Equivalency of Continuation and Optimization Methods to Determine Saddle-node and Limit-induced Bifurcations in Power Systems I: Transversality Conditions

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***Abstract*— This paper is the first part of a series of two papers which present a comprehensive and detailed study of an optimization-based approach to identify and analyze saddle-node bifurcations (SNB) and limit-induced bifurcations (LIB) of a power system model, which are known to be directly associated with voltage stability problems in these systems. Theoretical studies are presented, formally demonstrating that solution points obtained from an optimization model, which is based on complementarity constraints used to properly represent generators' voltage controls, correspond to either SNB or LIB points of this model; this is accomplished by showing that optimality conditions of these solution points yield the transversality conditions of the corresponding bifurcation points. A simple but realistic test system is used to numerically illustrate the theoretical discussions.**

***Index Terms*— Saddle-node bifurcations, limit-induced bifurcations, transversality conditions, optimization methods, voltage stability, maximum loadability.**

I. INTRODUCTION

VOLTAGE stability (VS) has become rather important in modern power systems, due to the fact that systems are being operated close to their VS limits, as demonstrated by many recent major blackouts which can be directly associated with VS problems. Furthermore, the implementation and application of open market principles have exacerbated this problem, since security margins are being reduced to respond to market pressures [1]–[3]. Consequently, the prediction, identification and avoidance of voltage instability points play a significant role in power system planning and operation. Nonlinear phenomena, particularly saddle-node bifurcations (SNB) and limit-induced bifurcations (LIB) have been shown to be directly associated with VS problems in power systems [4]. It is important to highlight the fact that other types of bifurcations in power systems, such as Hopf bifurcations (HB), associated with oscillatory instabilities [5], and Singularity-induced bifurcations (SIB), associated with differential-algebraic models [4], [6], [7], are not considered in this paper, since these types of bifurcations have not been shown in practice to be directly related to VS problems [4].

Continuation Power Flow (CPF) and Optimal-Power-Flow-based Direct Methods (OPF-DM) are two different techniques that are used in practice to compute VS margins. The most widely used method is the CPF, which is a technique that consists in increasing the loading level until a voltage, current or voltage stability limit is detected in a power flow model, and it is based on a predictor-corrector scheme to find the complete equilibrium profile or bifurcation manifold (PV curve) of a set of power flow equations with respect to a given scalar variable. This scalar parameter is typically referred to as the bifurcation parameter or loading factor, as it is used to model changes in system demand [8], [9]. In [10], it is shown that this method can be viewed as a Generalized Reduced Gradient (GRG) approach for solving a maximum loadability optimization problem.

The OPF-DM is an optimization-based method that consists in maximizing the loading factor, while satisfying the power flow equations, bus-voltage and generators' reactive power limits, and other operating limits of interest (e.g. transmission-line thermal limits) [11], [12]. A variety of OPF models based on this problem definition have been proposed; for example, the authors in [13]–[15] propose a multi-objective OPF for maximizing both the social welfare and the loading factor. This type of optimization problems can be solved by means of Interior-Point Methods (IPM), which have been shown to be computationally efficient for power system studies [16].

An important difference between the CPF and the most popular implementations of the OPF-DM is that, in the CPF, the voltage is kept constant at generation buses while their reactive power output is within limits (PV bus model). In the "standard" OPF-DM, generator voltages and reactive powers are allowed to change within limits, so that "optimal" operating conditions are obtained. These different approaches may lead to different solutions; an interesting discussion about this issue can be found in [2]. An OPF-DM model that is shown empirically to produce similar results as the CPF approach is presented and discussed in [17], where PV buses are modeled using complementarity constraints; the latter are shown here to be particularly important to demonstrate the equivalency of CPF and OPF-DM approaches.

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The current paper, as the first part of a series of two papers, presents a detailed theoretical analysis of the application of OPF-DM to the study of SNBs and LIBs in power systems. Previous works have formally shown that optimization methods can be used to compute SNBs in power system models, and that these methods are basically equivalent to more "classical" computational approaches [10]. Furthermore, some issues associated with the application of OPF-DMs to the computation of LIBs are discussed in [18]. However, up to now, to the authors' knowledge, the links between solutions of OPF-DMs and SNBs and LIBS have not yet been dealt with in the technical literature as formally and systematically as it is done here. Hence, the present paper concentrates on demonstrating that solution points obtained from a given OPF-DM model correspond to either SNB or LIB points; this is accomplished by showing that the optimality conditions of these solution points yield the transversality conditions of the corresponding bifurcation points. The second paper concentrates on formally demonstrating that formulas proposed and developed for the study of sensitivities at SNBs and LIBs can also be obtained through an OPF-DM approach, discussing as well the advantages that the use of optimization approaches present for the computation of a variety of sensitivities. A simple but realistic test system example is used to numerically illustrate the theoretical discussions presented in these two papers.

This paper is structured as follows: Section II presents a concise but thorough description of the VS problem, the power system model used to study it, and its mathematical characterization through bifurcation theory. The optimization models used for the OPF-DM studies of interest to this paper are discussed in detail in Section III. Section IV concentrates on formally showing that the solution points of an optimization model described in the previous section correspond to either SNB or LIB points, based on optimality conditions and the corresponding bifurcation transversality conditions. The theoretical discussions are illustrated with the help of a 6-bus test system in Section V. Finally, in Section VI, the main contributions of the paper are highlighted.

II. Definitions

Voltage stability is associated with the capability of a power system to maintain steady acceptable voltages at all buses, not only under normal operating conditions, but also after being subjected to a disturbance [19]. It is a well established fact that voltage collapse in power systems is associated with system demand increasing beyond certain limits, as well as with the lack or reactive power support in the system caused by limitations in the generation or transmission of reactive power. System contingencies such as generator or line unexpected outages exacerbate, if not trigger, the VS problems [4], [20]. Usually, VS analysis consist in determining the system conditions at which the equilibrium points of a dynamic model of the power system merge and disappear; these points have been associated with certain bifurcations of the corresponding system models [4].

A. System Models

Power systems are typically modeled with nonlinear differential-algebraic equations (DAE), which are a class of nonlinear systems, as follows:

$$\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} f(x,y,\lambda,p) \\ g(x,y,\lambda,p) \end{bmatrix} = F(z,\lambda,p) \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is a vector of the differential variables which represents the dynamic states of generators, loads and system controllers; $y \in \mathbb{R}^{n_y}$ is a vector of algebraic variables that typically results from neglecting fast dynamics, such as load bus voltages magnitudes and angles; $z = (x, y) \in \mathbb{R}^{n_z}$; $\lambda \in \mathbb{R}^+$ stands for a slow varying "uncontrollable" parameter, typically used to represent load changes that move the system from one equilibrium point to another; and $p \in \mathbb{R}^{n_p}$ represents "controllable" parameters associated with control settings, such as Automatic Voltage Regulator (AVR) set points. The function $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^+ \times \mathbb{R}^{n_p} \mapsto \mathbb{R}^{n_x}$ is a nonlinear vector field directly associated with the state variables x, and representing the system differential equations, such as those associated with the generator mechanical dynamics; and $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^+ \times \mathbb{R}^{n_p} \mapsto \mathbb{R}^{n_y}$ represents the system nonlinear algebraic constraints, such as the power flow equations, and algebraic constraints associated with the synchronous machine model.

If the Jacobian $\nabla_y^T g(\cdot)$ of the algebraic constraints is invertible, i.e. nonsingular along a "solution path" of (1), the behavior of the system is mainly defined by the following Ordinary Differential Equation (ODE) model

$$\dot{x} = f(x, y^{-1}(x,\lambda,p), \lambda, p)$$

were $y^{-1}(x,\lambda,p)$ results from applying the Implicit Function Theorem to the algebraic constraints along the system trajectories of interest [10], [21]. The interested reader is referred to [22] for a detailed discussion when $\nabla_y^T g(\cdot)$ is not guaranteed to be invertible; this problem is associated with SIBs, which are beyond the scope of the present paper, since this phenomenon is not directly related to VS problems in practice [4].

Equilibrium points $z_o = (x_o, y_o)$ of (1) are defined by the solutions of the nonlinear equations:

$$F(z_o,\lambda_o,p_o) = \begin{bmatrix} f(x_o,y_o,\lambda_o,p_o) \\ g(x_o,y_o,\lambda_o,p_o) \end{bmatrix} = 0$$

It is important to highlight the fact that the system equilibria are in practice obtained from a subset of equations:

$$G(\hat{z}_o,\lambda_o,\hat{p}_o) = G|_o = 0 \subset F(z_o,\lambda_o,p_o) = F|_o = 0 \tag{2}$$

where $G|_o = 0$ stands for the power flow equations; $G \subset g$; $\hat{z}_o \in \mathbb{R}^{n_{\hat{z}}} \subset z$ is the set of voltage and angles at all buses as well as the reactive power of the generator (PV) buses; and $\hat{p}_o \in \mathbb{R}^{n_{\hat{p}}} \subseteq p$ usually represents the voltage levels and "base" active power injections at PV buses, "base" active and reactive power injections at load buses, transformer fixed-tap settings and other controller settings.

Power flow models have been used in practice for voltage collapse studies, since these models form the basis for defining the actual system operating conditions [4]. However, one should be aware that the solutions of the power flow equations do not necessarily correspond to system equilibria, since a solution of $G|_o = 0$ does not imply that $F|_o = 0$, even though this is not a significant issue in practice. Therefore, in this paper, actual SNBs and LIBs of (1) are assumed to correspond to similar "bifurcation" points of the power flow equations, which is the case of certain power system models [23], [24]; thus, the paper concentrates on the analysis of SNBs and LIBs of (2).

B. Bifurcation Analysis

Bifurcation theory yields tools that are able to classify, study, and give qualitative and quantitative information about the behavior of a nonlinear system close to bifurcation or "critical" equilibrium points as system parameters change [25]. The parameters are assumed to change "slowly", so that the system can be assumed to "move" from equilibrium point to equilibrium point with these changes (quasi-static assumption). Hence, bifurcation analysis are usually associated with the study of equilibria of the nonlinear system model [4].

In power systems, SNBs and some types of LIBs are basically characterized by the local merging and disappearance of power flow solutions as certain system parameters, particularly system demand, slowly change; this phenomena has been associated with VS problems [4]. These kinds of bifurcations are also referred to in the technical literature as fold or turning points.

1) Saddle-Node Bifurcations (SNB): These types of bifurcations occur when two equilibrium points, typically one stable and one unstable in practice, merge and disappear as the parameter λ slowly changes, as illustrated in the PV curves of Figs. 1(a) and 1(b), where V_{G_i} and Q_{G_i} stand for a generator i's terminal voltage magnitude and reactive power, respectively. Mathematically, the SNB point for the power flow model (2) is a solution point $(\hat{z}_c, \lambda_c, \hat{p}_o)$ where the Jacobian $\nabla_{\hat{z}}^T G|_c$ has a simple and unique zero eigenvalue, with nonzero eigenvectors [24], [26]. The following *transversality conditions* can be used to characterize and detect SNBs [10]:

$$\nabla_{\hat{z}}^T G|_c \hat{v} = \nabla_{\hat{z}} G|_c \hat{w} = 0 \tag{3}$$

$$\nabla_\lambda G|_c \ \hat{w} \neq 0 \tag{4}$$

$$\hat{w}^T \left[\nabla_{\hat{z}}^{2^T} G|_c \hat{v} \right] \hat{v} \neq 0 \tag{5}$$

where $\hat{v}$ and $\hat{w} \in \mathbb{R}^{n_{\hat{z}}}$ are normalized right and left eigenvectors of the Jacobian $\nabla_{\hat{z}}^T G|_c$. The first condition implies that the Jacobian matrix is singular; the second and third conditions ensure that there is no equilibria near $(\hat{z}_c, \lambda_c, \hat{p}_o)$ for $\lambda > \lambda_c$ (or $\lambda < \lambda_c$, depending on the sign of (5)). Note that the subscript c is used throughout the paper to denote a bifurcation point.

