministically-bounded additive uncertainty set. More specifically, we will model the k^{th} user's channel as:

$$\mathbf{h}_k = \hat{\mathbf{h}}_k + \mathbf{e}_k, \tag{11}$$

where $\hat{\mathbf{h}}_k$ is the transmitter's estimate of the k^{th} user's channel, and $\mathbf{e}_k$ is the corresponding estimation error. In order to avoid making any assumptions on the statistics of $\mathbf{e}_k$, we will merely assume that it lies in the ball $\|\mathbf{e}_k\| \leq \delta_k$, for some given δ_k. This model is a convenient one for systems in which the channel state information is (vector) quantized at the receivers and fed back to the transmitter; e.g., [7]. In particular, if the quantizer is (almost) uniform, then the quantization cells in the interior of the quantization region can be approximated by balls of size δ_k. A similar bounded uncertainty model has been used in the context of generic beamforming systems [17], [18], where it is the error in the estimate of the steering vector that is being bounded, and in CDMA systems [19].

By using the vector $\underline{\mathbf{e}}_k = [\mathrm{Re}\{\mathbf{e}_k\}, \mathrm{Im}\{\mathbf{e}_k\}]$, the uncertainty set of each channel can be described by the following (spherical) region:

$$\mathcal{U}_k(\delta_k) = \{\underline{\mathbf{h}}_k \mid \underline{\mathbf{h}}_k = \underline{\hat{\mathbf{h}}}_k + \underline{\mathbf{e}}_k,\ \|\underline{\mathbf{e}}_k\| \leq \delta_k\}. \tag{12}$$

III. Precoding with QoS Constraints: Uncertain CSI Case

Given the model for the channel uncertainty in (12), our goal is to design a robust precoding matrix that minimizes the transmitted power required to ensure that the users' QoS requirements are satisfied for all channels $\underline{\mathbf{h}}_k$ within the uncertainty region $\mathcal{U}_k(\delta_k)$. Using the SOCP formulation in (10), this design problem can be formulated as the following semi-infinite SOCP[4]:

$$\min_{\underline{\mathbf{P}},\, t} \quad t \tag{13a}$$

$$\text{s. t.} \quad \big\|\mathrm{vec}\big([\underline{\mathbf{p}}_1, \ldots, \underline{\mathbf{p}}_K]\big)\big\| \leq t, \tag{13b}$$

$$\big\|[\underline{\mathbf{h}}_k \underline{\mathbf{P}}, \quad \sigma_{n_k}]\big\| \leq \beta_k \underline{\mathbf{h}}_k \underline{\mathbf{p}}_k, \qquad \forall\, \underline{\mathbf{h}}_k \in \mathcal{U}_k(\delta_k), \quad 1 \leq k \leq K. \tag{13c}$$

For later convenience, any precoder of finite power that satisfies (13c) will be said to provide a robust QoS guarantee.

[4]Observe that (13c) contains an infinite number of second-order cone constraints, one for each $\underline{\mathbf{h}}_k \in \mathcal{U}_k(\delta_k)$.

Since $\underline{\mathbf{h}}_k$ is present on both the left and right hand sides of each SOC constraint in (13c), the left and right hand sides of (13c) vary together and share the same ellipsoidal uncertainty region. That joint variation appears to make this problem difficult to solve, but the formal treatment of the computational tractability of this problem remains an open problem [9], [20]. Some insight can be obtained by using a standard transformation (via the Schur Complement Theorem [21]) to write the SOC constraint $\left\|[\underline{\mathbf{h}}_k\underline{\mathbf{P}}, \quad \sigma_{n_k}]\right\| \leq \beta_k \underline{\mathbf{h}}_k \underline{\mathbf{p}}_k$ as an equivalent linear matrix inequality (LMI) [22]

$$\mathbf{F}_k(\underline{\mathbf{P}}, \underline{\mathbf{h}}_k) = \begin{bmatrix} \beta_k \underline{\mathbf{h}}_k \underline{\mathbf{p}}_k & [\underline{\mathbf{h}}_k\underline{\mathbf{P}}, \quad \sigma_{n_k}] \\ [\underline{\mathbf{h}}_k\underline{\mathbf{P}}, \quad \sigma_{n_k}]^T & (\beta_k \underline{\mathbf{h}}_k \underline{\mathbf{p}}_k)\, \mathbf{I}_{(2K+1)} \end{bmatrix} \geq \mathbf{0}. \tag{14}$$

By substituting $\underline{\mathbf{h}}_k = \hat{\underline{\mathbf{h}}}_k + \underline{\mathbf{e}}_k$ in (14), the inequality $\mathbf{F}_k(\underline{\mathbf{P}}, \underline{\mathbf{h}}_k) \geq \mathbf{0}$ takes the form:

$$\overline{\mathbf{F}}_k(\underline{\mathbf{P}}, \hat{\underline{\mathbf{h}}}_k, \mathbf{M}_k) = \mathbf{F}_k(\underline{\mathbf{P}}, \hat{\underline{\mathbf{h}}}_k) + \mathbf{M}_k \mathbf{R}_k(\underline{\mathbf{P}}, \beta_k) + \mathbf{R}_k^T(\underline{\mathbf{P}}, \beta_k)\mathbf{M}_k^T \geq \mathbf{0}, \tag{15}$$

where the matrices $\mathbf{M}_k$ and $\mathbf{R}_k(\underline{\mathbf{P}}, \beta_k)$ are:

$$\mathbf{M}_k = \mathbf{I}_{(2K+2)} \otimes \underline{\mathbf{e}}_k, \tag{16}$$

$$\mathbf{R}_k(\underline{\mathbf{P}}, \beta_k) = \begin{bmatrix} \frac{1}{2}\beta_k \underline{\mathbf{p}}_k & [\underline{\mathbf{P}}, \quad \mathbf{0}] \\ \mathbf{0} & (\frac{1}{2}\beta_k)\mathbf{I}_{(2K+1)} \otimes \underline{\mathbf{p}}_k \end{bmatrix}. \tag{17}$$

From (16), we observe that the uncertainty matrix $\mathbf{M}_k$ belongs to a subspace $\mathcal{M}$ of block diagonal matrices with equal blocks. Specifically,

$$\mathcal{M} = \{\mathbf{M} \,|\mathbf{M} = \mathbf{I}_{(2K+2)} \otimes \underline{\mathbf{e}},\ \underline{\mathbf{e}} \in \mathbb{R}^{1\times 2N_t}\}. \tag{18}$$

Hence, the spectral norm of $\mathbf{M}_k$ is $\|\mathbf{M}_k\| = \|\underline{\mathbf{e}}_k\| \leq \delta_k$. Using (14)–(18), the robust QoS design problem in (13) can also be formulated as the following semi-infinite robust semidefinite program (SDP):

$$\min_{\underline{\mathbf{P}},\, t} \quad t \tag{19a}$$

$$\text{s. t.} \quad \left\|\text{vec}\big([\underline{\mathbf{p}}_1,\ \ldots,\ \underline{\mathbf{p}}_K]\big)\right\| \leq t, \tag{19b}$$

$$\overline{\mathbf{F}}_k(\underline{\mathbf{P}}, \hat{\underline{\mathbf{h}}}_k, \mathbf{M}_k) \geq \mathbf{0}, \qquad \forall\, \mathbf{M}_k \in \mathcal{M},\ \|\mathbf{M}_k\| \leq \delta_k, \quad 1 \leq k \leq K. \tag{19c}$$

A general instance of (19) is NP-hard for a general subspace $\mathcal{M}$, [23]; see also [20], [24]. This result and the undetermined tractability of the robust SOCP in (13) suggest that in order to obtain a robust design technique that is guaranteed to be computationally tractable, we will need to modify the formulation of (13) or (19). In the following section, we will present three conservative design approaches that yield convex and efficiently-solvable restrictions of (13) and (19). These approaches are conservative in the sense that they guarantee that the SINR constraints are satisfied for a larger set of channel uncertainties than that described in (12), and hence the resulting transmission power is larger than (or equal to) that of an optimal solution to (13), if such a solution could be found. The three approaches yield SDP formulations that offer different trade-offs between the degree of conservatism and the size of the resulting SDP (and hence its computational cost).

