Volumetric Path and Klee-Minty Constructions

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Abstract

By introducing redundant Klee-Minty examples, we have previously shown that the central path can be bent along the simplex path. In this paper, we seek for an analogous result for the *volumetric path* defined by the volumetric barrier function. Although we only have a complete proof in 2D, the evidence provided by some illustrations anticipates that a Klee-Minty construction exists for the volumetric path in general dimensions too.

Introduction

Introduced by Dantzig [1] in 1947, the *simplex method* is a powerful algorithm for linear optimization (LO) problems. In 1972 Klee and Minty [5] showed that the simplex method may take an exponential number of iterations.

Beginning with the seminal work of Karmarkar [4], the study of interior point methods (IPMs) as polynomial time algorithms has dominated the continuous optimization literature, in particular LO. The most practical class of IPMs is the class of path-following IPMs. By defining a self-concordant barrier function over the feasible region, path-following IPMs start at a *center* of the feasible region and follow an *interior path* to converge to the an optimal point; see e.g., [7]. Two well-known self concordant barrier functions are the logarithmic and the volumetric barrier [8] functions each defining a center and a path converging to an optimal point. For clarity in the rest of the paper, the centers and paths defined by the two different barriers are refereed to as "analytic center" and "central path" corresponding to the logarithmic barrier, and "volumetric center" and "volumetric path" corresponding to the volumetric barrier.

The iteration complexity upper-bound for central path-following IPMs are known to be polynomial with respect to the number of inequalities in the LO problems. In fact, using the logarithmic barrier function, the iteration complexity for LO problems is bounded above by $O(\sqrt{N}\log\frac{N}{\epsilon})$, where N and ϵ respectively denote the number of inequalities and the given accuracy. While with the volumetric barrier function, this complexity upper-bound is $O(\sqrt[4]{Nn^2}\log\frac{N}{\epsilon})$, where n is the dimension of the problem.

In [2], we showed that the central path of a redundant representation of the Klee-Minty n-cube may trace the path followed by the simplex method. In this construction, uniform distances for the redundant constraints have been chosen. This provided an $O(n^4 2^{9n})$ iteration complexity upper-bound and $2^n - 1$ iteration complexity lower-bound. In [3], by allowing the distances of the redundant constraints to the corresponding facets to decay geometrically, the bounds are significantly tightened with an $O(n^{2.5}2^n)$ iteration complexity upper-bound.

The results for the logarithmic barrier function encouraged us to seek for similar results for the volumetric barrier function which is known to be less sensitive to redundancy than the logarithmic one (as it will also be clarified in the sequel). Since the analogous construction did not work for the volumetric barrier, as a result of the search for different constructions, we came up with a new and even simpler construction. This new construction not only works for the volumetric barrier (at least for lower dimensions), but also provides tighter bounds with the logarithmic barrier function [6]. In fact, the iteration complexity upper-bound becomes $O(n^{1.5}2^n)$. The question why the previous constructions do not work might for the volumetric barrier be answered by investigating carefully the differences of the effects of redundancy on the two barriers.

1 Klee-Minty Cubes and Barriers

In this section, we first introduce our redundant Klee-Minty construction. Then, we define the logarithmic and the volumetric barrier functions. We describe the centers and the interior paths defined by the respective barriers. Finally, we explain some important properties of the barriers, and study the effects of redundancy on the centers and the paths defined by them. For any vector x, we denote $X = \text{diag}(x)$, $\ln x = (\ln x_1, \dots, \ln x_n)^T$ and the all-one vector is denoted by e.

Assume that $\tau > 0$ and h and d are positive integer-valued vectors. The redundant Klee-Minty construction, where the redundant constraints are placed parallel to the coordinate-planes, is defined as

$$\begin{array}{llllll} \min & x_n & & & & \\ \text{subject to} & s_k & = & x_k - \tau x_{k-1}, & s_k \geq 0 & k = 1, \dots, n, \\ & \bar{s}_k & = & 1 - x_k - \tau x_{k-1}, & \bar{s}_k \geq 0 & k = 1, \dots, n, \\ & \tilde{s}_k & = & d_k + x_k, & \tilde{s}_k \geq 0 & \text{repeated } h_k \text{ times, for } k = 1, \dots, n. \end{array}$$

When $h = 0$, this formulation corresponds to the non-redundant Klee-Minty example. The logarithmic and the volumetric barrier functions are denoted by F and V, respectively,

$$\begin{aligned} F(x) &= -e^T \ln s - e^T \ln \bar{s} - h^T \ln \tilde{s}, \\ V(x) &= \frac{1}{2} \ln \det \nabla^2 F(x). \end{aligned}$$