2) Limit-Induced Bifurcations (LIB): These types of bifurcations in power systems were first studied in detail in [27], and can be typically encountered in these systems, since as the load increases, reactive power demand generally increases and reactive power limits of generators or other voltage regulating devices being reached. These bifurcations result in reduced

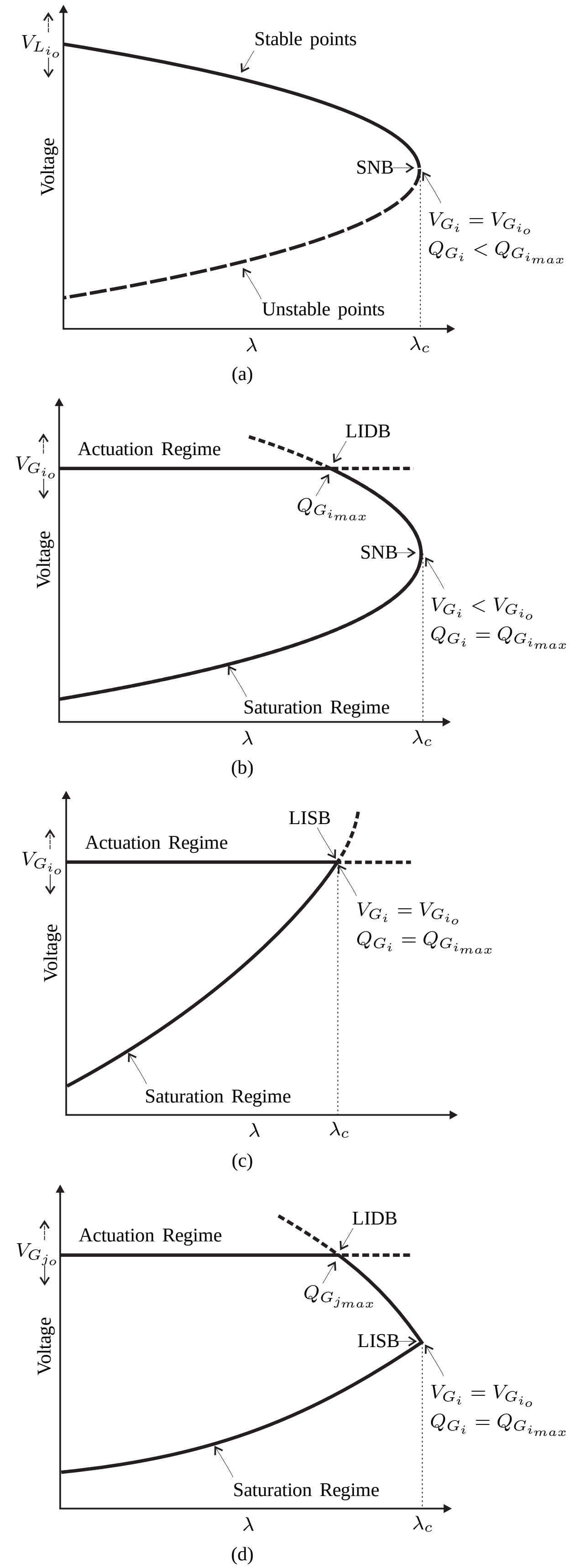

Fig. 1. Main bifurcations observed in PV curves: (a) SNB without Q_G limits; (b) LIDB followed by an SNB; (c) LISB; (d) LISB preceded by a LIDB.

VS margins, and in some cases the operating point "disappears" causing a voltage collapse [4], as illustrated in Fig. 1. Mathematically, the LIBs associated with power flow models are solution points $(\hat{z}_c, \lambda_c, \hat{p}_o)$ where all the eigenvalues of the corresponding Jacobian $\nabla_{\hat{z}}^T G|_c$ have nonzero real parts, i.e. the power flow Jacobian is nonsingular [28].

These bifurcations are divided into two types, namely, limit-induced dynamic bifurcations (LIDB), and limit-induced static bifurcations (LISB). In the case of LIDBs, the equilibrium points continue to exist after being reached as the bifurcation parameter λ changes, as illustrated in Figs. 1(b) and 1(d). On the other hand, LISBs are somewhat similar to SNBs in the sense that these correspond to points at which two solutions merge and disappear as the bifurcation parameter λ changes, as depicted in Fig. 1(c); thus, LISBs also are associated with maximum loadability margins in power flow models.

In general, the limits that triggered LIBs can be categorized into three basic types of limits, namely, actuation limits, state limits and switching limits [28]. The actuation limits appear when certain variables, which are functions of some of the state variables encounter a limit. These limits do not directly affect the state variables but the overall dynamics, and can be modeled through the use of actuation functions. In power systems models, actuation limits typically depend on only one state variable at a time, and one of these inequalities becomes an equality on encountering a limit. The state limits have a direct effect on the state variables, and occur when a state reaches its limit, which results in the system dimension dropping by one, since the state variable becomes a constant in the model. These kinds of limits can be modeled by setting the state derivative equal to zero when the limits are reached. Finally, the switching limits are followed by pre-established actions (e.g. relaying mechanisms or protective limiters in the physical system) that might result in a change in the whole system, and consequently in the states. These limits can be modeled, for instance, by introducing certain binary variables that represent the internal logic of a relay element.

For the power flow model, actuation limits can be directly associated with LIBs. Therefore, this paper focuses on these types of limits to analyze LIBs, using the following representation that results from the proper ordering of the power flow equations (2), and with similar notation as the one proposed in [28]:

$$G(\hat{z}, \lambda, \hat{p}) = \begin{bmatrix} \hat{g}(\tilde{z}, \hat{r}, \lambda, \hat{p}) \\ \hat{r} - \hat{s}(\tilde{z}, \lambda, \hat{p}) \end{bmatrix} = 0 \tag{6}$$

where $\tilde{z} \in \mathbb{R}^{n_{\tilde{z}}}$, $\hat{r} \in \mathbb{R}^{n_{\hat{r}}}$, $\hat{z} = (\tilde{z}, \hat{r})$, and the actuation limits are modeled as:

$$\hat{r}_i = \begin{cases} \hat{r}_{i_{min}}, & \text{if } \hat{s}_i(\tilde{z}, \lambda, \hat{p}) < \hat{r}_{i_{min}} \\ \hat{s}_i(\tilde{z}, \lambda, \hat{p}), & \text{if } \hat{r}_{i_{min}} \leq \hat{s}_i(\tilde{z}, \lambda, \hat{p}) \leq \hat{r}_{i_{max}} \\ \hat{r}_{i_{max}}, & \text{if } \hat{s}_i(\tilde{z}, \lambda, \hat{p}) > \hat{r}_{i_{max}} \end{cases} \tag{7}$$

Since in power flow models, LIBs of interest are typically associated with generators reaching their maximum reactive power limits, at a LIB point $(\hat{z}_c, \lambda_c, \hat{p}_o) = (\tilde{z}_c, \hat{r}_c, \lambda_c, \hat{p}_o)$, the following two sets of equations apply:

$$G_a(\hat{z}_c, \lambda_c, \hat{p}_o) = \begin{bmatrix} \hat{g}(\tilde{z}_c, \hat{r}_c, \lambda_c, \hat{p}_o) \\ \hat{r}_{k_c} - \hat{s}_k(\tilde{z}_c, \lambda_c, \hat{p}_o) \quad \forall k \neq i \\ \hat{r}_{i_c} - \hat{s}_i(\tilde{z}_c, \lambda_c, \hat{p}_o) \end{bmatrix} = 0 \tag{8}$$

$$G_b(\hat{z}_c, \lambda_c, \hat{p}_o) = \begin{bmatrix} \hat{g}(\tilde{z}_c, \hat{r}_c, \lambda_c, \hat{p}_o) \\ \hat{r}_{k_c} - \hat{s}_k(\tilde{z}_c, \lambda_c, \hat{p}_o) \quad \forall k \neq i \\ \hat{r}_{i_c} - \hat{r}_{i_{max}} \end{bmatrix} = 0 \tag{9}$$

where (8) corresponds to the system equations "before" a limit is reached, and (9) represents the system "after" a limit is reached as λ increases. These system conditions can be referred to as the system in actuation regime and in saturation regime, respectively, as depicted in Fig. 1. Notice that a "critical" solution or bifurcation point must satisfy both sets of equations, and that the difference between (8) and (9) is only the equation corresponding to actuation limit i, since a LIB occurs when a single generator i reaches its maximum reactive power limit.

The transversality conditions for LIBs may then be defined as follows [28]:

1) $G_a|_c = G_b|_c = 0$
2) Jacobians $J_a^i = \nabla_{\hat{z}}^T G_a|_c$ and $J_b^i = \nabla_{\hat{z}}^T G_b|_c$ have nonzero real parts, i.e.

$$\det(J_a^i) \neq 0 \quad \text{and} \quad \det(J_b^i) \neq 0 \tag{10}$$

3) The index:

$$\alpha = \frac{\det J_a^i}{\det J_b^i} \neq 0 \tag{11}$$

defines the type of LIB; thus, $\alpha > 0$ for a LISB, and $\alpha < 0$ for a LIDB.

III. OPF-BASED DIRECT METHOD (OPF-DM)

Optimization methods may be used to compute maximum loadability points of power flow models, which are directly associated with SNBs and LISBs of the corresponding model equations, as initially proposed in [11]. Thus, based on the aforementioned SNB and LIB definitions, the bifurcation point directly corresponds to the solution of the following optimization model, as formally demonstrated in Section IV:

$$\max_{\tilde{z}, \hat{r}, \lambda} \quad \lambda \tag{12a}$$

$$\text{s.t.} \quad \hat{g}(\tilde{z}, \hat{r}, \lambda, \hat{p}_o) = 0 \tag{12b}$$

$$\hat{h}(\tilde{z}, \hat{r}, \lambda, \hat{p}_o) = 0 \tag{12c}$$

$$\hat{r}_{min} \leq \hat{r} \leq \hat{r}_{max} \tag{12d}$$

where the nonlinear function $\hat{h}$ is used to represent the actuation limit equations introduced in (6), since in these optimization models, the actuation limits are typically not represented explicitly, as illustrated below. The issue of how constraints (12c) are actually represented in this model, and the effect of this modeling on the solution of optimization problem (12) is discussed in detail below. Note that (12d) basically corresponds to (7).

A. OPF-DM in Standard Form

For a typical power flow model, let $\tilde{z} = (\delta, V_L, K_G)$, $\hat{r} = (Q_G, V_G)$, and $\hat{p} = (P_S, P_D)$. In this case, δ stands for all the bus voltage phasor angles but one (slack bus); V_L and V_G correspond to the load and generator bus voltage phasor magnitudes, respectively, and Q_G represents the generator reactive power output. The variables P_S and P_D define the change in generation and demand powers, respectively, as follows:

$$\begin{aligned} P_G &= P_{G_o} + (\lambda + K_G) P_S \\ P_L &= P_{L_o} + \lambda\, P_D \\ Q_L &= Q_{L_o} + \lambda\, K_L P_D \end{aligned} \tag{13}$$

where P_{G_o}, P_{L_o} and Q_{L_o} stand for the "base" generation and load levels, thus defining an "initial" operating point; K_G is a variable used to model a distributed slack bus; and K_L is a constant used to represent a constant power factor load.

Based on the aforementioned variable definition and if the actuation functions (12c) are omitted, the model can be restated as:

$$\begin{aligned} \max \quad & \lambda && (14a) \\ \text{s.t.} \quad & \widehat{G}(\delta, V_L, K_G, Q_G, V_G, \lambda, P_S, P_D) = 0 && (14b) \\ & Q_{G_{i_{min}}} \le Q_{G_i} \le Q_{G_{i_{max}}} \quad \forall i \in \mathcal{G} && (14c) \\ & V_{G_{i_{min}}} \le V_{G_i} \le V_{G_{i_{max}}} \quad \forall i \in \mathcal{G} && (14d) \end{aligned}$$

where $\widehat{G}$ stand for the classical active and reactive power balanced equations for each generator and load bus (two for every system bus), and basically contains $\hat{g}$ and $\hat{h}$, as defined in (12); and $\mathcal{G}$ is the set of generation buses. It is important to highlight the fact that in this optimization model no other limits such as load bus voltage magnitude limits, generator active power limits, or power transfer limits, which are typical operating limits considered in such OPF models, are represented here. The reason for this is that these are "hard" limits and not actuation limits, i.e. limits that basically define "undesirable" operating conditions which may be associated with system protections rather than system controls, and hence do not lead to LIBs; these limits would only clutter the theoretical analyzes presented in the next section, without adding much to the discussions.

It has been shown that if no limits become active, the sufficient Karush-Kuhn Tucker (KKT) optimality conditions evaluated at the solution point of (14) are equivalent to the transversality conditions (3) and (4) for SNBs [10]; however, it has not yet been formally shown that the third transversality condition (5) is also met, which is an issue addressed here. It can also be argued that this model may provide a different maximum loading point different from that obtained using the CPF technique if reactive power limits become active [17]. This is due to the fact the objective of the optimization model (14) is to "optimize" the generator voltage and reactive power levels so that the loading factor is maximized, and hence there is no guarantee that the voltage at generation buses would be maintained constant while the reactive power output at such buses is within its limits, which is the typical representation of the generator voltage regulation controls in the power flow models used in CPF techniques.