IV. Conservative Approaches to Robust Precoder Design with QoS Constraints

A. Robust SOCP with Independent Uncertainty

In this section, we will work directly with the robust SOCP formulation in (13). The presence of $\underline{\mathbf{h}}_k$ on both the left and right hand sides of each SOC constraint in (13c) means that these terms vary together and share the same ellipsoidal uncertainty region. We will obtain a conservative robust design by assuming independent uncertainties for $\underline{\mathbf{h}}_k$ on the left and right hand sides of (13c). Relaxing the common uncertainty structure in this way will result in a tractable restriction of (13) that can be formulated as an SDP. To obtain that SDP, we will use the following lemma [20]:

Lemma 1: Consider the robust SOCP:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} \quad & \|\mathbf{A}\mathbf{x}+\mathbf{b}]\| \le \mathbf{f}^T\mathbf{x}+g \quad \forall \mathbf{A} \in \mathcal{Y}, \mathbf{f} \in \mathcal{W}, \end{aligned}$$

where the ellipsoidal uncertainty regions $\mathcal{Y} = \{\mathbf{A} \mid \mathbf{A} = \mathbf{A}^0 + \sum_{j=1}^{y} \theta_i \, \mathbf{A}^j, \ \|\boldsymbol{\theta}\| \le 1\}$ and $\mathcal{W} = \{\mathbf{f} \mid \mathbf{f} = \mathbf{f}^0 + \sum_{j=1}^{w} \phi_i \, \mathbf{f}^j, \ \|\boldsymbol{\phi}\| \le 1\}$ are independent. This robust SOCP is equivalent to the following SDP:

$$\begin{aligned} \min_{\mathbf{x},\mu,\lambda} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} \quad & \begin{bmatrix} \lambda-\mu & \mathbf{0} & (\mathbf{A}^0\mathbf{x}+\mathbf{b})^T \\ \mathbf{0} & \mu\mathbf{I} & [\mathbf{A}^1\mathbf{x} \ \dots \ \mathbf{A}^y\mathbf{x}]^T \\ \mathbf{A}^0\mathbf{x}+\mathbf{b} & [\mathbf{A}^1\mathbf{x} \ \dots \ \mathbf{A}^y\mathbf{x}] & \lambda\mathbf{I} \end{bmatrix} \ge \mathbf{0}, \\ & \begin{bmatrix} {\mathbf{f}^0}^T\mathbf{x}+g-\lambda & [{\mathbf{f}^1}^T\mathbf{x} \ \dots \ {\mathbf{f}^w}^T\mathbf{x}] \\ [{\mathbf{f}^1}^T\mathbf{x} \ \dots \ {\mathbf{f}^w}^T\mathbf{x}]^T & ({\mathbf{f}^0}^T\mathbf{x}+g-\lambda)\mathbf{I} \end{bmatrix} \ge \mathbf{0}. \end{aligned}$$

$\square$

By writing $\underline{\mathbf{h}}_k = \hat{\underline{\mathbf{h}}}_k + \underline{\mathbf{e}}_k = \hat{\underline{\mathbf{h}}}_k + \delta_k \mathbf{u}$, $\|\mathbf{u}\| \leq 1$, and invoking Lemma 1, we obtain the following SDP formulation of a conservative version of (13):

$$\min_{\underline{\mathbf{P}}, \boldsymbol{\mu}, \boldsymbol{\lambda}, t} \quad t \tag{22a}$$

$$\text{s. t.} \quad \left\|\text{vec}\left([\underline{\mathbf{p}}_1, \ldots, \underline{\mathbf{p}}_K]\right)\right\| \leq t, \tag{22b}$$

$$\mathbf{A}_k(\underline{\mathbf{P}}, \lambda_k, \mu_k, \delta_k) = \begin{bmatrix} \lambda_k - \mu_k & \mathbf{0} & [\hat{\underline{\mathbf{h}}}_k \underline{\mathbf{P}}, \ \sigma_{n_k}] \\ \mathbf{0} & \mu_k \mathbf{I}_{2N_t} & \delta_k [\underline{\mathbf{P}}, \ \mathbf{0}] \\ [\hat{\underline{\mathbf{h}}}_k \underline{\mathbf{P}}, \ \sigma_{n_k}]^T & \delta_k [\underline{\mathbf{P}}, \ \mathbf{0}]^T & \lambda_k \mathbf{I}_{(2K+1)} \end{bmatrix} \geq \mathbf{0}, \quad 1 \leq k \leq K, \tag{22c}$$

$$\mathbf{B}_k(\underline{\mathbf{P}}, \lambda_k, \delta_k, \beta_k) = \begin{bmatrix} \beta_k \hat{\underline{\mathbf{h}}}_k \underline{\mathbf{p}}_k - \lambda_k & \delta_k \beta_k \underline{\mathbf{p}}_k^T \\ \delta_k \beta_k \underline{\mathbf{p}}_k & (\beta_k \hat{\underline{\mathbf{h}}}_k \underline{\mathbf{p}}_k - \lambda_k) \mathbf{I}_{2N_t} \end{bmatrix} \geq \mathbf{0}, \quad 1 \leq k \leq K. \tag{22d}$$

This problem can be efficiently solved using general purpose implementations of interior point methods; e.g., SeDuMi [25].

B. Robust SDP with Unstructured Uncertainty

In this section and the following one, we will obtain two conservative robust designs from the SDP formulation in (19) of the original design problem. The difficulty of solving (19) arises from the particular structure that the matrix $\mathbf{M}_k$ must possess. In this section we will show that if we restrict the robust design so that the SINR targets are to be satisfied for all $\|\mathbf{M}_k\| \leq \delta_k$ rather than just those $\mathbf{M}_k \in \mathcal{M}$ that satisfy $\|\mathbf{M}_k\| \leq \delta_k$, then one can obtain an efficiently-solvable problem. That is, we will show that by replacing (19c) by

$$\overline{\mathbf{F}}_k(\underline{\mathbf{P}}, \hat{\underline{\mathbf{h}}}_k, \mathbf{M}_k) \geq \mathbf{0}, \qquad \forall \, \|\mathbf{M}_k\| \leq \delta_k, \quad 1 \leq k \leq K, \tag{23}$$

one can obtain a restriction of (19) that can be efficiently solved.

Although (23) is simpler than (19c), it still represents an infinite set of LMIs, one for each $\mathbf{M}_k$ that satisfies $\|\mathbf{M}_k\| \leq \delta_k$. However, by using the following lemma, which is a special case of a more general result in [24], this semi-infinite LMI constraint can be precisely transformed into a single LMI.

