Line and Plane Arrangements with Large Average Diameter

Antoine Deza*

Department of Computing and Software
McMaster University
Hamilton, Ontario, Canada
deza@mcmaster.ca

Feng Xie†

Department of Computing and Software
McMaster University
Hamilton, Ontario, Canada
xief@mcmaster.ca

Abstract: Let $\Delta_{\mathcal{A}}(n,d)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by n hyperplanes in dimension d. We have $\Delta_{\mathcal{A}}(n,2) \leq 2 + \frac{2}{n-1}$ in the plane, and $\Delta_{\mathcal{A}}(n,3) \leq 3 + \frac{4}{n-1}$ in dimension 3. In general, the average diameter of a bounded cell of a simple arrangement is conjectured to be less than the dimension; that is, $\Delta_{\mathcal{A}}(n,d) \leq d$. We propose line and plane arrangements with large average diameter yielding $\Delta_{\mathcal{A}}(n,2) \geq 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$ and $\Delta_{\mathcal{A}}(n,3) \geq 3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)}$.

Keywords: line and plane arrangements, bounded cell, average diameter, lower bounds

1 Introduction

Let $\mathcal{A}$ be a simple arrangement formed by n hyperplanes in dimension d. We recall that an arrangement is called simple if $n \geq d+1$ and any d hyperplanes intersect at a unique distinct point. The number of bounded cells (bounded connected component of the complement of the hyperplanes) of $\mathcal{A}$ is $I = \binom{n-1}{d}$. Let $\delta(\mathcal{A})$ denote the average diameter of a bounded cell P_i of $\mathcal{A}$; that is,

$$\delta(\mathcal{A}) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I}.$$

where $\delta(P_i)$ denotes the diameter of P_i, i.e., the smallest number such that any two vertices of P_i can be connected by a path with at most $\delta(P_i)$ edges. Let $\Delta_{\mathcal{A}}(n,d)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by n inequalities in dimension d. Deza, Terlaky and Zinchenko [3] conjectured that $\Delta_{\mathcal{A}}(n,d) \leq d$, and showed that if the conjecture of Hirsch holds for polytopes in dimension d, then $\Delta_{\mathcal{A}}(n,d)$ would satisfy $\Delta_{\mathcal{A}}(n,d) \leq d + \frac{2d}{n-1}$. In dimension 2 and 3, they showed that $\Delta_{\mathcal{A}}(n,2) \leq 2 + \frac{2}{n-1}$ and $\Delta_{\mathcal{A}}(n,3) \leq 3 + \frac{4}{n-1}$. We recall that a polytope is a bounded polyhedron and that the conjecture of Hirsch, formulated in 1957 and reported in [1], states that the diameter of a polyhedron defined by n inequalities in dimension d is not greater than $n-d$. The conjecture does not hold for unbounded

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polyhedra. A simple line arrangement with average diameter equal to $2 - \frac{2}{n-1}$ was given in [3]. We propose a line arrangement with average diameter $2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$ and a plane arrangement with average diameter $3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)}$ yielding $2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)} \leq \Delta_{\mathcal{A}}(n,2) \leq 2 + \frac{2}{n-1}$ and $3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)} \leq \Delta_{\mathcal{A}}(n,3) \leq 3 + \frac{4}{n-1}$. For polytopes and arrangements, we refer to the books of Grünbaum [5] and Ziegler [7] and the references therein.

2 Line Arrangements with Large Average Diameter

For $n \geq 4$, we consider the simple line arrangement $\mathcal{A}^o_{n,2}$ made of the 2 lines h_1 and h_2 forming respectively the x and y axis and an additional $n-2$ lines defined by their intersection with h_1 and h_2. We have $h_k \cap h_1 = \{1+(k-3)\varepsilon, 0\}$, and $h_k \cap h_2 = \{0, 1-(k-3)\varepsilon\}$ for $k = 3, 4, \ldots, n-1$, and $h_n \cap h_1 = \{2, 0\}$ and $h_n \cap h_1 = \{0, 2+\varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < \frac{1}{n}$. See Figure 1 for an arrangement combinatorially equivalent to $\mathcal{A}^o_{7,2}$.