B. OPF-DM with Complementarity Constraints

An optimization model that has been empirically shown to yield the same SNB or LISB points as a CPF technique has been proposed in [17]. The authors in this paper propose an optimization model that is based upon the idea that many problems encountered in engineering, physics or economics, which behave according to different rules under different circumstances, can be modeled using complementarity constraints, since these constraints can be used to model a change in system behavior. Thus, the change from a PV to a PQ bus, when a generation reactive power limit is reached, can be modeled using these type of constraints in the OPF problem as follows [29]:

$$\begin{aligned} 0 \le (Q_{G_k} - Q_{G_{k_{min}}}) \perp V_{a_k} \ge 0 \\ \Rightarrow\ (Q_{G_k} - Q_{G_{k_{min}}}) V_{a_k} = 0 \end{aligned}$$

$$\begin{aligned} 0 \le (Q_{G_{k_{max}}} - Q_{G_k}) \perp V_{b_k} \ge 0 \\ \Rightarrow\ (Q_{G_k} - Q_{G_{k_{max}}}) V_{b_k} = 0 \end{aligned}$$

where V_a and V_b are auxiliary, nonnegative variables that allow increasing or decreasing the generator voltage set point, depending on the state of Q_G. Thus:

$$\begin{aligned} \text{if } Q_{G_k} = Q_{G_{k_{min}}} &\Rightarrow V_{a_k} \ge 0 \text{ and } V_{b_k} = 0 \\ \text{if } Q_{G_{k_{min}}} < Q_{G_k} < Q_{G_{k_{max}}} &\Rightarrow V_{a_k} = V_{b_k} = 0 \\ \text{if } Q_{G_k} = Q_{G_{k_{max}}} &\Rightarrow V_{a_k} = 0 \text{ and } V_{b_k} \ge 0 \end{aligned}$$

This yields the following Mixed Complementarity Problem (MCP) [17]:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & \widehat{G}(\delta, V_L, K_G, Q_G, V_G, \lambda, P_S, P_D) = 0 && (15a) \\ & (Q_{G_k} - Q_{G_{k_{min}}}) V_{a_k} = 0 \quad \forall k \in \mathcal{G} && (15b) \\ & (Q_{G_k} - Q_{G_{k_{max}}}) V_{b_k} = 0 \quad \forall k \in \mathcal{G} && (15c) \\ & V_{G_k} = V_{G_{k_o}} + V_{a_k} - V_{b_k} \quad \forall k \in \mathcal{G} && (15d) \\ & Q_{G_{k_{min}}} \le Q_{G_k} \le Q_{G_{k_{max}}} \quad \forall k \in \mathcal{G} && (15e) \\ & V_{a_k}, V_{b_k} \ge 0 \quad \forall k \in \mathcal{G} && (15f) \end{aligned}$$

where V_{G_o} is the generator voltage regulator set point, i.e. the generator terminal voltage level if Q_G is within limits; and the constraints (15b)-(15d), associated with the auxiliary variables V_a and V_b, are used to model the actuation limits associated with the generator voltage regulators. Hence, in this model, $\tilde{z} = (\delta, V_L, K_G, V_G)$, $\hat{r} = (Q_G, V_a, V_b)$, $\hat{p} = (P_S, P_D, V_{G_o})$, and $\hat{g}$ and $\hat{h}$ are contained within constraints (15a)-(15d); the actual representation of these two vector functions is discussed in detail in Section IV. Observe that generator bus voltage limits are not included in this model, since these limits would basically correspond to "hard" operating limits, such as load bus voltage limits, for this particular model, i.e. these limits cannot be associated with actuation limits that result from the modeling of system controls, and lead to LIBs; hence, these limits are not considered in the formulation to simplify the theoretical analysis without loss of generality.

IV. Theoretical Analysis of the OPF-DM

In this section, it is formally shown that a solution to the OPF-DM model (15) corresponds to either an SNB or a LISB, by demonstrating that the transversality conditions of the corresponding bifurcations are met, based on the necessary and sufficient optimality conditions of the optimal solution. Only LIBs associated with maximum reactive power limits are analyzed here, since VS problems in practice are typically associated with generators reaching these limits as the demand in the system increases.

The following theorem shows that an optimal solution of (15), at which a given generator is at its reactive power limit while its terminal voltage is at its regulator set point, corresponds to a LISB and cannot be a LIDB, which is something one can intuitively deduce from Fig. 1(d), if the following assumptions are met [30]:

- Regularity and strict complementarity conditions must be met at the optimal point, i.e. there must not be degeneracy of the optimization problem at the solution point.
- The solution point must be in a convex region, with the constraints being C^2 and convex at this point.

These assumptions are referred throughout the rest of the paper as optimality solution (OS) assumptions for convenience. It is important to highlight the fact that there is no guarantee that all possible solutions of (15) would meet these OS assumptions. However, from numerical results reported in [11] and [17], where these types of optimization problems are solved for a variety of small and large systems, these solutions are shown to be robust, thus basically meeting these conditions in practice.

Theorem 1: Let $(\hat{z}_c, \lambda_c)$, $\hat{z}_c = (\tilde{z}_c, \hat{r}_c)$, be a local optimum of (15) that meets the aforementioned OS assumptions for $\hat{p} = \hat{p}_o$, where a given generator i satisfies:

$$\left.\begin{array}{c} Q_{G_{i_c}} = Q_{G_{i_{max}}} \\ V_{G_{i_c}} = V_{G_{i_o}} \end{array}\right\} \;\Rightarrow\; V_{a_i} = V_{b_i} = 0 \tag{16}$$

while some other generators $j \neq i \in \mathcal{G}_j \subset \mathcal{G}$ satisfy:

$$\left.\begin{array}{c} Q_{G_{j_c}} = Q_{G_{j_{max}}} \\ V_{G_{j_c}} < V_{G_{j_o}} \end{array}\right\} \;\Rightarrow\; \left\{\begin{array}{l} V_{a_j} = 0 \\ V_{b_j} > 0 \end{array}\right. \tag{17}$$

and the rest of the generators $\bar{\jmath} \neq j \neq i \in \mathcal{G}_{\bar{\jmath}} \subset \mathcal{G}$ are not at their reactive power limits, i.e.

$$\left.\begin{array}{c} Q_{G_{\bar{\jmath}_{min}}} < Q_{G_{\bar{\jmath}_c}} < Q_{G_{\bar{\jmath}_{max}}} \\ V_{G_{\bar{\jmath}_c}} = V_{G_{\bar{\jmath}_o}} \end{array}\right\} \;\Rightarrow\; V_{a_{\bar{\jmath}}} = V_{b_{\bar{\jmath}}} = 0 \tag{18}$$

(Assumptions (17) generalizes the case where a LISB occurs after a LIDB in λ space, as depicted in Fig. 1(d).) Then, $(\hat{z}_c, \lambda_c, \hat{p}_o)$ is a LISB of the power flow model defined by equations (15a)-(15d).

The formal proof to Theorem 1 can be found in Appendix I. This theorem basically proves that a given local optimum of (15) can be a LISB and not a LIDB, and that it can be preceded by some generators reaching reactive power limits, i.e. LIDBs. The following theorem shows that this local optimum can also be an SNB.

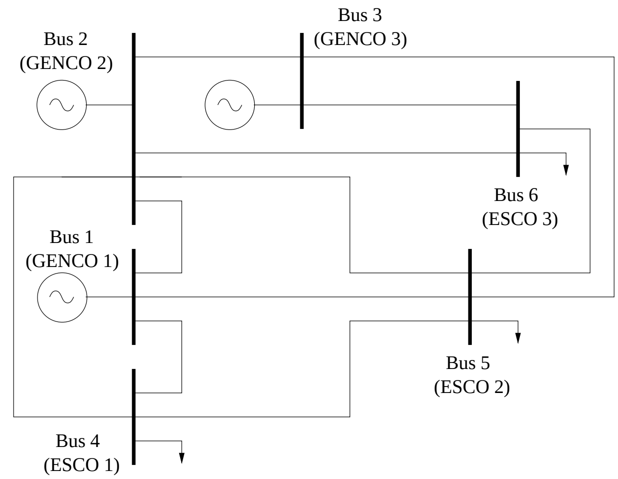

Fig. 2. Six-bus test system.

Theorem 2: Let $(\hat{z}_c, \lambda_c)$ be local optimum of (15) that meets the abovementioned OS assumptions for $\hat{p} = \hat{p}_o$, where some generators $j \in \mathcal{G}_j \subset \mathcal{G}$ satisfy:

$$\left.\begin{array}{c} Q_{G_{j_c}} = Q_{G_{j_{max}}} \\ V_{G_{j_c}} < V_{G_{j_o}} \end{array}\right\} \;\Rightarrow\; \left\{\begin{array}{l} V_{a_j} = 0 \\ V_{b_j} > 0 \end{array}\right. \tag{19}$$

while the rest of the generators $\bar{\jmath} \neq j \in \mathcal{G}_{\bar{\jmath}} \subset \mathcal{G}$, $\mathcal{G} = \mathcal{G}_{\bar{\jmath}} \cup \mathcal{G}_j$, are not at their reactive power limits, i.e.

$$\left.\begin{array}{c} Q_{G_{\bar{\jmath}_{min}}} < Q_{G_{\bar{\jmath}_c}} < Q_{G_{\bar{\jmath}_{max}}} \\ V_{G_{\bar{\jmath}_c}} = V_{G_{\bar{\jmath}_o}} \end{array}\right\} \;\Rightarrow\; V_{a_{\bar{\jmath}}} = V_{b_{\bar{\jmath}}} = 0 \tag{20}$$

(Assumptions (19) and (20) generalize the case where an SNB occurs after a LIDB in λ space, as depicted in Fig. 1(b).) Then, $(\hat{z}_c, \lambda_c, \hat{p}_o)$ is an SNB of the power flow model defined by equations (15a)-(15d).

The formal proof to Theorem 2 can found in Appendix II. Finally, the following corollary argues that an optimum of (15) can only be a LISB or an SNB. The proof to this corollary can be found in Appendix III.

Corollary 1: Any solution point $(\hat{z}_c, \lambda_c, \hat{p}_o)$ of (15) that meets the aforementioned OS assumptions is either a LISB or an SNB.

V. Numerical Examples

This section presents a numerical comparison between the OPF-DM and the CPF method to illustrate some of the theoretical issues discussed in the previous section. Thus, the maximum loading factor, voltage and reactive power levels obtained from solving (15) are compared with those obtained using the a standard CPF, for a variety of test cases for the 6-bus system shown in Fig. 2 [13], where the generators' voltage set points and reactive power limits are assumed to be $V_{G_o} = 1.05$ p.u. and $Q_G = \pm 1.5$ p.u., respectively.

A. Practical Implementation Issues

The OPF-DM with complementarity constraints can be implemented in AMPL, using the `complements` operator [31], [32], which allows complementarity conditions to be directly

TABLE I
OPF-DM VS CPF FOR THE 6-BUS TEST SYSTEM

	LISB		SNB (Q limits)		SNB (no Q limits)	
	OPF-DM	CPF	OPF-DM	CPF	OPF-DM	CPF
V_{G_1}	1.0500	1.0500	0.9648	0.9657	1.0500	1.0500
V_{G_2}	1.0025	1.0026	1.0500	1.0500	1.0500	1.0500
V_{G_3}	1.0029	1.0029	1.0500	1.0500	1.0500	1.0500
V_{L_4}	0.8458	0.8458	0.6027	0.6048	0.5360	0.5360
V_{L_5}	0.8546	0.8545	0.8586	0.8591	0.7129	0.7125
V_{L_6}	0.8687	0.8686	0.9465	0.9466	0.7679	0.7677
Q_{G_1}	1.5	1.5	1.5	1.5	3.1588	3.1600
Q_{G_2}	1.5	1.5	0.9577	0.9511	6.2724	6.2734
Q_{G_3}	1.5	1.5	1.4712	1.4682	3.5828	3.5856
λ_c	4.4966	4.5049	1.9046	1.9081	11.1141	11.1330

specified in the constraint declarations, and then solved using solvers specifically designed for complementarity problems such as KNITRO [33]. Alternatively, the complementarity constraints can be specified as nonsmooth constraints as in (15), solving the optimization problem with nonlinear programming solvers such as LOQO, KNITRO or IPOPT; this is the approach used here to obtain the numerical results discussed in this section. On the other hand, UWPFLOW [34], which is a popular and well-tested software tool with a robust implementation of a CPF technique, was used to obtain PV curves for illustrative and comparison purposes. For both techniques, the generation and load variations were assumed to be defined by (13).