Lemma 2: Let $\mathbf{F}(\mathbf{x})$ be a symmetric matrix, and let $\mathbf{F}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ be affine functions of $\mathbf{x}$. Then

$$\overline{\mathbf{F}}(\mathbf{x}, \mathbf{M}) = \mathbf{F}(\mathbf{x}) + \mathbf{M}\mathbf{R}(\mathbf{x}) + \mathbf{R}^T(\mathbf{x})\mathbf{M}^T \geq \mathbf{0}, \quad \forall \, \|\mathbf{M}\| \leq \delta \tag{24}$$

if and only if there exists a scalar τ such that

$$\begin{bmatrix} \mathbf{F}(\mathbf{x}) - \tau\mathbf{I} & \mathbf{R}^T(\mathbf{x}) \\ \mathbf{R}(\mathbf{x}) & \tau\delta^{-2}\mathbf{I} \end{bmatrix} \geq \mathbf{0}.$$

□

By applying the result of Lemma 2 to the LMIs in (23), we obtain the following efficiently-solvable SDP formulation for a conservative approach to the robust precoder design problem:

$$\min_{\underline{\mathbf{P}}, t, \tau_1, \ldots, \tau_K} \quad t \tag{25a}$$

$$\text{s. t.} \quad \left\|\text{vec}\big([\underline{\mathbf{p}}_1, \; \ldots, \; \underline{\mathbf{p}}_K]\big)\right\| \leq t, \tag{25b}$$

$$\begin{bmatrix} \mathbf{F}_k(\underline{\mathbf{P}}, \underline{\hat{\mathbf{h}}}_k) - \tau_k \, \mathbf{I}_{(2K+2)} & \mathbf{R}^T(\underline{\mathbf{P}}, \beta_k) \\ \mathbf{R}(\underline{\mathbf{P}}, \beta_k) & \tau_k \delta_k^{-2} \mathbf{I}_q \end{bmatrix} \geq \mathbf{0}, \quad 1 \leq k \leq K, \tag{25c}$$

where $\mathbf{F}_k(\underline{\mathbf{P}}, \underline{\hat{\mathbf{h}}}_k)$ and $\mathbf{R}(\underline{\mathbf{P}}, \beta_k)$ were defined in (14) and (17), respectively, and $q = 4N_t(K+1)$.

C. Robust SDP with Structured Uncertainty

The efficiently-solvable conservative formulation in (25) was obtained by relaxing the structural constraint $\mathbf{M}_k \in \mathcal{M}$ in (19). In this section we will obtain a less conservative formulation of (19) that results in an SDP that retains this structural constraint.

We begin with a definition. Given an arbitrary subspace of matrices $\tilde{\mathcal{M}}$, let $\mathcal{B}_{\tilde{\mathcal{M}}}$ denote the following set of matrices associated with $\tilde{\mathcal{M}}$:

$$\mathcal{B}_{\tilde{\mathcal{M}}} = \{(\mathbf{S}, \mathbf{T}, \mathbf{G}) | \, \mathbf{S}\mathbf{M} = \mathbf{M}\mathbf{T}, \; \mathbf{G}\mathbf{M} = -\mathbf{M}^T\mathbf{G}^T, \quad \forall \, \mathbf{M} \, \in \, \tilde{\mathcal{M}}\}. \tag{26}$$

For the subspace $\mathcal{M}$ in (18), applying (26) yields:

$$\mathbf{T} = \mathbf{S} \otimes \mathbf{I}_{2N_t}, \qquad \mathbf{G} = \mathbf{0}, \tag{27}$$

where $\mathbf{S} \in \mathbb{R}^{(2K+2)\times(2K+2)}$. Although the construction of $\mathcal{B}_{\mathcal{M}}$ may appear to be arbitrary, it enables us to develop an SDP formulation of a conservative design that retains the structure $\mathbf{M}_k \in \mathcal{M}$. To do so, we will use the following result, which is a special case of a more general result in [24].

Lemma 3: Consider the following robust SDP problem:

$$\min_{\mathbf{x}} \quad \mathbf{c}^T\mathbf{x}$$

$$\text{s. t.} \quad \overline{\mathbf{F}}(\mathbf{x}, \mathbf{M}) = \mathbf{F}(\mathbf{x}) + \mathbf{M}\mathbf{R}(\mathbf{x}) + \mathbf{R}^T(\mathbf{x})\mathbf{M}^T \geq \mathbf{0}, \quad \forall \, \mathbf{M} \in \tilde{\mathcal{M}}, \; \|\mathbf{M}\| \leq \delta,$$

where $\mathbf{F}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ are affine functions of $\mathbf{x}$, and the subspace $\tilde{\mathcal{M}}$ is arbitrary. Let $\mathcal{B}_{\tilde{\mathcal{M}}}$ denote the set of matrices in (26) associated with $\tilde{\mathcal{M}}$. An upper bound on the optimal value of this robust SDP and a corresponding solution $\mathbf{x}$ can be obtained by solving the following SDP:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{S},\mathbf{T},\mathbf{G}} \quad & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} \quad & (\mathbf{S},\mathbf{T},\mathbf{G}) \in \mathcal{B}_{\mathcal{M}}, \\ & \mathbf{S} \geq \mathbf{0}, \\ & \begin{bmatrix} \mathbf{F}(\mathbf{x}) - \mathbf{S} & \mathbf{R}^T(\mathbf{x}) + \mathbf{G} \\ \mathbf{R}(\mathbf{x}) - \mathbf{G} & \delta^{-2}\mathbf{T} \end{bmatrix} \geq \mathbf{0}. \end{aligned}$$

□

By applying Lemma 3 to (19), using the characterization of $\mathcal{B}_{\mathcal{M}}$ in (27), it can be shown that the solution of the following SDP generates a conservative solution to the original design problem:

$$\min_{\substack{\underline{\mathbf{P}},t \\ \mathbf{S}_k=\mathbf{S}_k^T,\, 1\leq k\leq K}} \quad t \tag{30a}$$

$$\text{s. t.} \quad \|\text{vec}\big([\underline{\mathbf{p}}_1, \, \ldots, \, \underline{\mathbf{p}}_K]\big)\| \leq t, \tag{30b}$$

$$\begin{bmatrix} \mathbf{F}_k(\underline{\mathbf{P}},\hat{\underline{\mathbf{h}}}_k) - \mathbf{S}_k & \mathbf{R}^T(\underline{\mathbf{P}},\beta_k) \\ \mathbf{R}(\underline{\mathbf{P}},\beta_k) & \delta_k^{-2}\mathbf{S}_k \otimes \mathbf{I}_{2N_t} \end{bmatrix} \geq \mathbf{0}, \quad 1 \leq k \leq K, \tag{30c}$$

where $\mathbf{F}_k(\underline{\mathbf{P}},\hat{\underline{\mathbf{h}}}_k)$ and $\mathbf{R}(\underline{\mathbf{P}},\beta_k)$ are as defined in the previous section, and we have exploited the fact that (30c) implies that $\mathbf{S}_k \otimes \mathbf{I} \geq \mathbf{0}$ and hence that $\mathbf{S}_k \geq \mathbf{0}$, which eliminates the constraints that would have been generated by the constraint $\mathbf{S} \geq \mathbf{0}$ in Lemma 3. We would like to point out that the SDP in (25) is the special case of the SDP in (30) that is obtained when $\mathbf{S}_k = \tau_k\mathbf{I}$. Therefore, the solution of (30) yields a tighter upper bound on the minimum power required to solve the original problem than the solution of (25).