We have $\nabla^2 F(x) = AS^{-2}A^T + \bar{A}\bar{S}^{-2}\bar{A}^T + \tilde{S}^{-2}H$, where A and $\bar{A}$ are the following $n \times n$ matrices:

$$A = \begin{pmatrix} -1 & \tau & & & \\ & -1 & \tau & & \\ & & \ddots & \ddots & \\ & & & -1 & \tau \\ & & & & -1 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 1 & \tau & & & \\ & 1 & \tau & & \\ & & \ddots & \ddots & \\ & & & 1 & \tau \\ & & & & 1 \end{pmatrix}.$$

Since F and V are strictly convex functions over a closed convex set, they have unique minimizers that identify the *analytic center* and the *volumetric center*, respectively. Therefore, the centers are unique solutions of the following nonlinear equation systems:

$$\begin{aligned} \nabla F(x) &= -AS^{-1}e - \bar{A}\bar{S}^{-1}e - \tilde{S}^{-1}He = 0, \\ \nabla V(x) &= -AS^{-3}\sigma - \bar{A}\bar{S}^{-3}\bar{\sigma} - \tilde{S}^{-3}H\tilde{\sigma} = 0, \end{aligned}$$

where, for $k = 1, \dots, n$,

$$\begin{aligned} \sigma_k &= a_k^T \left(\nabla^2 F(x)\right)^{-1} a_k, \\ \bar{\sigma}_k &= \bar{a}_k^T \left(\nabla^2 F(x)\right)^{-1} \bar{a}_k, \\ \tilde{\sigma}_k &= e_k^T \left(\nabla^2 F(x)\right)^{-1} e_k. \end{aligned}$$

The unique minimizers of the strictly convex functions $x_n + \mu F(x)$ and $x_n + \mu V(x)$ over the feasible region, for each $\mu > 0$, define the *central path* $\{x^F(\mu) : \ \mu > 0\}$ and the *volumetric path* $\{x^V(\mu) : \ \mu > 0\}$, respectively. These paths

- are defined in the interior of the feasible region.
- are analytic curves, i.e., infinitely many times differentiable.
- converge to the unique optimal point for the Klee-Minty example.

Our goal is to provide such redundant representations of Klee-Minty examples for which, analogous to the central path, the volumetric path visits a small neighborhood of all the vertices of the Klee-Minty cubes.

2 Comparison of the Barriers and Illustrations

In this section, we outline some of the interesting effects of redundancy on the barriers.

1. Let the feasible region be the line segment $0 \leq x \leq 1$, where $x \geq 0$ is repeated h times. We have $F(x) = -h \ln x - \ln(1-x)$ and $V(x) = \frac{1}{2} \ln(hx^{-2} + (1-x)^{-2})$. Then, the analytic center is $1 - 1/(1+h)$ while the volumetric center is $1 - 1/(1+\sqrt[3]{h})$. Therefore, the volumetric center is less affected by redundancy than the analytic center.
2. Let the feasible region be $x_i \geq 0$ repeated h_i times, for $i = 1, \ldots, n$. Thus, $F(x) = -h^T \ln x$ and $V(x) = -e^T \ln x + \frac{1}{2} e^T \ln h$. Obviously, redundancy does not affect the Newton direction for the volumetric barrier, while it does for the logarithmic barrier.
3. Adding redundant constraints to the Klee-Minty problem , as described in Section 1, always pushes the analytic center while it may push or attract the volumetric center, see Figures 1 and 2. In the figures, star-, circle-, square-, and heart-labeled curves are, respectively, the paths with no redundancy, with redundant constraints placed at the distances $d_1 = 5, 4, 3$ and $d_2 = 0$. We used $\tau = 0.1$, and added $h_1 = 10, 10^6$ redundant constraints for the central path and for the volumetric path, respectively, while keeping $h_2 = 0$.

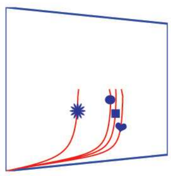


Figure 1: Central paths

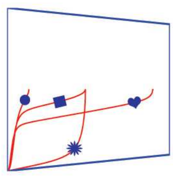


Figure 2: Volumetric paths

4. The analytic center is always displaced by the addition of redundant constraints. However, it may not have any effect on the volumetric center, see Figure 2.
5. As the number of redundant constraints increases, the analytic center approaches to a boundary point of the feasible region while the volumetric center may approach to an interior point, see Figure 4. The solid and dotted curves are, respectively, the curves with no redundancy and with some redundant constraints placed at the distance $d_1 = d_2 = 0$. We used $\tau = 0.1$, and added $h_2 = 10^3, 10^{30}$ redundant constraints for central path and the volumetric path, respectively, while keeping $h_1 = 0$.

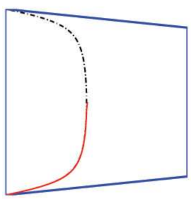


Figure 3: Central paths

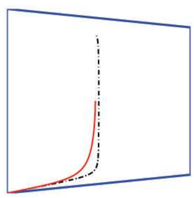


Figure 4: Volumetric paths

6. Analogous to the central path [2, 3, 6], the volumetric path can be bent along the simplex path at least in lower dimensions, see Figures 5, 6, and 8. In Figures 5 and 6, we used $h = (10^2, 10^3)$ and $d = (3, 1)$ for the central path and $h = (2 \times 10^9, 5 \times 10^{10})$ and $d = (100, 1)$ for the volumetric path applied on the Klee-Minty example with $\tau = \delta = 0.1$.