It is important to highlight the fact that the initial operating point is rather important, since it is used to define the generator voltage set points for the optimization problem, as well as the starting point for the CPF, and it must be obtained by running an initial power flow simulation. The auxiliary variables used in the definition of the complementarity constraints must be initialized to zero.

B. Numerical Results

The PV curves in Fig. 3 present three bifurcation profiles under different operating conditions: Fig. 3(a) shows a LISB at $\lambda_c = 4.5049$ p.u., preceded by LIDBs, for the base system topology; Fig. 3(b) shows an SNB at $\lambda_c = 1.9081$ p.u., preceded by LIDBs, when line 2-4 is removed from the system; and Fig. 3(c) shows another SNB at a $\lambda_c = 11.1330$ p.u. when Q-limits are ignored for the base system. Observe in these plots that the bifurcations in the first two cases are preceded by some LIDBs in λ space; also, in the last case, the SNB occurs at a larger loading factor, with the voltages at generator buses remaining constant. Notice as well the sharp “edge” of the bifurcation manifold at the maximum loading point defined by λ_c, which is a characteristic of LISBs, and the “quadratic” shape of the manifolds around the SNBs, which is also typical.

Table I presents a comparison of the solutions obtained using the optimization model (15) as well as the equivalent results obtained from the CPF, depicted in Fig. 3. The results

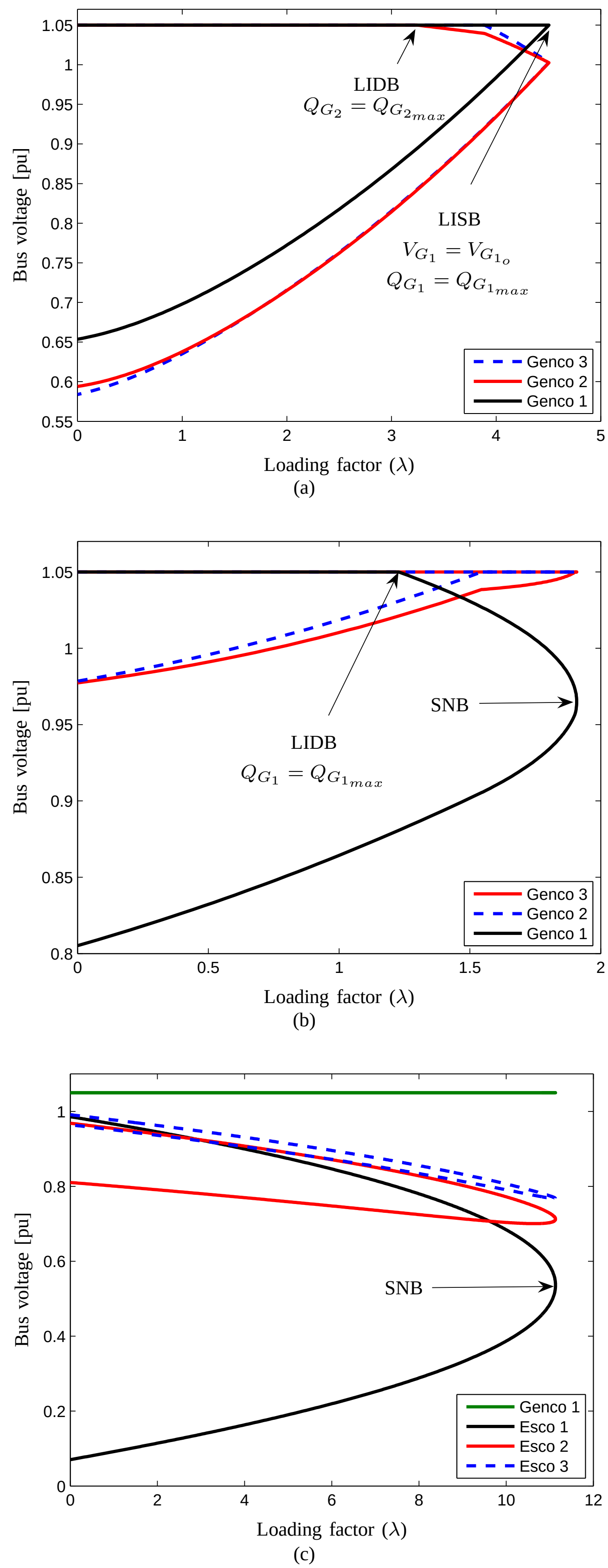

Fig. 3. Generators' PV curves for the 6-bus test system: (a) Base case (LISB preceded by LIDBs); (b) line 2-4 outage (SNB preceded by LIDBs); (c) base system neglecting reactive power limits (SNB).

presented in the first and second columns correspond to the base case, and show that GENCO 1 satisfies the LISB condition $Q_{G_{1_c}} = Q_{G_{1_{max}}}$ and $V_{G_{1_c}} = V_{G_{1_o}}$ at λ_c, while GENCO 2 and GENCO 3 are at their reactive power limits with their voltages below the corresponding set points, i.e. the system has undergone 2 LIDBs before reaching a LISB in λ space, as clearly illustrated in Fig. 3(a). The results in the third and fourth columns, obtained by removing line 2-4, show GENCO 2 and GENCO 3 within their reactive power limits and at their corresponding voltage set points, whereas GENCO 1 has reached its maximum reactive power limit and its voltage is below its set point, indicating the occurrence of a LIDB before the SNB in λ space, as depicted in Fig. 3(b). Finally, the results presented in the last two columns, which correspond to the base system without generator reactive power limits, show all generators at their voltage set points as well as large reactive power outputs, i.e. there are no LIDBs before the SNB in λ space. This table shows that both techniques basically give the same solution; the small differences can be basically attributed to numerical approximations, particularly in the case of the CPF. The execution time for the OPF-DM was in the range of 0.12 sec, which was faster than the CPF; the reader is referred to [17] for more numerical comparisons in larger systems.

The sequence of generators reaching the maximum reactive power limit can be also obtained from the OPF-DM, by ranking in descending order the difference $\Delta V_{G_{i_c}} = V_{G_{i_o}} - V_{G_{i_c}}$. Thus, the largest difference corresponds to the first generator reaching its maximum reactive power limit, and so on. If the difference is negative, then the generator would have reached the minimum reactive power limit. For example, the ranked differences for the base case are: $\Delta V_{G_{2_c}} = 0.0475$, $\Delta V_{G_{3_c}} = 0.0471$ and $\Delta V_{G_{1_c}} = 0$, which agrees with what is observed in Fig. 3(a).

A test was carried out to study the effect of setting the upper voltage limits at generation buses at their corresponding set points for the model without complementarity constraints (14), with the objective that by defining $V_{G_{max}} = V_{G_o}$, the optimization solution process would "fix" the generator voltages at their initial maximum voltage levels, thus yielding similar results as those obtained solving (15), since voltages at generation buses, if not fixed, typically increase when increasing the load. It is interesting to notice that this approach generated the same results as those depicted in Table I. However, this was not the case for other test systems, since (14) does not necessarily guarantees that generators are going to be at their maximum voltage values (voltage set points in this case), if their reactive power limits have not been reached, which is a condition in (15).

VI. Conclusions

This paper has presented a detailed, theoretical study of an optimization method able to determine two types of fold bifurcations directly associated with voltage instabilities in power systems. It was demonstrated that the necessary and sufficient optimality conditions yield the transversality conditions for SNBs and LISBs, thus showing that the solution of the studied optimization problem yields the same results as those obtained with the more popular CPF techniques used to analyze these types of bifurcations in power systems.

The advantages of stating the SNB/LIB problem as an optimization problem is that optimization solution techniques can be computational more effective than CPF methods for maximum loadability studies, particularly when using well-tested and efficient solution techniques such as Interior Point Methods. Furthermore, optimization approaches are more versatile than CPF techniques, since the problem can be readily restated so that optimal control parameter values can be calculated to increase the maximum loadability margins of a system, or readily carry out a variety of sensitivity studies, as demonstrated on the second part of this paper.

Appendix I
Proof of Theorem 1

Proof: Let $Q_G = (Q_{\overline{G}}, Q_{G_i})$, i.e. the generator reactive power variables are ordered so that generator i is the last variable; similarly for V_G, V_a and V_b. Hence, the Lagrangian function of (15) may then be expressed as:

$$\begin{aligned}\mathcal{L} = \lambda &- \hat{\mu}_1^T \widehat{G}_S(\hat{z}_c, \lambda_c, \hat{p}_o) - \hat{\mu}_2^T \widehat{G}_{Q_{\overline{G}}}(\hat{z}_c, \lambda_c, \hat{p}_o) \\ &-\hat{\mu}_3 \widehat{G}_{Q_{G_i}}(\hat{z}_c, \lambda_c, \hat{p}_o) - \hat{\mu}_4^T (Q_{\overline{G}} - Q_{\overline{G}_{min}}) V_{\bar{a}} \\ &-\hat{\mu}_5^T (Q_{\overline{G}} - Q_{\overline{G}_{max}}) V_{\bar{b}} - \hat{\mu}_6 (Q_{G_i} - Q_{G_{i_{min}}}) V_{a_i} \\ &-\hat{\mu}_7 (Q_{G_i} - Q_{G_{i_{max}}}) V_{b_i} - \hat{\mu}_8^T \left(Q_{\overline{G}_{min}} - Q_{\overline{G}}\right) \\ &-\hat{\mu}_9^T \left(Q_{\overline{G}} - Q_{\overline{G}_{max}}\right) - \hat{\mu}_{10} \left(Q_{G_{i_{min}}} - Q_{G_i}\right) \\ &-\hat{\mu}_{11} \left(Q_{G_i} - Q_{G_{i_{max}}}\right) - \hat{\mu}_{12}^T \left(V_{\overline{G}} - V_{\overline{G}_o} - V_{\bar{a}} + V_{\bar{b}}\right) \\ &-\hat{\mu}_{13} \left(V_{G_i} - V_{G_{i_o}} - V_{a_i} + V_{b_i}\right) - \hat{\mu}_{14}^T (-V_{\bar{a}}) \\ &-\hat{\mu}_{15}^T (-V_{\bar{b}}) - \hat{\mu}_{16} (-V_{a_i}) - \hat{\mu}_{17} (-V_{b_i})\end{aligned}$$

where the functions $\widehat{G}_S$, $\widehat{G}_{Q_{\overline{G}}}$ and $\widehat{G}_{Q_{G_i}}$ are appropriately defined subsets of $\widehat{G}$; and the $\hat{\mu}$s correspond to the Lagrange multipliers of (15).