D. Some Comparisons

As we have just pointed out, the structured robust SDP in (30) yields a tighter upper bound on the minimum transmission power than the unstructured SDP in (25). Furthermore, our numerical experiments suggest that the unstructured SDP in (25) provides a tighter upper bound than that obtained from the robust SOCP with independent uncertainties in (22). Given this performance hierarchy, it is of interest to examine the relative size and structure of each of the proposed formulations, and that of the design

TABLE I

A COMPARISON OF THE SIZE AND STRUCTURE OF VARIOUS DESIGN METHODS

Method	Number of Variables		Number of Constraints	
	Core	Additional	SOC	LMI
Perfect CSI (10)	$2N_tK+1$		1 of size $2N_tK+1$ K of size $K+2$ plus K linear equalities	
Robust SOCP (22)	$2N_tK+1$	$2K$	1 of size $2N_tK+1$	K of size $2N_t+2K+2$ K of size $2N_t+1$
Robust SDP, Unstruct. (25)	$2N_tK+1$	K	1 of size $2N_tK+1$	K of size $2(K+1)(2N_t+1)$
Robust SDP, Struct. (30)	$2N_tK+1$	$K(K+1)(2K+3)$	1 of size $2N_tK+1$	K of size $2(K+1)(2N_t+1)$

problem for the case of perfect CSI; c.f. (10). We have collected this information in Table I, where the "core" design variables are the $2N_tK$ unique elements of $\underline{\mathbf{P}}$ and the scalar t. In the robust SOCP formulation, each (unique) element of $\underline{\mathbf{P}}$ enters all of the LMIs in (22c) and one of the LMIs in (22d), and in the robust SDP formulations, each element of $\underline{\mathbf{P}}$ enters all the LMIs. The additional variables in the robust SOCP formulation are the $2K$ scalars, λ_k and μ_k. Each λ_k is involved in 2 LMIs (one from the set in (22c) and one from the set in (22d)) and each μ_k is involved in only one. The additional variables in the unstructured robust SDP formulation are the K scalars τ_k, and each one is involved in only one LMI. In the structured robust SDP formulation, the additional variables take the form of the K symmetric matrices $\mathbf{S}_k$, each of which is of size $(2K+2)\times(2K+2)$ and is involved in only one LMI. In addition to the structure of the additional variables, Table I also emphasizes the fact that the constraints in the two robust SDP approaches have the same structure, while that of the robust SOCP approach is somewhat simpler. These observations show that the improved bounds provided by the robust SDP approaches do incur an increase of the size of the SDP. However, our numerical experiments in Section VI suggest that in some applications the improved performance will justify this increase in size.

E. Maximum Allowable Uncertainty Size

Up until this point, we have focused on problems in which the goal has been to minimize the transmission power required to guarantee that the SINR of each user exceeds the required value for every channel uncertainty in the bounded set in (12). As mentioned in Section II, for the class of systems with full row rank channel matrices, $\mathbf{H}$, the QoS requirements can always be satisfied in the absence of channel uncertainty, but this is not the case when the transmitter's model for the channel is inaccurate. This fact raises the question of whether one can determine, for a given set of channel estimates, the largest uncertainty set for which the robust QoS guarantee can be made. That is, find the largest value of δ, namely δ_{max}, such that (13) (or (19)) has a finite solution. This problem is of interest in the design of quantization codebooks for feeding back estimates of the channel to the transmitter. In particular, one may wish to choose the rate of the codebooks to be large enough (and the quantization cells small enough) so that it is possible to design a robust precoder with finite power.[5] As we will point out below, we can obtain efficiently solvable formulations for lower bounds on δ_{max} by using the conservative approaches to the robust QoS design problem.

Using the first conservative approach in Section IV-A, it can be shown that the optimal value of the following problem is a lower bound on δ_{max}:

$$\max_{\underline{\mathbf{P}},\boldsymbol{\mu},\boldsymbol{\lambda},\rho} \quad \rho \tag{31a}$$

$$\text{s. t.} \quad \mathbf{A}_k(\underline{\mathbf{P}},\lambda_k,\mu_k,\rho) \geq \mathbf{0}, \quad 1 \leq k \leq K, \tag{31b}$$

$$\mathbf{B}_k(\underline{\mathbf{P}},\lambda_k,\rho,\beta_k) \geq \mathbf{0}, \quad 1 \leq k \leq K. \tag{31c}$$

where $\mathbf{A}_k(\underline{\mathbf{P}},\lambda_k,\mu_k,\rho)$ and $\mathbf{B}_k(\underline{\mathbf{P}},\lambda_k,\rho,\beta_k)$ are as defined in (22c) and (22d), respectively. Although similar to (22), the above problem is not jointly convex in $\underline{\mathbf{P}}$ and ρ, since the constraints (31b) and (31c) are not jointly affine. However, this problem is quasi-convex (c.f. [22]), and an optimal solution can be efficiently found using a one-dimensional bisection search on ρ in which the problem solved at each step is the convex feasibility problem corresponding to (31) with a fixed value for ρ.

Using the structured robust SDP approach in Section IV-C, it can be shown that $\delta_{\text{max}} \geq (\alpha^*)^{-1/2}$, where α^* is the optimal value of the following quasi-convex optimization problem

$$\min_{\substack{\underline{\mathbf{P}},\alpha \\ \mathbf{S}_k=\mathbf{S}_k^T,\, 1\leq k\leq K}} \quad \alpha \tag{32a}$$

$$\text{s. t.} \quad \begin{bmatrix} \mathbf{F}_k(\underline{\mathbf{P}},\hat{\underline{\mathbf{h}}}_k) - \mathbf{S}_k & \mathbf{R}^T(\underline{\mathbf{P}},\beta_k) \\ \mathbf{R}(\underline{\mathbf{P}},\beta_k) & \alpha\mathbf{S}_k \otimes \mathbf{I}_{2N_t} \end{bmatrix} \geq \mathbf{0}, \quad 1 \leq k \leq K. \tag{32b}$$

(The unstructured robust SDP approach leads to the special case in which $\mathbf{S}_k = \tau_k\mathbf{I}$.) This problem can be solved using bisection search on α. Furthermore, by observing that the constraints in (32b) can be

[5]The minimum distance between two codewords (and hence δ) is bounded by a function of the size of the codebook and the dimension of the space [26], [27].

written as

$$\alpha \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_k \otimes \mathbf{I}_{2N_t} \end{bmatrix} - \begin{bmatrix} -\mathbf{F}_k(\underline{\mathbf{P}}, \hat{\underline{\mathbf{h}}}_k) + \mathbf{S}_k & -\mathbf{R}^T(\underline{\mathbf{P}}, \beta_k) \\ -\mathbf{R}(\underline{\mathbf{P}}, \beta_k) & \mathbf{0} \end{bmatrix} \geq \mathbf{0}, \quad 1 \leq k \leq K, \tag{33}$$

one can show that (32) is equivalent to minimizing the largest generalized eigen value of a pair of (block diagonal) symmetric matrices that depend affinely on the decision variables [24], [28]. Identifying (32) as lying within this class of problems is of interest because efficient algorithms that exploit the structure of the constituent matrices in (33) are available for such problems; c.f. [28], [29].

V. Fair SINR Maximization

In the previous section, we considered problems that required each user to be provided with an SINR that is at least as large as a given SINR requirement, even in the presence of uncertainty. In this section, we consider the related problem of maximizing the SINR of the "weakest" user subject to a transmitted power constraint, in scenarios with uncertain CSI. Problems of this form are sometimes called max-min fair SINR problems; e.g., [5], [6]. While most formulations of max-min fair SINR problems have focussed on the case of perfect CSI, under the bounded uncertainty model in (12) the robust max-min fair SINR problem can be stated as[6]

$$\max_{\underline{\mathbf{P}}, \gamma_0} \quad \gamma_0 \tag{34a}$$

$$\text{s. t.} \quad \text{SINR}_k \geq \gamma_0, \qquad \forall\, \underline{\mathbf{h}}_k \in \mathcal{U}_k(\delta_k), \quad 1 \leq k \leq K, \tag{34b}$$

$$\frac{1}{2}\text{tr}(\underline{\mathbf{P}}\underline{\mathbf{P}}^T) \leq P_{\text{total}}. \tag{34c}$$

By defining $\beta_0 = \sqrt{1 + 1/\gamma_0}$ and using the SOC characterization of the QoS constraints in (10c), this problem can be cast as the following (semi-infinite) quasi-convex optimization problem (see [6] for a related formulation for the case of perfect CSI)

$$\min_{\underline{\mathbf{P}}, \beta_0} \quad \beta_0 \tag{35a}$$

$$\text{s. t.} \quad \big\| [\underline{\mathbf{h}}_k \underline{\mathbf{P}}, \quad \sigma_{n_k}] \big\| \leq \beta_0 \underline{\mathbf{h}}_k \underline{\mathbf{p}}_k, \qquad \forall\, \underline{\mathbf{h}}_k \in \mathcal{U}_k(\delta_k), \quad 1 \leq k \leq K, \tag{35b}$$

$$\big\| \text{vec}\big([\underline{\mathbf{p}}_1, \ \ldots, \ \underline{\mathbf{p}}_K] \big) \big\| \leq P_{\text{total}}. \tag{35c}$$

[6]Although we will not discuss them here, the "per-antenna" and "shaping" power constraints discussed in the Appendix can be easily incorporated into the proposed framework.