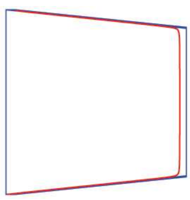


Figure 5: Central path

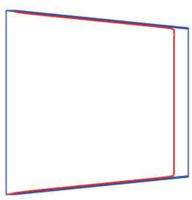


Figure 6: Volumetric path

3 Goal and Results

In this section, we discuss about the volumetric path and how it can be bent along the simplex path (the edge-path followed by the simplex method). The volumetric center is the unique minimizer of the strictly convex function V over a compact convex set. Thus, it can be identified by the solution of the nonlinear system of equations $\nabla V(x) = 0$, or equivalently

$$\begin{cases} \frac{\sigma_k}{s_k^3} - \frac{\tau\sigma_{k+1}}{s_{k+1}^3} - \frac{\bar{\sigma}_k}{\bar{s}_k^3} - \frac{\tau\bar{\sigma}_{k+1}}{\bar{s}_{k+1}^3} + \frac{h_k\tilde{\sigma}_k}{\tilde{s}_k^3} = 0 & \text{for } k = 1, \dots, n-1 \\ \frac{\sigma_n}{s_n^3} - \frac{\bar{\sigma}_n}{\bar{s}_n^3} + \frac{h_n\tilde{\sigma}_n}{\tilde{s}_n^3} = 0 & \\ s_k > 0, \bar{s}_k > 0, \tilde{s}_k > 0 & \text{for } k = 1, \dots, n. \end{cases} \tag{1}$$

For given $\mu > 0$, the μ-center $x^V(\mu)$, which is the unique minimizer of the strictly convex function $x_n + \mu V(x)$, satisfies all equalities of (1) except the last one whose right hand side has to be changed to $\frac{-1}{\mu}$.

The goal is to show that the volumetric center is in a δ-neighborhood of the vertex with the largest objective value and the volumetric path stays in the δ-neighborhood of the simplex path. These can equivalently be stated as

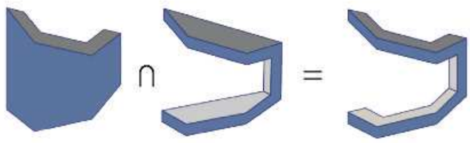
- The volumetric center satisfies: $\bar{s}_n < \delta,\ s_{n-1} < \delta,\ \dots,\ s_1 < \delta$,
- For $k = 1, \dots, n-1$, any point on the volumetric path satisfies:

$$s_{k+1} \geq \delta,\ \bar{s}_{k+1} \geq \delta \quad \Rightarrow \quad \bar{s}_k < \delta,\ s_{k-1} < \delta,\ \dots,\ s_1 < \delta.$$

The latter statement means that any μ-center on the volumetric path belong, for all $k = 1, \dots, n-1$, to the sets

$$A_\delta^{k+1} = \{x : \ s_{k+1} < \delta \ \text{ or } \ \bar{s}_{k+1} < \delta \ \text{ or } \ \bar{s}_k < \delta, s_{k-1} < \delta, \dots, s_1 < \delta\},$$

thus belong to $P_\delta = \bigcap_{k=2}^n A_\delta^k$, which characterizes the δ-neighborhood of the simplex path as illustrated in Figure 7.



$A_\delta^2 \quad \bigcap \quad A_\delta^3 \quad = \quad P_\delta$

Figure 7: The set P_δ for the Klee-Minty 3-cube.

Theorem 3.1. *Assume that $n = 2$, $d = \left(\frac{10}{\tau}, 1\right)$, and $h = \left(\frac{2(10+\tau)^3}{\tau^3\delta^3}, \frac{(10+\tau)^5}{2\tau^3\delta^3}\right)$. Then, the volumetric center is in the δ-neighborhood of the vertex with the largest objective value and the volumetric path stays in the δ-neighborhood of the simplex path. In other words,*

- *The volumetric center satisfies: $\bar{s}_2 < \delta,\ s_1 < \delta$.*
- *Any point on the volumetric path satisfies: $s_2 \geq \delta,\ \bar{s}_2 \geq \delta \quad \Rightarrow \quad \bar{s}_1 < \delta$.*

Generalization. Theorem 3.1 and numerical evidence in 2 and 3 dimensions (see Figures 6 and 8) strengthen the belief that volumetric path can be bent along the simplex path of the Klee-Minty cubes. However, the number of required redundant inequalities is significantly higher.

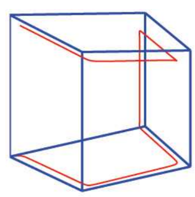


Figure 8: Volumetric path

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