The KKT optimality conditions state that the gradient of the Lagrangian function must be equal to zero at the optimum [30]. Thus:

$$\begin{aligned}\nabla_\delta \mathcal{L}|_c &= -\nabla_\delta \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_\delta \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\ &\quad -\nabla_\delta \widehat{G}_{Q_{G_i}}|_c \hat{\mu}_{3_c} = 0 \qquad (21)\end{aligned}$$

$$\begin{aligned}\nabla_{V_L} \mathcal{L}|_c &= -\nabla_{V_L} \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_{V_L} \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\ &\quad -\nabla_{V_L} \widehat{G}_{Q_{G_i}}|_c \hat{\mu}_{3_c} = 0 \qquad (22)\end{aligned}$$

$$\nabla_{K_G} \mathcal{L}|_c = -\nabla_{K_G} \widehat{G}_S|_c \hat{\mu}_{1_c} = 0 \qquad (23)$$

$$\nabla_{Q_{\overline{G}}} \mathcal{L}|_c = -\hat{\mu}_{2_c} - M_{\bar{a}_c} \hat{\mu}_{4_c} - M_{\bar{b}_c} \hat{\mu}_{5_c} + \hat{\mu}_{8_c} - \hat{\mu}_{9_c} = 0 \qquad (24)$$

$$\nabla_{Q_{G_i}} \mathcal{L}|_c = -\hat{\mu}_{3_c} - V_{a_{i_c}} \hat{\mu}_{6_c} - V_{b_{i_c}} \hat{\mu}_{7_c} + \hat{\mu}_{10_c} - \hat{\mu}_{11_c} = 0 \qquad (25)$$

$$\begin{aligned}\nabla_{V_{\overline{G}}} \mathcal{L}|_c &= -\nabla_{V_{\overline{G}}} \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_{V_{\overline{G}}} \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\ &\quad -\nabla_{V_{\overline{G}}} \widehat{G}_{Q_{G_i}}|_c \hat{\mu}_{3_c} - \hat{\mu}_{12_c} = 0 \qquad (26)\end{aligned}$$

$$\begin{aligned}\nabla_{V_{G_i}} \mathcal{L}|_c &= -\nabla_{V_{G_i}} \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_{V_{G_i}} \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\ &\quad -\nabla_{V_{G_i}} \widehat{G}_{Q_{G_i}}|_c \hat{\mu}_{3_c} - \hat{\mu}_{13_c} = 0 \qquad (27)\end{aligned}$$

$$\begin{aligned}\nabla_\lambda \mathcal{L}|_c &= -\nabla_\lambda \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_\lambda \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\ &\quad -\nabla_\lambda \widehat{G}_{Q_{G_i}}|_c \hat{\mu}_{3_c} + 1 = 0 \qquad (28)\end{aligned}$$

$$\nabla_{V_{\bar{a}}} \mathcal{L}|_c = -M_{Q_{\overline{G}_{min_c}}} \hat{\mu}_{4_c} + \hat{\mu}_{12_c} + \hat{\mu}_{14_c} = 0 \qquad (29)$$

$$\nabla_{V_{\bar{b}}}\mathcal{L}|_c = -M_{Q_{\overline{G}_{max_c}}}\hat{\mu}_{5_c} - \hat{\mu}_{12_c} + \hat{\mu}_{15_c} = 0 \quad (30)$$
$$\nabla_{V_{a_i}}\mathcal{L}|_c = -(Q_{G_{i_c}} - Q_{G_{i_{min}}})\hat{\mu}_{6_c} + \hat{\mu}_{13_c} + \hat{\mu}_{16_c} = 0 \quad (31)$$
$$\nabla_{V_{b_i}}\mathcal{L}|_c = -(Q_{G_{i_c}} - Q_{G_{i_{max}}})\hat{\mu}_{7_c} - \hat{\mu}_{13_c} + \hat{\mu}_{17_c} = 0 \quad (32)$$

where $M_{\bar{a}_c} = \text{diag}(V_{\bar{a}_c})$, $M_{\bar{b}_c} = \text{diag}(V_{\bar{b}_c})$, $M_{Q_{\overline{G}_{min_c}}} = \text{diag}(Q_{\overline{G}_c} - Q_{\overline{G}_{min}})$, and $M_{Q_{\overline{G}_{max_c}}} = \text{diag}(Q_{\overline{G}_c} - Q_{\overline{G}_{max}})$ are diagonal matrices. Also, the equality constraints must be equal to zero and the inequality constraints are less than or equal to zero at the optimum, i.e. this point must be feasible.

The complementarity slackness condition provides an indication of whether an inequality constraint is active or not. Hence, based on the regularity and strict complementarity OS assumptions, which imply that $\mu_c = (\mu_{1_c}, \ldots, \mu_{17_c}) \neq 0$ is unique, and $\mu_{l_c} > 0 \ \forall l \in \{\text{Active Constraint Set}\}$ [30], it follows from (16)-(18) that:

$$\hat{\mu}_{8_{k_c}}(Q_{G_{k_{min}}} - Q_{G_{k_c}}) = 0 \Rightarrow \hat{\mu}_{8_{k_c}} = 0 \ \forall k \in \bar{\mathcal{G}} \quad (33)$$
$$\hat{\mu}_{9_{\bar{\jmath}_c}}(Q_{G_{\bar{\jmath}_c}} - Q_{G_{\bar{\jmath}_{max}}}) = 0 \Rightarrow \hat{\mu}_{9_{\bar{\jmath}_c}} = 0 \ \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}} \quad (34)$$
$$\hat{\mu}_{9_{j_c}}(Q_{G_{j_c}} - Q_{G_{j_{max}}}) = 0 \Rightarrow \hat{\mu}_{9_{j_c}} > 0 \ \forall j \in \mathcal{G}_j \quad (35)$$
$$\hat{\mu}_{10_c}(Q_{G_{i_{min}}} - Q_{G_{i_c}}) = 0 \Rightarrow \hat{\mu}_{10_c} = 0 \quad (36)$$
$$\hat{\mu}_{11_c}(Q_{G_{i_c}} - Q_{G_{i_{max}}}) = 0 \Rightarrow \hat{\mu}_{11_c} > 0 \quad (37)$$
$$\hat{\mu}_{14_{k_c}}(-V_{a_{k_c}}) = 0 \Rightarrow \hat{\mu}_{14_{k_c}} > 0 \ \forall k \in \bar{\mathcal{G}} \quad (38)$$
$$\hat{\mu}_{15_{\bar{\jmath}_c}}(-V_{b_{\bar{\jmath}_c}}) = 0 \Rightarrow \hat{\mu}_{15_{\bar{\jmath}_c}} > 0 \ \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}} \quad (39)$$
$$\hat{\mu}_{15_{j_c}}(-V_{b_{j_c}}) = 0 \Rightarrow \hat{\mu}_{15_{j_c}} = 0 \ \forall j \in \mathcal{G}_j \quad (40)$$
$$\hat{\mu}_{16_c}(-V_{a_{i_c}}) = 0 \Rightarrow \hat{\mu}_{16_c} > 0 \quad (41)$$
$$\hat{\mu}_{17_c}(-V_{b_{i_c}}) = 0 \Rightarrow \hat{\mu}_{17_c} > 0 \quad (42)$$

where $\bar{\mathcal{G}} = \mathcal{G}_{\bar{\jmath}} \cup \mathcal{G}_j$.

Now, based on (16)-(18), the following actuation regime and saturation regime equations, evaluated at the solution point $(\hat{z}_c, \lambda_c, \hat{p}_o)$, are the minimum subsets of constraints (15a)-(15f) that uniquely define $\hat{z}_c$ for a given $(\lambda_c, \hat{p}_o)$, since the number of equations and unknowns is the same, i.e. $N = 2n_b + n_{\mathcal{G}}$, where n_b is the number of system buses and $n_{\mathcal{G}}$ is the number of generators:

$$G_a|_c = \begin{bmatrix} \widehat{G}(\delta_c, V_{L_c}, K_{G_c}, Q_{G_c}, V_{G_c}, \lambda_c, P_{S_o}, P_{D_o}) \\ V_{G_{\bar{\jmath}_c}} - V_{G_{\bar{\jmath}_o}} \quad \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}} \\ Q_{G_{j_c}} - Q_{G_{j_{max}}} \quad \forall j \in \mathcal{G}_j \\ V_{G_{i_c}} - V_{G_{i_o}} \end{bmatrix} = 0 \quad (43)$$

$$G_b|_c = \begin{bmatrix} \widehat{G}(\delta_c, V_{L_c}, K_{G_c}, Q_{G_c}, V_{G_c}, \lambda_c, P_{S_o}, P_{D_o}) \\ V_{G_{\bar{\jmath}_c}} - V_{G_{\bar{\jmath}_o}} \quad \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}} \\ Q_{G_{j_c}} - Q_{G_{j_{max}}} \quad \forall j \in \mathcal{G}_j \\ Q_{G_{i_c}} - Q_{G_{i_{max}}} \end{bmatrix} = 0 \quad (44)$$

Notice that these equations have a similar form as (8) and (9), respectively, where $\tilde{z}_c = (\delta_c, V_{L_c}, K_{G_c}, V_{G_c})$, $\hat{r}_c = Q_{G_c}$, $\hat{p}_o = (P_{S_o}, P_{D_o}, V_{G_o})$, $\hat{g}|_c = \widehat{G}|_c$, and

$$\hat{r}_c - s|_c \equiv \begin{bmatrix} V_{G_{\bar{\jmath}_c}} - V_{G_{\bar{\jmath}_o}} \quad \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}} \\ Q_{G_{j_c}} - Q_{G_{j_{max}}} \quad \forall j \in \mathcal{G}_j \\ V_{G_{i_c}} - V_{G_{i_o}} \end{bmatrix}$$

Observe that in this case, some of the actuation limit functions are implicit instead of explicit functions of the corresponding variables $\hat{r}$. Hence, for the optimal solution to be a LISB, one first must prove that the Jacobians J_a^i and J_b^i associated with (43) and (44) are nonsingular.

Let first prove that J_b^i is not singular. Hence, from (21)-(32) and with the proper ordering of variables and equations in (44), and assuming that $V_{\overline{G}} = (V_{G_{\bar{\jmath}}} \ \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}}, V_{G_j} \ \forall j \in \mathcal{G}_j)$, and similarly for $Q_{\overline{G}}$, it can be shown that:

$$J_b^{i^T} \hat{x}_b = \hat{b}_b \quad (45)$$

where

$$J_b^{i^T} = \quad (46)$$
$$\left[\begin{array}{ccccc|c} \nabla_\delta \widehat{G}_S|_c & \nabla_\delta \widehat{G}_{Q_{\overline{G}}}|_c & \nabla_\delta \widehat{G}_{Q_{G_i}}|_c & 0 & 0 & 0 \\ \nabla_{V_L} \widehat{G}_S|_c & \nabla_{V_L} \widehat{G}_{Q_{\overline{G}}}|_c & \nabla_{V_L} \widehat{G}_{Q_{G_i}}|_c & 0 & 0 & 0 \\ \nabla_{K_G} \widehat{G}_S|_c & 0 & 0 & 0 & 0 & 0 \\ \nabla_{V_{\overline{G}}} \widehat{G}_S|_c & \nabla_{V_{\overline{G}}} \widehat{G}_{Q_{\overline{G}}}|_c & \nabla_{V_{\overline{G}}} \widehat{G}_{Q_{G_i}}|_c & U & 0 & 0 \\ 0 & I_{n_{\bar{\mathcal{G}}}} & 0 & 0 & W & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \hline \nabla_{V_{G_i}} \widehat{G}_S|_c & \nabla_{V_{G_i}} \widehat{G}_{Q_{\overline{G}}}|_c & \nabla_{V_{G_i}} \widehat{G}_{Q_{G_i}}|_c & 0 & 0 & 0 \end{array}\right]$$
$$= \left[\begin{array}{c|c} A^T & e \\ \hline c^T & 0 \end{array}\right]$$

$$\hat{x}_b = \left[\begin{array}{c} \hat{\mu}_{1_c} \\ \hat{\mu}_{2_c} \\ \hat{\mu}_{3_c} \\ \hat{\mu}_{12_{\mathcal{G}_{\bar{\jmath}c}}} \\ \hat{\mu}_{9_{\mathcal{G}_{jc}}} \\ \hline \hat{\mu}_{11_c} \end{array}\right]$$

$$\hat{b}_b = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -W\hat{\mu}_{12_{\mathcal{G}_{jc}}} \\ -M_{\bar{a}_c}\hat{\mu}_{4_c} - M_{\bar{b}_c}\hat{\mu}_{5_c} + \hat{\mu}_{8_c} - U\hat{\mu}_{9_{\mathcal{G}_{\bar{\jmath}c}}} \\ -V_{a_{i_c}}\hat{\mu}_{6_c} - V_{b_{i_c}}\hat{\mu}_{7_c} + \hat{\mu}_{10_c} \\ \hline \hat{\mu}_{13_c} \end{array}\right]$$

and $\hat{\mu}_9 = (\hat{\mu}_{9_{\mathcal{G}_{\bar{\jmath}}}}, \hat{\mu}_{9_{\mathcal{G}_j}})$, $\hat{\mu}_{12} = (\hat{\mu}_{12_{\mathcal{G}_{\bar{\jmath}}}}, \hat{\mu}_{12_{\mathcal{G}_j}})$,

$$U = \begin{bmatrix} I_{n_{\mathcal{G}_{\bar{\jmath}}}} \\ 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 \\ I_{n_{\mathcal{G}_j}} \end{bmatrix}$$

where I_n is an $n \times n$ identity matrix.