Efficient formulations for precoders that minimize upper bounds on β_0 (and hence maximize lower bounds on γ_0) can be obtained by applying the conservative approaches of Section IV to (35). In particular, by applying the robust SOCP approach in Section IV-A, one obtains the following quasi-convex problem:

$$\min_{\underline{\mathbf{P}},\boldsymbol{\mu},\boldsymbol{\lambda},\beta_0} \quad \beta_0 \tag{36a}$$

$$\text{s. t.} \quad \mathbf{A}_k(\underline{\mathbf{P}},\lambda_k,\mu_k,\delta_k) \geq \mathbf{0}, \quad 1 \leq k \leq K, \tag{36b}$$

$$\mathbf{B}_k(\underline{\mathbf{P}},\lambda_k,\delta_k,\beta_0) \geq \mathbf{0}, \quad 1 \leq k \leq K, \tag{36c}$$

$$\big\|\text{vec}\big([\underline{\mathbf{p}}_1,\ \ldots,\ \underline{\mathbf{p}}_K]\big)\big\| \leq P_{\text{total}}, \tag{36d}$$

where $\mathbf{A}_k(\underline{\mathbf{P}},\lambda_k,\mu_k,\delta_k)$ and $\mathbf{B}_k(\underline{\mathbf{P}},\lambda_k,\delta_k,\beta_0)$ were defined in (22c) and (22c), respectively. This problem can be efficiently solved by using a bisection search on β_0 in which problem solved at each step is the convex feasibility problem generated by (36) with a fixed value of β_0. Similarly, the structured robust SDP approach of Section IV-C yields the following quasi-convex problem that can also be efficiently solved using a bisection search on β_0:

$$\min_{\substack{\underline{\mathbf{P}},\beta_0 \\ \mathbf{S}_k=\mathbf{S}_k^T,\,1\leq k\leq K}} \quad \beta_0 \tag{37a}$$

$$\text{s. t.} \quad \begin{bmatrix} \mathbf{F}_k(\underline{\mathbf{P}},\hat{\underline{\mathbf{h}}}_k)-\mathbf{S}_k & \mathbf{R}^T(\underline{\mathbf{P}},\beta_0) \\ \mathbf{R}(\underline{\mathbf{P}},\beta_0) & \delta_k^{-2}\mathbf{S}_k\otimes\mathbf{I}_{2N_t} \end{bmatrix} \geq \mathbf{0}, \quad 1 \leq k \leq K, \tag{37b}$$

$$\big\|\text{vec}\big([\underline{\mathbf{p}}_1,\ \ldots,\ \underline{\mathbf{p}}_K]\big)\big\| \leq P_{\text{total}}. \tag{37c}$$

VI. Simulation Studies

In this section we will compare the performance of the three proposed approaches to robust QoS precoding, namely the robust SOCP design (RSOCP) with independent uncertainty in Section IV-A, the unstructured robust SDP (RSDP-Unstruct.) in Section IV-B, and the robust SDP that preserves structure (RSDP-Struct.) in Section IV-C. We will also provide performance comparisons with some existing approaches, namely the robust autocorrelation matrix approach in [3], [4] (Robust Correl. Appr.), and the robust downlink power loading approach in [8]. The approach in [8] requires the beamforming vectors to be specified, and we will consider two choices: the columns of the pseudo-inverse of $\hat{\mathbf{H}}$ (Robust Power Load. 1); and the beamforming vectors obtained by assuming that $\hat{\mathbf{H}}$ is the actual channel and using the existing methods for QoS precoding with perfect CSI [1]–[6] (Robust Power Load. 2). The approaches in [3], [4] and [8] are based on uncertainty models that are different from the one in (12), and from each other. The approach in [3], [4] considers a model in which the spectral norm of the error in the (deterministic) autocorrelation matrix $\mathbf{C}_k = \mathbf{h}_k^H \mathbf{h}_k$ is bounded, and in the approach in [8] the Frobenius norm of the error in $\mathbf{C}_k$ is bounded. However, by bounding these norms of $\mathbf{C}_k$ in terms of the norm of $\mathbf{e}_k$, a comparable uncertainty set can be generated.[7] We will compare these schemes in an environment with $N_t = 3$ transmit antennas and $K = 3$ users. In our experiments, we will evaluate performance statistics for the standard case of independent Rayleigh fading channels in which the coefficients of the fading channels are modeled as being independent proper complex Gaussian random variables with zero-mean and unit variance, and the receivers' noise sources are modeled as zero-mean, additive, white, and Gaussian with unit variance.

In our first experiment, we randomly generated 2000 realizations of the set of channel estimates $\{\hat{\mathbf{h}}_k\}_{k=1}^K$ and examined the performance of each method in the presence of equal uncertainty, $\delta_k = \delta$, $\forall k$. The QoS requirement of each user is that the SINR is at least 10 dB. For each set of channel estimates and for each value of δ we determined whether each design method is able to generate a precoder (of finite power) that guarantees that the SINR constraints are satisfied in the presence of the modeled uncertainty. In Fig. 1 we provide the percentage of the 2000 channel realizations for which each method generated a precoder with finite power, as a function of the size of the uncertainty. From this figure, it is clear that the RSDP approach that preserves the structure of the uncertainty is able to provide robust QoS guarantees for a significantly larger percentage of the channels and for significantly larger uncertainty sets than the other methods. The unstructured SDP approach provides a reasonable degree of robustness to channel uncertainty compared to that provided by the RSOCP approach, the robust autocorrelation approach in [3], [4], and the robust power loading approach in [8].

In our second experiment, we selected those sets of channel estimates from the 2000 sets used in the first experiment for which all design methods were able to provide robust QoS guarantees for all uncertainties with $\delta \leq 0.015$. We calculated the average, over the 609 such channel environments, of the transmitted power required to achieve these robust QoS guarantees and we have plotted the results for different values of δ in Fig. 2(a). The average transmitted power approaches infinity for a certain value of δ when for one (or more) of the channel estimates the method under consideration cannot provide the

[7] A bound on the spectral norm of the error in the matrix $\mathbf{C}_k$ can be obtained as follows: $\|(\hat{\mathbf{h}}_k + \mathbf{e}_k)^H(\hat{\mathbf{h}}_k + \mathbf{e}_k) - \mathbf{h}_k^H \mathbf{h}_k\| = \|\hat{\mathbf{h}}_k^H \mathbf{e}_k + \mathbf{e}_k^H \hat{\mathbf{h}}_k + \mathbf{e}_k^H \mathbf{e}_k\| \leq \|\hat{\mathbf{h}}_k^H \mathbf{e}_k\| + \|\mathbf{e}_k^H \hat{\mathbf{h}}_k\| + \|\mathbf{e}_k^H \mathbf{e}_k\| = 2\|\hat{\mathbf{h}}_k\|\|\mathbf{e}_k\| + \|\mathbf{e}_k\|^2$. The same bound also holds for the Frobenius norm, since the matrices on the immediate right hand side of the inequality are all rank one. Furthermore, the uncertainty $\mathbf{e}_k = \delta_k \hat{\mathbf{h}}_k / \|\hat{\mathbf{h}}_k\|$ achieves this upper bound with equality for both norms. (See also [30].)