Now, from (25), (36) and (37):

$$\hat{\mu}_{3_c} = -\hat{\mu}_{11_c} \neq 0 \quad (47)$$

From (35)

$$\hat{\mu}_{9_{\mathcal{G}_{jc}}} \neq 0 \quad (48)$$

And from (32) and (42)

$$\hat{\mu}_{13_c} = \hat{\mu}_{17_c} \neq 0 \quad (49)$$

Hence, from (47)-(49), it follows that:

$$\hat{x}_b \neq 0 \quad \text{and} \quad \hat{b}_b \neq 0$$

and are both unique. Therefore, one can conclude from (45) that J_b^i is nonsingular, i.e.

$$\det(J_b^i) \neq 0 \quad (50)$$

With similar arguments, it can be readily shown that:

$$J_a^{i^T} \hat{x}_a = \hat{b}_a \tag{51}$$

where

$$J_a^{i^T} = \left[\begin{array}{c|c} A^T & 0 \\ \hline c^T & 1 \end{array}\right] \tag{52}$$

$$\hat{x}_a = \left[\begin{array}{c} \hat{\mu}_{1_c} \\ \hat{\mu}_{2_c} \\ \hat{\mu}_{3_c} \\ \hat{\mu}_{12_{\mathcal{G}_{\bar{j}c}}} \\ \hat{\mu}_{9_{\mathcal{G}_{jc}}} \\ \hline \hat{\mu}_{13_c} \end{array}\right]$$

$$\hat{b}_a = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -W\hat{\mu}_{12_{\mathcal{G}_{jc}}} \\ -M_{\bar{a}_c}\hat{\mu}_{4_c} - M_{\bar{b}_c}\hat{\mu}_{5_c} + \hat{\mu}_{8_c} - U\hat{\mu}_{9_{\mathcal{G}_{\bar{j}c}}} \\ -V_{a_{i_c}}\hat{\mu}_{6_c} - V_{b_{i_c}}\hat{\mu}_{7_c} + \hat{\mu}_{10_c} - \hat{\mu}_{11_c} \\ \hline 0 \end{array}\right]$$

Therefore, from (47)-(49), it follows that:

$$\hat{x}_a \neq 0 \quad \text{and} \quad \hat{b}_a \neq 0$$

and are both unique, yielding from (51) a nonsingular J_a^i, i.e.

$$\det(J_a^i) \neq 0 \tag{53}$$

Thus, from (50) and (53), it is clear that the solution point $(\hat{z}_c, \lambda_c, \hat{p}_o)$ meets transversality conditions (10).

The second transversality condition (11) simply states that the ratio of the determinants of J_a^i and J_b^i must be positive for $(\hat{z}_c, \lambda_c, \hat{p}_o)$ to be a LISB. Thus, from (46) and (52), and based on Schur's Complements [35], it follows that:

$$\begin{aligned} \det(J_a^i) &= \det(A) \\ \det(J_b^i) &= -e^T A^{-1} c \det(A) \end{aligned}$$

Terefore:

$$\alpha = \frac{\det(J_a^i)}{\det(J_b^i)} = \frac{1}{-e^T A^{-1} c} \tag{54}$$

Now, from (43), it follows that:

$$\nabla_{\hat{z}}^T G_a|_c \, \mathrm{d}\hat{z} + \nabla_{V_{G_{i_o}}}^T G_a|_c \, \mathrm{d}V_{G_{i_o}} = 0$$

which from (52) can be rewritten as:

$$\left[\begin{array}{c|c} A & c \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} \mathrm{d}\bar{z} \\ \hline \mathrm{d}\bar{V}_{G_i} \end{array}\right] - \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] \mathrm{d}V_{G_{i_o}} = 0$$

where $\hat{z} = (\bar{z}, V_{G_i})$. This yields:

$$\begin{aligned} \mathrm{d}\bar{z} &= -A^{-1} c \, \mathrm{d}V_{G_i} &(55) \\ \mathrm{d}V_{G_i} &= \mathrm{d}V_{G_{i_o}} &(56) \end{aligned}$$

On the other hand, from (44) and (46), one has that:

$$\left[\begin{array}{c|c} A & c \\ \hline e^T & 0 \end{array}\right] \left[\begin{array}{c} \mathrm{d}\bar{z} \\ \hline \mathrm{d}V_{G_i} \end{array}\right] - \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] \mathrm{d}Q_{G_{i_{max}}} = 0$$

which yields (55) as well as:

$$\mathrm{d}Q_{G_i} = e^T \, \mathrm{d}\bar{z} = \mathrm{d}Q_{G_{i_{max}}} \tag{57}$$

Thus, from (55), (56) and (57), it follows that:

$$\left.\frac{\mathrm{d}Q_{G_{i_{max}}}}{\mathrm{d}V_{G_{i_o}}}\right|_c = -e^T A^{-1} c$$

which, from (54), leads to:

$$\alpha = \left.\frac{\mathrm{d}V_{G_{i_o}}}{\mathrm{d}Q_{G_{i_{max}}}}\right|_c \tag{58}$$

Now, from the optimization model (15), the sensitivities of the objective function with respect to $Q_{G_{i_{max}}}$ and and $V_{G_{i_o}}$ evaluated at the optimal point can be stated as [36]:

$$\begin{aligned} \hat{\mu}_{11_c} &= \left.\frac{d\lambda}{dQ_{G_{i_{max}}}}\right|_c \\ \hat{\mu}_{13_c} &= \left.\frac{d\lambda}{dV_{G_{i_o}}}\right|_c \end{aligned}$$

Hence, from (37), (49), and (58), it follows that:

$$\alpha = \frac{\hat{\mu}_{11_c}}{\hat{\mu}_{13_c}} > 0 \tag{59}$$

which satisfies the second transversality condition (11). Therefore, the optimal solution $(\hat{z}_c, \lambda_c, \hat{p}_o)$ which meets the given OS assumptions is a LISB.

Finally, observe that, at a LIDB, assumptions (16)-(18) are also met. However, (59) rules out the possibility of a LIDB being a solution of (15). ■

APPENDIX II
PROOF OF THEOREM 2

Proof: Following a similar approach to the proof of Theorem 1, let $Q_G = (Q_{\overline{G}}, Q_{\widetilde{G}})$, where $Q_{\overline{G}} = (Q_{G_{\bar{j}}} \ \forall \bar{j} \in \mathcal{G}_{\bar{j}}, Q_{G_j} \ \forall j \in \mathcal{G}_j)$, and similarly for V_G, V_a and V_b. Hence, the Lagrangian function of (15) may then be expressed as:

$$\begin{aligned} \mathcal{L} = \lambda &- \hat{\mu}_1^T \widehat{G}_S(\hat{z}_c, \lambda_c, \hat{p}_o) - \hat{\mu}_2^T \widehat{G}_{Q_{\overline{G}}}(\hat{z}_c, \lambda_c, \hat{p}_o) \\ &- \hat{\mu}_3 \widehat{G}_{Q_{\widetilde{G}}}(\hat{z}_c, \lambda_c, \hat{p}_o) - \hat{\mu}_4^T (Q_{\overline{G}} - Q_{\overline{G}_{min}}) V_{\bar{a}} \\ &- \hat{\mu}_5^T (Q_{\overline{G}} - Q_{\overline{G}_{max}}) V_{\bar{b}} - \hat{\mu}_6^T (Q_{\widetilde{G}} - Q_{\widetilde{G}_{min}}) V_{\tilde{a}} \\ &- \hat{\mu}_7^T (Q_{\widetilde{G}} - Q_{\widetilde{G}_{max}}) V_{\tilde{b}} - \hat{\mu}_8^T \left(Q_{\overline{G}_{min}} - Q_{\overline{G}}\right) \\ &- \hat{\mu}_9^T \left(Q_{\overline{G}} - Q_{\overline{G}_{max}}\right) - \hat{\mu}_{10}^T \left(Q_{\widetilde{G}_{min}} - Q_{\widetilde{G}}\right) \\ &- \hat{\mu}_{11}^T \left(Q_{\widetilde{G}} - Q_{\widetilde{G}_{max}}\right) - \hat{\mu}_{12}^T \left(V_{\overline{G}} - V_{\overline{G}_o} - V_{\bar{a}} + V_{\bar{b}}\right) \\ &- \hat{\mu}_{13} \left(V_{\widetilde{G}} - V_{\widetilde{G}_o} - V_{\tilde{a}} + V_{\tilde{b}}\right) - \hat{\mu}_{14}^T (-V_{\bar{a}}) \\ &- \hat{\mu}_{15}^T (-V_{\bar{b}}) - \hat{\mu}_{16}^T (-V_{\tilde{a}}) - \hat{\mu}_{17}^T (-V_{\tilde{b}}) \end{aligned}$$

From the KKT optimality conditions, it follows that:

$$\begin{aligned}
\nabla_\delta \mathcal{L}|_c &= -\nabla_\delta \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_\delta \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\
&\quad -\nabla_\delta \widehat{G}_{Q_{\widetilde{G}}}|_c \hat{\mu}_{3_c} = 0 && (60)\\
\nabla_{V_L} \mathcal{L}|_c &= -\nabla_{V_L} \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_{V_L} \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\
&\quad -\nabla_{V_L} \widehat{G}_{Q_{\widetilde{G}}}|_c \hat{\mu}_{3_c} = 0 && (61)\\
\nabla_{K_G} \mathcal{L}|_c &= -\nabla_{K_G} \widehat{G}_S|_c \hat{\mu}_{1_c} = 0 && (62)\\
\nabla_{Q_{\overline{G}}} \mathcal{L}|_c &= -\hat{\mu}_{2_c} - M_{\bar{a}_c}\hat{\mu}_{4_c} - M_{\bar{b}_c}\hat{\mu}_{5_c} + \hat{\mu}_{8_c} - \hat{\mu}_{9_c} = 0 && (63)\\
\nabla_{Q_{\widetilde{G}}} \mathcal{L}|_c &= -\hat{\mu}_{3_c} - M_{\tilde{a}_c}\hat{\mu}_{6_c} - M_{\tilde{b}_c}\hat{\mu}_{7_c} + \hat{\mu}_{10_c} - \hat{\mu}_{11_c} = 0 && (64)\\
\nabla_{V_{\overline{G}}} \mathcal{L}|_c &= -\nabla_{V_{\overline{G}}} \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_{V_{\overline{G}}} \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\
&\quad -\nabla_{V_{\overline{G}}} \widehat{G}_{Q_{\widetilde{G}}}|_c \hat{\mu}_{3_c} - \hat{\mu}_{12_c} = 0 && (65)\\
\nabla_{V_{\widetilde{G}}} \mathcal{L}|_c &= -\nabla_{V_{\widetilde{G}}} \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_{V_{\widetilde{G}}} \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\
&\quad -\nabla_{V_{\widetilde{G}}} \widehat{G}_{Q_{\widetilde{G}}}|_c \hat{\mu}_{3_c} - \hat{\mu}_{13_c} = 0 && (66)\\
\nabla_\lambda \mathcal{L}|_c &= -\nabla_\lambda \widehat{G}_S|_c \hat{\mu}_{1_c} - \nabla_\lambda \widehat{G}_{Q_{\overline{G}}}|_c \hat{\mu}_{2_c} \\
&\quad -\nabla_\lambda \widehat{G}_{Q_{\widetilde{G}}}|_c \hat{\mu}_{3_c} + 1 = 0 && (67)\\
\nabla_{V_{\bar{a}}} \mathcal{L}|_c &= -M_{Q_{\overline{G}_{min_c}}} \hat{\mu}_{4_c} + \hat{\mu}_{12_c} + \hat{\mu}_{14_c} = 0 && (68)\\
\nabla_{V_{\bar{b}}} \mathcal{L}|_c &= -M_{Q_{\overline{G}_{max_c}}} \hat{\mu}_{5_c} - \hat{\mu}_{12_c} + \hat{\mu}_{15_c} = 0 && (69)\\
\nabla_{V_{\tilde{a}}} \mathcal{L}|_c &= -M_{Q_{\widetilde{G}_{min_c}}} \hat{\mu}_{6_c} + \hat{\mu}_{13_c} + \hat{\mu}_{16_c} = 0 && (70)\\
\nabla_{V_{\tilde{b}}} \mathcal{L}|_c &= -M_{Q_{\widetilde{G}_{max_c}}} \hat{\mu}_{7_c} - \hat{\mu}_{13_c} + \hat{\mu}_{17_c} = 0 && (71)
\end{aligned}$$

where $M_{\bar{a}_c} = \text{diag}(V_{\bar{a}_c})$, and similarly for $M_{\bar{b}_c}$, $M_{\tilde{a}_c}$, $M_{\tilde{b}_c}$; and $M_{Q_{\overline{G}_{min_c}}} = \text{diag}(Q_{\overline{G}_c} - Q_{\overline{G}_{min}})$, and similarly for $M_{Q_{\overline{G}_{max_c}}}$, $M_{Q_{\widetilde{G}_{min_c}}}$, and $M_{Q_{\widetilde{G}_{max_c}}}$. Furthermore, all the equality constraints must be equal to zero, while the inequality constraints must be less than or equal to zero.