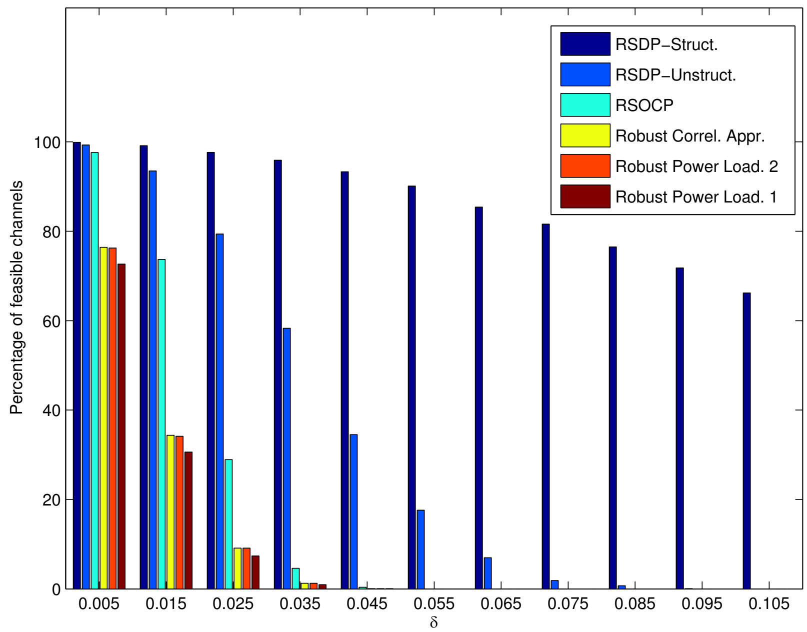
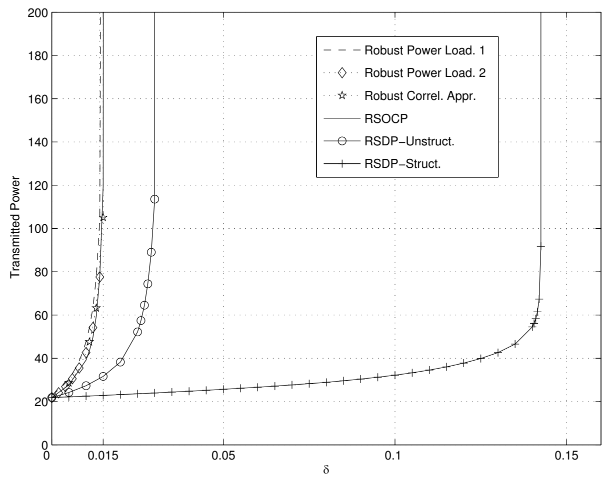


Fig. 1. Percentage of channel realizations for which the robust QoS guarantee can be made against the uncertainty size δ, for a system with $N_t = 3$ and $K = 3$.

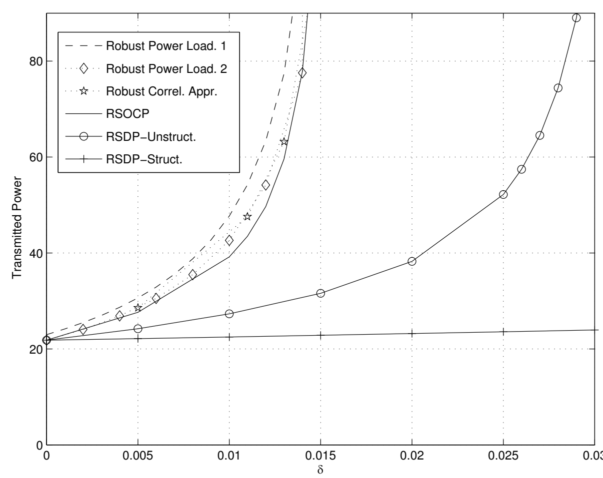
robust QoS guarantee with finite power. The excellent performance of the structured RSDP method and the good performance of the unstructured RSDP method that were apparent in Fig. 1 are also apparent in Fig. 2(a). In Fig. 2(b), we provide a detail of Fig. 2(a) in order to demonstrate the relative difference in the performance of the RSOCP approach, the robust autocorrelation approach in [3], [4], and the robust power loading approach in [8].

In the third experiment, we used the 2000 randomly generated realizations of the estimates of the channel environments from the first experiment, and for each scenario we used the methods in Section IV-E to find lower bounds on the value of the uncertainty, $\delta_{\max}$, above which each design method is unable guarantee the SINR requirements in the presence of the modeled uncertainty. In these experiments the size of uncertainty was the same for each user (i.e., $\delta_k = \delta$, $\forall k$), and the minimum SINR requirement of each user was 10 dB. In Fig. 3 we plot the cumulative distribution function (CDF) of the lower bound on $\delta_{\max}$ for each method. From this figure, it is clear that the relative performance of each method under this performance metric is similar to that established from the first two experiments.

In the fourth experiment, we examine the performance of the 2000 randomly generated realizations of the set of channel estimates $\{\hat{\mathbf{h}}_k\}_{k=1}^{K}$ in the presence of equal uncertainty, $\delta_k = \delta = 0.05$, $\forall k$. The SINR requirements of the three users are equal and varied from 0 to 25 dB. For each set of channel estimates



(a)



(b)

Fig. 2. Average of the transmitted power tr($\mathbf{P}^H\mathbf{P}$), on a linear scale, versus uncertainty size δ for a system with $N_t = 3$ and $K = 3$. Part (b) is a detail of part (a).

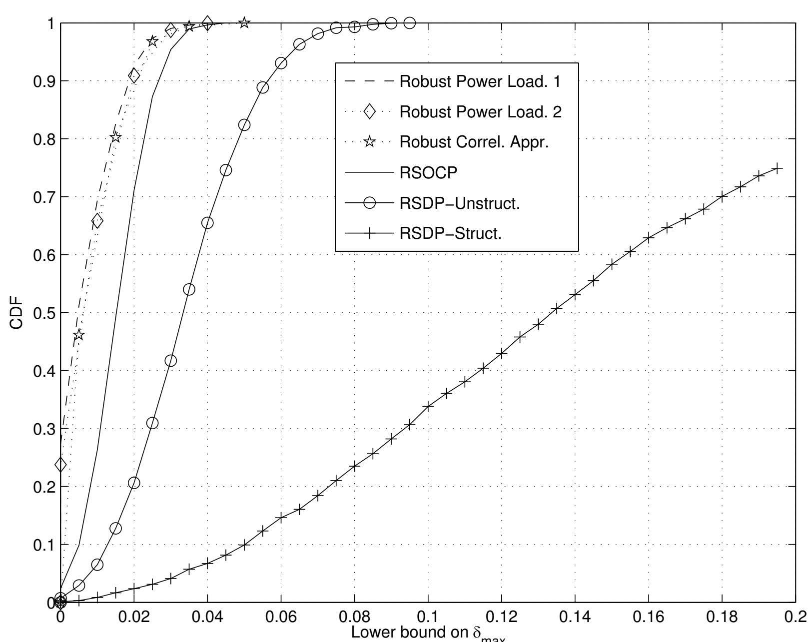


Fig. 3. The (empirical) cumulative distribution function (CDF) of lower bounds on δ_{max} for a system with $N_t = 3$ and $K = 3$.

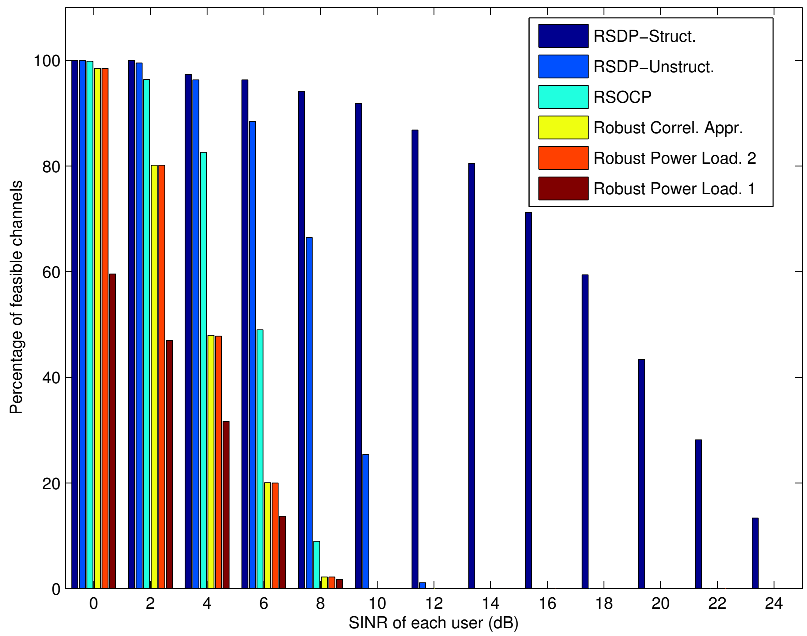


Fig. 4. Percentage of channel realizations for which the robust QoS guarantee can be made against the required SINRs, for a system with $N_t = 3$ and $K = 3$.

and for each value of the required SINR we determined whether each design method is able to generate a precoder (of finite power) that guarantees the required SINRs. In Fig. 4 we provide a histogram of the fraction of the 2000 channel realizations for which each method generated a precoder with finite power. From this figure, it is clear that both robust SDP approaches are able to provide robust QoS guarantees for a wider range of required SINRs than the other methods, with the structured SDP approach having a significant advantage.

In the fifth experiment, we selected all the sets of channel estimates from the 2000 sets used in the previous experiment for which all design methods were able to provide robust QoS guarantees for all SINRs less than or equal to 6dB. We calculated the average, over the 264 such channel environments, of the transmitted power required to achieve these robust QoS guarantees. We have plotted the equal SINR requirement of each user versus the average transmitted power in Fig. 5. In order to assess the additional power required to achieve robustness, we have included the corresponding curve for the case of perfect CSI at the transmitter; c.f. [1]–[6] and (10). This figure illustrates the saturation effect that channel uncertainty imposes on the growth of the SINR of each user with the transmitted power. This effect was observed in [7] for non-robust linear precoding on the MISO downlink with quantized CSI. Fig. 5 also illustrates the role that robust precoding can play in extending the SINR interval over which

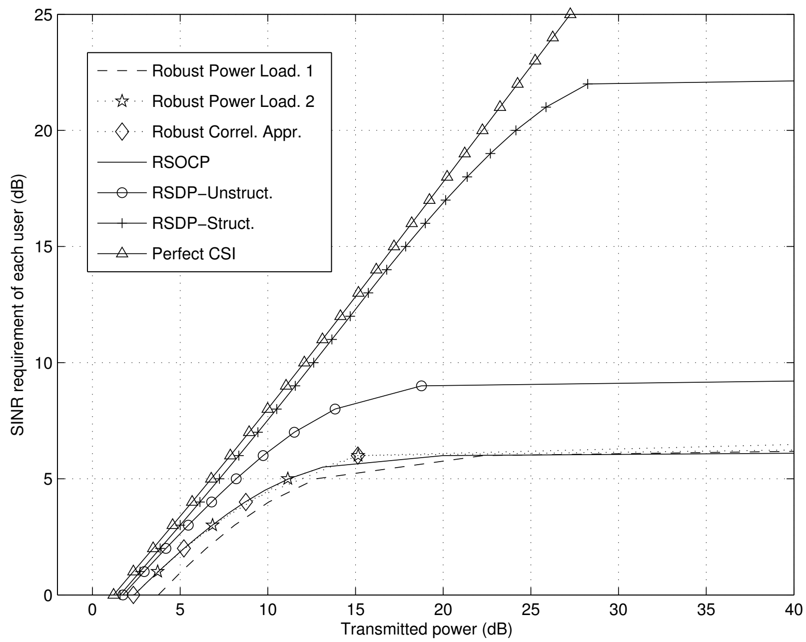


Fig. 5. Maximum achievable (equal) SINRs against the average transmitted power, for a system with $N_t = 3$ and $K = 3$.

linear growth with the transmitted power can be achieved. This is particularly evident for the robust SDP approaches.

In the sixth experiment, we examine the performance of the 2000 randomly generated realizations of the set of channel estimates $\{\hat{\mathbf{h}}_k\}_{k=1}^{K}$ in the presence of equal uncertainty, $\delta_k = \delta = 0.05$, $\forall k$. The SINR requirements of the three users are equal and varied from 0 to 25 dB. For each set of channel estimates, we determine the maximum value of the SINR, $\mathrm{SINR}_{\max}$, above which each design method is unable to guarantee the SINR requirements. In Fig. 6 we plot the CDF of $\mathrm{SINR}_{\max}$ for each method. From this figure, it is clear that the three proposed approaches are able to provide SINR guarantees for significantly larger values of SINR than the robust power loading approaches in [8] and the robust autocorrelation approach in [3], [4].

In the last experiment, we assess the degree of conservatism of each method by studying the statistics of the actual received SINRs for channel realizations within a given uncertainty bound. Scenarios in which the actual SINRs are likely to be significantly higher than the requested SINRs indicate that the transmitter adopts a more conservative approach that requires higher transmitted power. Ideally, when perfect CSI is available at the transmitter, the actual received SINRs are equal to the requested ones, i.e., all QoS constraints are achieved with equality [1]–[6]. In this experiment, we selected the sets of

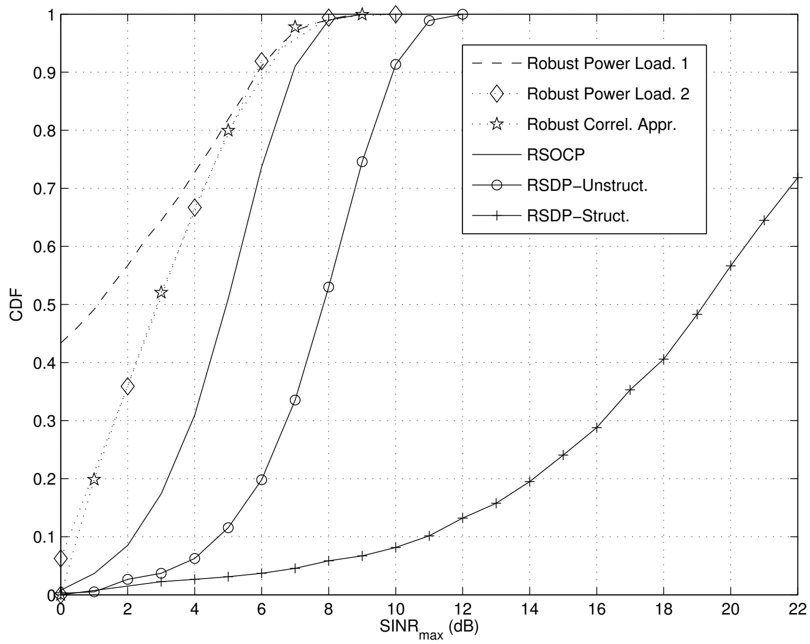


Fig. 6. The (empirical) cumulative distribution function (CDF) of SINR_{max} for a system with $N_t = 3$ and $K = 3$.