From the regularity and strict complementarity OS assumptions, which imply a unique $\mu_c = (\mu_{1_c}, \ldots, \mu_{17_c}) \neq 0$, with $\mu_{l_c} > 0 \ \forall l \in \{\text{Active Constraint Set}\}$, it follows from (19) and (20) that:

$$\begin{aligned}
\hat{\mu}_{8_{\bar{\jmath}c}}(Q_{G_{\bar{\jmath}min}} - Q_{G_{\bar{\jmath}c}}) = 0 \ &\Rightarrow \ \hat{\mu}_{8_{\bar{\jmath}c}} = 0 \ \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}} && (72)\\
\hat{\mu}_{9_{\bar{\jmath}c}}(Q_{G_{\bar{\jmath}c}} - Q_{G_{\bar{\jmath}max}}) = 0 \ &\Rightarrow \ \hat{\mu}_{9_{\bar{\jmath}c}} = 0 \ \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}} && (73)\\
\hat{\mu}_{10_{jc}}(Q_{G_{jmin}} - Q_{G_{jc}}) = 0 \ &\Rightarrow \ \hat{\mu}_{10_{jc}} = 0 \ \forall j \in \mathcal{G}_j && (74)\\
\hat{\mu}_{11_{jc}}(Q_{G_{jc}} - Q_{G_{jmax}}) = 0 \ &\Rightarrow \ \hat{\mu}_{11_{jc}} > 0 \ \forall j \in \mathcal{G}_j && (75)\\
\hat{\mu}_{14_{\bar{\jmath}c}}(-V_{a_{\bar{\jmath}c}}) = 0 \ &\Rightarrow \ \hat{\mu}_{14_{\bar{\jmath}c}} > 0 \ \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}} && (76)\\
\hat{\mu}_{15_{\bar{\jmath}c}}(-V_{b_{\bar{\jmath}c}}) = 0 \ &\Rightarrow \ \hat{\mu}_{15_{\bar{\jmath}c}} > 0 \ \forall \bar{\jmath} \in \mathcal{G}_{\bar{\jmath}} && (77)\\
\hat{\mu}_{16_{jc}}(-V_{a_{jc}}) = 0 \ &\Rightarrow \ \hat{\mu}_{16_{jc}} > 0 \ \forall j \in \mathcal{G}_j && (78)\\
\hat{\mu}_{17_{jc}}(-V_{b_{jc}}) = 0 \ &\Rightarrow \ \hat{\mu}_{17_{jc}} = 0 \ \forall j \in \mathcal{G}_j && (79)
\end{aligned}$$

Now, based on (19) and (20), the following equations, evaluated at the solution point $(\hat{z}_c, \lambda_c, \hat{p}_o)$, form the minimum subset of constraints (15a)-(15f) that uniquely define $\hat{z}_c$ for a given $(\lambda_c, \hat{p}_o)$, since the number of equations and unknowns is the same, i.e. N:

$$G|_c = \begin{bmatrix} \widehat{G}(\delta_c, V_{L_c}, K_{G_c}, Q_{G_c}, V_{G_c}, \lambda_c, P_{S_o}, P_{D_o}) \\ V_{\overline{G}_c} - V_{\overline{G}_o} \\ Q_{\widetilde{G}_c} - Q_{\widetilde{G}_{max}} \end{bmatrix} = 0 \quad (80)$$

Hence, for the optimal solution to be an SNB, one first must prove that the Jacobian $J = \nabla_{\hat{z}}^T G|_c$ is singular with unique nonzero eigenvectors, where $\hat{z} = (\delta, V_L, K_G, V_G, Q_G)$.

From (60)-(71) and with the proper ordering of variables and equations in (44), it can be shown that:

$$\nabla_{\hat{z}} G|_c \ \hat{w} = \hat{b} \quad (81)$$

where,

$$\nabla_{\hat{z}} G|_c = \quad (82)$$
$$\begin{bmatrix}
\nabla_\delta \widehat{G}_S|_c & \nabla_\delta \widehat{G}_{Q_{\overline{G}}}|_c & \nabla_\delta \widehat{G}_{Q_{\widetilde{G}}}|_c & 0 & 0 \\
\nabla_{V_L} \widehat{G}_S|_c & \nabla_{V_L} \widehat{G}_{Q_{\overline{G}}}|_c & \nabla_{V_L} \widehat{G}_{Q_{\widetilde{G}}}|_c & 0 & 0 \\
\nabla_{K_G} \widehat{G}_S|_c & 0 & 0 & 0 & 0 \\
\nabla_{V_{\overline{G}}} \widehat{G}_S|_c & \nabla_{V_{\overline{G}}} \widehat{G}_{Q_{\overline{G}}}|_c & \nabla_{V_{\overline{G}}} \widehat{G}_{Q_{\widetilde{G}}}|_c & I_{n_{\mathcal{G}_{\bar{\jmath}}}} & 0 \\
\nabla_{V_{\widetilde{G}}} \widehat{G}_S|_c & \nabla_{V_{\widetilde{G}}} \widehat{G}_{Q_{\overline{G}}}|_c & \nabla_{V_{\widetilde{G}}} \widehat{G}_{Q_{\widetilde{G}}}|_c & 0 & 0 \\
0 & I_{n_{\mathcal{G}_{\bar{\jmath}}}} & 0 & 0 & 0 \\
0 & 0 & I_{n_{\mathcal{G}_j}} & 0 & I_{n_{\mathcal{G}_j}}
\end{bmatrix}$$

$$\hat{w} = \begin{bmatrix} \hat{\mu}_{1_c} \\ \hat{\mu}_{2_c} \\ \hat{\mu}_{3_c} \\ \hat{\mu}_{12_c} \\ \hat{\mu}_{11_c} + M_{\tilde{b}_c}\hat{\mu}_{7_c} \end{bmatrix} \quad (83)$$
$$\hat{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\hat{\mu}_{13_c} \\ -M_{\bar{a}_c}\hat{\mu}_{4_c} - M_{\bar{b}_c}\hat{\mu}_{5_c} + \hat{\mu}_{8_c} - \hat{\mu}_{9_c} \\ -M_{\tilde{a}_c}\hat{\mu}_{6_c} + \hat{\mu}_{10_c} \end{bmatrix}$$

Now, from (71) and (79):

$$\hat{\mu}_{13_c} = \hat{\mu}_{17_c} = 0 \quad (84)$$

From (20), (72) and (73):

$$-M_{\bar{a}_c}\hat{\mu}_{4_c} - M_{\bar{b}_c}\hat{\mu}_{5_c} + \hat{\mu}_{8_c} - \hat{\mu}_{9_c} = 0 \quad (85)$$

From (19) and (74):

$$-M_{\tilde{a}_c}\hat{\mu}_{6_c} + \hat{\mu}_{10_c} = 0 \quad (86)$$

Hence, from (84)-(86), it follows that:

$$\nabla_{\hat{z}} G|_c \ \hat{w} = 0$$

Finally, since, from the regularity and strict complementarity OS assumptions, it follows that $\mu_{1_c} \neq 0$, $\mu_{2_c} \neq 0$, and $\mu_{3_c} \neq 0$, as $\hat{\mu}_c \neq 0$ and is unique Hence, $\hat{w} \neq 0$ and is unique, from which it can be concluded that the optimum $(\hat{z}_c, \lambda_c, \hat{p}_o)$ meets the SNB transversality condition (3).

Now, from (67), (80) and (83), it follows that:

$$\nabla_\lambda \mathcal{L}|_c = -\nabla_\lambda G|_c \ \hat{w} + 1 = 0$$
$$\Rightarrow \ \nabla_\lambda G|_c \ \hat{w} \neq 0$$

which corresponds to the SNB transversality condition (4).

The third SNB transversality condition (5) can be verified by means of the second order conditions for optimality. Thus, from assumptions (19) and (20) regarding the optimum $(\hat{z}_c, \lambda_c, \hat{p}_o)$, and from (80), as well as based on the previous analysis, the optimization model (15) can be restated as follows, since it would yield the same optimal solution:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & G(\hat{z}, \lambda, \hat{p}_o) = 0 \end{aligned}$$

Thus, the corresponding Lagrangian function may be defined as:

$$\mathcal{L}(\hat{z}, \lambda, \hat{p}_o, \hat{\mu}) = \lambda - \hat{\mu}^T G(\hat{z}, \lambda, \hat{p}_o)$$

which, based on the KKT optimality conditions, leads to:

$$\nabla_{\hat{z}} \mathcal{L}|_c = -\nabla_{\hat{z}} G|_c \hat{\mu}_c = -\nabla_{\hat{z}} G|_c \hat{w} = 0 \tag{88}$$

$$\nabla_{\hat{w}} \mathcal{L}|_c = -G|_c = 0 \tag{89}$$

$$\nabla_{\lambda} \mathcal{L}|_c = -\nabla_{\lambda} G|_c \hat{w} + 1 = 0 \tag{90}$$

From the second order optimality conditions [30], which states that the Hessian of the Lagrangian function is positive definite, it follows that:

$$\hat{\rho}^T \nabla_Z^2 \mathcal{L}(\hat{z}_c, \lambda_c, \hat{p}_o, \hat{w}) \hat{\rho} > 0 \quad \forall \hat{\rho} \neq 0 \tag{91}$$

where $Z = (\hat{z}, \lambda, \hat{\mu})$.

Now, from (88)-(90), it follows that:

$$\nabla_Z^2 \mathcal{L}|_c = \begin{bmatrix} \nabla_{\hat{z}}^2 G|_c \hat{w} & \nabla_{\hat{z}} G|_c & \nabla_{\lambda \hat{z}}^2 G|_c \hat{w} \\ \nabla_{\hat{z}}^T G|_c & 0 & \nabla_{\lambda}^T G|_c \\ \hat{w}^T \nabla_{\hat{z}\lambda}^{2^T} G|_c & \nabla_{\lambda} G|_c & \nabla_{\lambda}^2 G|_c \hat{w} \end{bmatrix} \tag{92}$$

Hence, for a chosen $\hat{\rho} = (\hat{v}, 0, 0) \neq 0$, from (91) one has that:

$$\hat{v}^T \left[\nabla_{\hat{z}}^2 G|_c \hat{w} \right] \hat{v} > 0$$

Taking the transpose and considering the properties of tensor products:

$$\hat{w}^T \left[\nabla_{\hat{z}}^{2^T} G|_c \hat{v} \right] \hat{v} > 0$$

This corresponds to the third SNB transversality condition (5). ∎

APPENDIX III
PROOF OF COROLLARY 1

Proof: Observe that Theorem 1 proofs that a LIDB cannot be a solution of (15). Now, notice that all possible limit conditions of the inequality constraints of (15) are considered in assumptions (16)-(18) and (19)-(20) of Theorems 1 and 2, respectively; thus, the cases of none or all generators reaching their limits are simply particular cases of these assumptions. Hence, any feasible solution of (15), would either meet assumptions (16)-(18) or (19)-(20). Therefore, the solution point $(\hat{z}_c, \lambda_c, \hat{p}_o)$ can only be a LISB or an SNB. ∎

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AUTHORS’ RESPONSE

We would like to thank the Editor and Reviewers for their thoughtful comments, which we have taken into consideration to improve the paper. Below, please find a detail explanation of how each one of the reviewers' concerns has been addressed in the revised paper.

>
>
>Review Number 1.
>*****************
>
>Comments to the Author
>----------------------
>
>This paper presents an interesting demonstration
>of equivalency of continuation power flow (CPF)
>and optimal power flow based directed method
>(OPF-DM) in terms of occurrence of saddle-node
>and limit-induced bifurcation in power systems. The major
contributions
>include proofs of satisfaction of a complete set
>of transversality conditions when a saddle-node
>or LISB bifurcation occurs. For the saddle-node
>bifurcation, it is an extension of reference
>[7], which fails to address the transversality
>condition (5) in this manuscript.

>General Comments:
>
>1. CPF can be used computes PV curves for the
>increased loading factor by keeping the voltage
>constant when the reactive power generation is
>within the limit. The OPF-DM is set up to
>maximize the loading factors, and it will also maintain the voltage
level
>if the reactive power is within the range after
>introducing a complementarity constraint. It is
>thus not surprising that the same type of
>bifurcation is produced at the maximum loading
>factor. The complementarity constraint modeling is critical for
>the equivalency of CPF and OPF-DM and should be
>mentioned the abstract/introduction.

Authors’ response:

A comment has been included in the Abstract and Introduction (4th paragraph) to emphasize the fact that the OPF-DM model is based on the use of complementarity constraints.

>2. Another question can be asked is whether the
>CPF and the OPF-DM are equivalent in producing
>other types of bifurcation such as a Hopf
>bifurcation. It will be good to have at least
>some discussions about this issue.