channel estimates from the 2000 sets used in the first experiment for which all design methods were able to provide robust QoS guarantees of 10 dB for all users for the uncertainty bound $\delta = 0.015$. For each of the 609 such channel environments, we randomly generated 100 channel uncertainties that were uniformly distributed in direction and whose norms were equal to 0.01. In Fig. 7 we plot the CDF of the actual received SINRs for each design method. To help interpret this figure, in Tab. II we have provided the average transmission powers of each design method. It is apparent from Fig. 7 and Tab. II that the proposed approaches are able to save transmission power by reducing the likelihood that a user's SINR requirement is substantially over-satisfied.

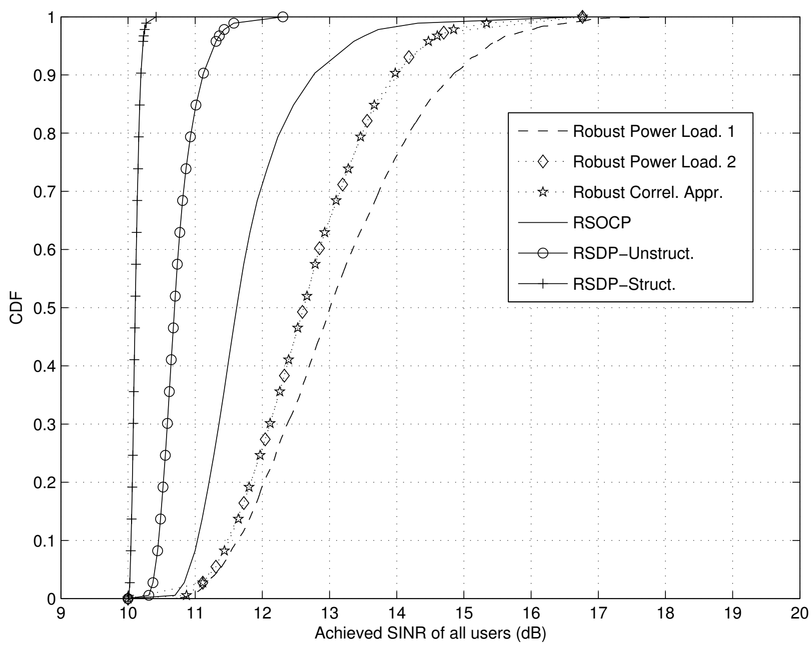


Fig. 7. The (empirical) cumulative distribution function (CDF) of the actual received SINRs for a system with $N_t = 3$ and $K = 3$ and a target SINR of 10 dB.

TABLE II

TRANSMISSION POWERS FOR FIG. 7

Approach	Transmission Power, $\mathrm{tr}(\mathbf{P}^H\mathbf{P})$
Robust Power Load. 1	44.70
Robust Power Load. 2	42.64
Robust Autocorr.	42.60
RSOCP	39.19
RSDP-Unstruct	27.31
RSDP-Struct	22.49

VII. Conclusion

We have considered linear precoding (beamforming) for broadcast channels with QoS constraints in the presence of uncertain CSI at the transmitter. We studied the design of a robust linear precoder that minimizes the total transmitted power while satisfying the users' QoS constraints for all channel realizations within a bounded uncertainty region around the transmitter's estimate of each user's channel.[8] Since that problem is not known to be computationally tractable, we presented three conservative design approaches that yield convex and computationally-efficient restrictions of the original design problem. We also showed how the conservative design approach could be used to obtain efficiently-solvable quasi-convex restrictions of some related problems, including the robust counterpart of the problem of maximizing the minimum SINR subject to a given power constraint. As illustrated by the simulations, the proposed approaches can satisfy the users' QoS requirements for a significantly larger set of uncertainties than existing methods, and require less transmission power to do so.

Appendix

In this appendix we show how different power constraints can be incorporated in our formulations. Consider a set of per-antenna power constraints, $\mathrm{E}\{|x_n|^2\} \leq P_n$, one for each $1 \leq n \leq N_t$, where P_n is the bound on the power transmitted from the n^{th} antenna. Each of these constraints can be written as

$$\sum_{k=1}^{K} [\underline{\mathbf{p}}_k]_n^2 + [\underline{\mathbf{p}}_k]_{n+N_t}^2 \leq P_n, \tag{38}$$

where $[\,\cdot\,]_n$ denotes the n^{th} element of a vector. This is a convex quadratic constraint on the elements of $\underline{\mathbf{P}}_k$, and can formulated as a second order cone constraint and directly accommodated in (10) and all the subsequent robust counterparts.

The shaping constraint $\mathrm{E}\{\mathbf{x}^H \mathbf{Q}(\theta)\mathbf{x}\} \leq P_{\text{shape}}(\theta)$ can be written as

$$\sum_{k=1}^{K} \underline{\mathbf{p}}_k{}^T \underline{\mathbf{Q}}(\theta)\, \underline{\mathbf{p}}_k \leq P_{\text{shape}}(\theta), \qquad \forall\, \theta \,\in\, \Theta, \tag{39}$$

where $\underline{\mathbf{Q}}(\theta)$ is defined analogously to (8). A convenient way in which this constraint can be incorporated into (10) is to write

$$\|\mathrm{vec}\big(\underline{\mathbf{Q}}(\theta)^{1/2}[\underline{\mathbf{p}}_1,\, \ldots,\, \underline{\mathbf{p}}_K]\big)\| \leq \sqrt{P_{\text{shape}}(\theta)}, \qquad \forall\, \theta \,\in\, \Theta. \tag{40}$$

[8] If a certain probability of outage can be tolerated, then a so-called chance constrained formulation might be appropriate; see, e.g., [31] for an application in multiple access systems. We have extended the approach taken in this paper to the chance constrained framework, and the results will appear in due course.

Whenever the set Θ is discrete and finite, this set of SOCs constraints can be easily incorporated in (10) without compromising our approach. Integral constraints of the form

$$\int_{\theta_1}^{\theta_2} \mathrm{E}\{\mathbf{x}^H \mathbf{Q}(\theta)\mathbf{x}\} d\theta \leq \int_{\theta_1}^{\theta_2} P_{\text{shape}}(\theta) d\theta \tag{41}$$

can be accommodated in a similar way.

The power constraints considered above all have the SOCP formulations, but they all fall into the more general class of shaping constraints

$$\underline{\mathbf{P}}^T \underline{\mathbf{Q}}\, \underline{\mathbf{P}} \leq \mathbf{C}_{\text{shape}}, \tag{42}$$

for given $\underline{\mathbf{Q}} > \mathbf{0}$ and $\mathbf{C}_{\text{shape}}$, that have been previously studied for the single user case [10]. Using the Schur Complement Theorem [21], this constraint is equivalent to the LMI

$$\mathbf{C}(\underline{\mathbf{P}}) = \begin{bmatrix} \mathbf{C}_{\text{shape}} & \underline{\mathbf{P}}^T \\ \underline{\mathbf{P}} & \underline{\mathbf{Q}}^{-1} \end{bmatrix} \geq \mathbf{0}, \tag{43}$$

and hence constraints of the form in (42) can be easily incorporated into our approach.

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