Authors’ response:

Other types of bifurcations in power systems, such as Hopf bifurcations (associated with oscillatory instabilities) or Singularity-induced bifurcations (associated with DAE systems), are out of the scope of this paper, since it has been shown that SNBs and LIBs are the bifurcations of most interest for voltage stability studies in practice. This has been clearly stated on the first paragraph of the Introduction.

>3. OS assumptions in page 5 play a very
>important role. It is mentioned in the
>manuscript that there is no guarantee that all
>possible solutions of equation (15) would meet
>the OS assumptions. It would help if further
>illustration is provided in this topic.

Authors’ response:

Numerical results from practical implementations of these optimization models have been widely reported in the literature, showing that solutions are robust and that these conditions are basically met in practice. A comment and a reference have been added to Section IV (2nd paragraph) to highlight this fact.

>Specific Comments:
>
>1. Please carefully check the English in the
>section of Introduction. There are a number of
>places that need to be fixed or rewritten. An
>incomplete set of the problems include:
>   (1). "Mayor" in line 7 under Section I Introduction should be major;

>   (2). "has" in line 10 under section I Introduction should be have;

>   (3). "consists on ..." in the last line
> of column 1 and the first line of paragraph 2 in column 2, both in Page 1.

>   (4). The first setence of paragraph 3
> in column 2 of page 1 is too long and please rephrase.
>   ...

Authors’ response:

All of these typos have been corrected.

>2. In equation (2), G|_o and
>g(x_o,y_o,lambda_o,p_o) both represent the power
>flow. Please spell out whether they are the same.

Authors’ response:

Please note that G is a subset of g. This is now formally stated below equation (2).

>3. w in equation (4), it should be w_hat.

>4. Page 5, the second paragraph under Section
>IV, "... shows that a an ...". "a" should be removed.

>5. Column 1 in page 7, immediately under
>equation "b^hat_b=...", \mu^hat_9 and \mu^hat_{12} should be fixed.

Authors' response:

These errors have been corrected.

>Review Number 2.
>*****************
>
>Comments to the Author
>-----------------------
>
>
>The paper under review presents a theoretical
>study showing that solutions points obtained
>from an optimization model correspond to either
>SNB or LIB points. The authors provide a sound
>mathematical framework where demonstrations are well supported.
>

>Recommended Revisions
>
>1.	The proofs of the theorems are not
>generally used in the paper in subsequent
>sections. The authors should consider moving
>the proofs into an appendix to enhance the "readability of the paper".

Authors' response:

The proofs have been moved to appendices, as suggested by the reviewer.

>2.	A citation(s) should be added to the first sentence of Section II.B

Authors' response:

An appropriate reference has been added, as suggested.

>3.	A citation(s) should be added to: page 4, column 1, line 2 or
>the authors should
>provide a enhanced explanation of the actuation function.

Authors' response:

Please note that the explanation of the types of limits and their association to LIBs comes basically from reference [28], as mentioned at the beginning of the paragraph the reviewer refers to. Please observe as well that the actuation limits and (implicit) functions are defined in Section III.

>4. Page 2 , column 2 , the authors should
>better explain the inverse of y used in the unnumbered equation?

Authors' response:

The paragraph where the inverse of y is mentioned has been rewritten and additional references have been added to better explain its meaning.

>5. Fig.1: Actuation and saturation regime
>should be better defined/discussed in the
>text. Also a vertical line at \lambda_c may enhance the presentation.

Authors' response:

A vertical line to clearly shown the meaning of \lambda_c in Fig.1 has been added as suggested. Please note that these regimes are formally defined below eqs. (8) and (9); a reference to Fig.1 has been added to clarify these definitions.

>6. Equation 6 and 7, "^" and "~"
>superscripts variables should be better defined.
>Also q,l,c superscripts should be better defined.

Authors' response:

In both cases "^" and "~" are used to reorder the variables in subsets, which are now formally defined immediately after eq. (6). Th-e subscripts "q" and "l" have been replaced by the less cryptic "max" and "min" subscripts, and the subscript "c" is now formally defined at the end of Section II.B.1.

>7. Equations 8 and 9: An enhanced discussed
>of the exclusion of i/th/ generator should be added.

Authors' response:

An explanation of why a single generator "i" is considered has been added to the paragraph below eqs. (8) and (9).

>8. Page 4, column 1, 5th line under
>equation 9, the sentence should be revised to clarify the authors' intent.

Authors' response:

This sentence has been revised to clarify our meaning.

>9. An explanation should be added for equation 12d.

Authors' response:

We have added a sentence to the explanation below eqs. (12), defining (12d).

>10. Page 5, column 2, section IV paragraph 3
>(two bullet points) should be revised to clarify the authors' intent/thought.

Authors' response:

The paragraph after the two bullet point has been revised to clarify the meaning and validity of these assumptions in practice. Please note that these two assumptions are formal conditions, which are explained in great detail in [30], required of the optimal solution points.

>11. Page 8, column 1, theorem 1 and 2: using
>word "may be" is not suitable in a theorem.

Authors' response:

The words "may be" have been changed to "can be". Please note that the optimal solution can be either an SNB or a LISB.

>12. Fig.3 should be better labeled, for
>example it is not shown which point is LISB or
>LIDB. This would enhance the clarity of the numerical result section.

Authors' response:

Labels for the LIDB and LISB points have been added to this figure as suggested.

>13. The main contribution should be enhanced
>or explained more clearly. Using "may be" in the
>conclusion does not seem suitable and strong.

Authors' response:

The words "may be" have been changed to "can be" to strengthen our concluding remarks. However, in practice, one cannot guarantee that the OPF-DM method discussed would always generate better results than a standard CPF method.

>Editorial suggestions/corrections
>
>1. page 1 , column 1 , first paragraph,
>line 7 : "major blackouts" instead of "mayor blackouts".

>2. Page 3 , column 1 , first paragraph,
>line 2 : "analysis" instead of "analyses"

>3. Page 5, column 1, "\hat{G} stands for" instead of "\hat{G} stand for"

>4. Equation 4: should it read \hat{\omega} instead of \omega ?

>5. Page 5, column 1, paragraph 3, the

>spacing between lines is not in correct format.

>6.	Page 5, column 2, section IV paragraph
>2, revise the first line, specifically "a an"

Authors' response:

All of these errors have been corrected.

>General Comment on Editorial suggestions: The
>paper should be reviewed thoroughly for simple
>editorial errors before consideration is made for resubmission.
>
>
>
>
>
>Review Number 3.
>****************
>
>Comments to the Author
>---------------------
>
>Review of the paper "Equivalency of Continuation and Optimization
>Methods to Determine Saddle-node and Limit-induced Bifurcations in
>Power Systems I: Transversality Conditions" by Avalos, Canizares,
>Milano and Conejo.
>---
>
>The above paper was written not for a wide audience. The paper makes
>often references to other papers even when the notation is considered.
>This makes the exposition of main idea of the paper to be rather
>difficult to follow. Besides, there are several typos and grammar
>errors in the paper. The authors often use very long sentences which
>makes their arguments to be blurred and not well exposed.

Authors' response:

Please note that the paper is mainly directed to people who have particular interest on the VS issues discussed here. The reason why we need to refer to other papers is because there are concepts which are out of the scope of this paper, as well as space limitations. The paper has been thoughtfully revised for typos and grammar errors, and several long sentences have been rephrased.

>Here are some specific remarks/errors:
>
>(1)In the Introduction I found expressions: "consists on increasing",
>"consists on maximizing", "control actions of a the generators'
>voltage." Later in section III: "nonnegative variables that allows,"
>and in Section IV: "shows that a an optimal solution."

Authors' response:

This has been corrected.

>(2)In the Definitions (section II): there is no clear explanation of

>the notation used, particularly the many variables denoted by the
>letter z, z sub 0, z hat sub 0, z tilde, z hat sub c, z tilde sub c,
>etc. I was lost when trying to follow authors' notation. The g hat in
>(6) and h hat in (12c) are not defined.

Authors' response:

All the variables associated with z are clearly and formally defined as they appear. For example, z is defined at the beginning of Section II.A, and z hat is defined below eq.(2). The need for this notation is to clearly and formally distinguish among the different models used in VS studies, i.e. dynamic versus power flow models, which is an important issue. Standard non-linear notation to define equilibrium points (o), critical or bifurcation points (c), etc., was used throughout the paper. The need to rather formally distinguish among the different models referred to throughout the paper, as needed, forced us to use a wide variety of notation, all of which is properly and formally defined. Please note that g hat is formally defined in eq. (6) and a detailed explanation of what h hat stands for is provided below eqs. (12).

>(3)The concept of equilibria of DAEs defined in (2) is much more
>subtle than that discussed by the authors and includes also singular
>equilibria (see for example [1,2] below). Are singular equilibria not
> included in the authors' discussion of the SNBs and LIBs? It is not
>clear why only a subset of F is used to compute the system
>equilibria in (2). Which equations in F are not used in such a
>computation? Also, in any analysis of systems of DAEs it is important
>to discuss the concept of index of DAEs, as the index plays a very
>important role in all numerical methods for DAEs. The authors ignored
>this issue completely in their paper.

Authors' response:

A brief reference to SIBs and why these bifurcations are not considered in this paper has been added to the first paragraph of the Introduction. Please note, as discussed in great detail in [1], that only SNBs and LIBs are of relevance for VS analysis in practice.

Please observe that the fourth paragraph in Section II.A clearly states that G (a subset of F) does not necessarily yield system equilibria. In practice, LIBs and SNBs in power systems are studied using only G (power flow equations). Thus, as mentioned in the Introduction and explained in some detail in Section II.A, the paper concentrates on the analysis of SNBs and LIBs of eq. (2), given their use in practice to study VS problems.

References to DAE systems and associated issues pertinent to power system studies, which are beyond the scope of the present paper, are appropriately provided. The pertinent papers mentioned by the reviewer have been added to the bibliography.

>(4)The authors use G in (2) but G is given much later for the LIBs
>(formula (6)).

Authors' response:

Please note that G is formally defined in (2), whereas (6) is used to define g hat and the necessary actuation limits for LIB studies.

>(5)The many long sentences make the paper rather difficult to read.
>For example, in Section III the authors begin Part A with "For a
>typical power flow model..." and end this sentence with "...a constant
>power factor load." The whole sentence is 14 lines long and also
>contains equations (13) inside. Another similar example: in Part B of
>Section III I found the following sentence: "Thus the typical power
>flow model..." that ends with "...has reached a limit or not." The
total
>length of this sentence is 10 lines plus two formulas inside the
>sentence.

Authors' response:

The specific sentences that the reviewer refers to have been rephrased. Since we believe that the rest of the paper is properly written, no other sentences have been revised, with the exception of another sentence revision requested by Reviewer 1.

>(6)The two major theorems in the paper and particularly those huge
>systems of algebraic equations used in the proofs of the theorems are
>extremely unreadable and difficult to follow, particularly systems
>(19) through (30), (60) through (71) and (72) through (79). I think
>that the authors should change the proofs and write them in a much
>more reader-friendly way.

Authors' response:

Please note that, as mentioned below by the Reviewer, extreme care should be exercised in the proofs. This is the reason why we have been careful in clearly stating all of our assumptions as the theorems are presented and the proofs are developed step by step. We have not been able to devise a better way to carefully and formally prove the proposed theorems and corollary.

>(7)It is difficult using the current presentation to realize how
>computationally difficult it is to numerically compare the OPF-DM and
>CPF methods for even a small power system (the one considered in
>Section V). It is my opinion that it would be beneficial to the
>readers to see some analysis of the complexity of two methods in terms
>of the size of power systems.

Authors' response:

A comment and references have been added to Section V.B to address this specific issue.

>---
>
>Final remarks: I do not think that the paper should be published in
>its current version. Many improvements are needed to make the paper
>and its whole idea to be more reader-friendly. Extreme care should be

>exercised in the section of Definitions as well as in the proofs of
>two main theorems.

Authors' response:

As previously mentioned, extreme care has been exercised in stating and proving the theorems, with all assumptions being clearly stated. The proofs have been moved to appendices, as suggested by Reviewer 2, to improve readability.

>--

>[1] R. Riaza, S. L. Campbell, and W. Marszalek, "On singular
>equilibria of index-I DAEs," Circuits, Systems, and Signal Processing,
>vol.19, no.2, pp.1-27, 2000.
>
>[2] W. Marszalek, Z. Trzaska, "Singularity induced bifurcations in
>power systems," IEEE Trans. Power Systems, vol. 20, no.1, pp.312-320.
>Feb. 2005.