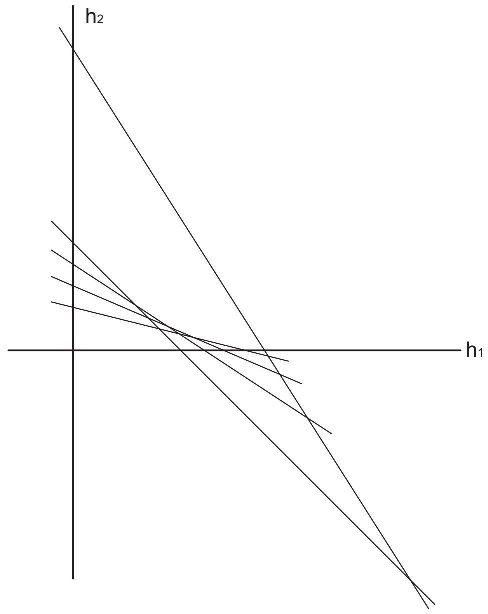

Figure 1: An arrangement combinatorially equivalent to $\mathcal{A}^o_{7,2}$

Proposition 1 *For $n \geq 4$, the bounded cells of the arrangement $\mathcal{A}^o_{n,2}$ consists of $n-2$ triangles, $\frac{(n-1)(n-4)}{2}$ 4-gons, and one n-gon.*

PROOF: The first $n-1$ lines of $\mathcal{A}^o_{n,2}$ clearly form a simple line arrangement which bounded cells are $n-3$ triangles and $\binom{n-3}{2}$ 4-gons. The last line h_n adds one n-gons, one triangle and $n-4$ 4-gons. $\square$

Corollary 2 *For $n \geq 4$, we have $\delta(\mathcal{A}^o_{n,2}) = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$.*

PROOF: Since the triangles have diameter 1, the 4-gons have diameter 2, and the n-gon has diameter $\lfloor \frac{n}{2} \rfloor$, we have $\delta(\mathcal{A}^o_{n,2}) = 2 - 2\frac{(n-2)-(\lfloor \frac{n}{2} \rfloor -2)}{(n-1)(n-2)} = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$. $\square$

Remark 3 *As there is only one combinatorial type of simple line arrangement for* $n = 4$, *we have* $\Delta_{\mathcal{A}}(4,2) = \delta(\mathcal{A}^o_{4,2}) = \frac{4}{3}$. *For* $n = 5$, *there is* 6 *combinatorial types of simple line arrangement and* $\delta(\mathcal{A}^o_{5,2})$ *is among the ones with maximal average diameter with* $\Delta_{\mathcal{A}}(5,2) = \delta(\mathcal{A}^o_{5,2}) = \frac{3}{2}$. *We believe that* $\Delta_{\mathcal{A}}(n,2) = \delta(\mathcal{A}^o_{n,2}) = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$ *for* $n \geq 4$.

A facet of an hyperplane arrangement belongs to either zero, one or two bounded cells. We call a facet external if it belongs to exactly one bounded cell and believe that arrangements with large average diameter have few external facets. The first $n-1$ lines of $\mathcal{A}^o_{n,2}$ form the line arrangement $\mathcal{A}^*_{n-1,2}$ proposed in [3]. The arrangement $\mathcal{A}^*_{n,2}$ has $3(n-2)$ external facets and average diameter $\delta(\mathcal{A}^*_{n,2}) = 2 - \frac{2}{n-1}$. The arrangement $\mathcal{A}^o_{n,2}$ has $2(n-1)$ external facets We believe that, in addition of maximizing the average diameter, $\mathcal{A}^o_{n,2}$ minimizes the number of external facets. Note that the envelope of the bounded cells of $\mathcal{A}^o_{n,2}$ has only one reflex vertex. Following the same approach, we generalize $\mathcal{A}^o_{n,2}$ to dimension 3 in Section 3 by considering a generalization of $\mathcal{A}^*_{n-1,2}$ and adding one plane to reduce the number of external facets.

3 Plane Arrangements with Large Average Diameter

For $n \geq 5$, we consider the simple plane arrangement $\mathcal{A}^o_{n,3}$ made of the 3 plane h_1, h_2 and h_3 forming, respectively, the plane $z = 0$, $y = 0$ and $x = 0$ and an additional $n-3$ plane defined by their intersection with $h_1 \cap h_2$, $h_1 \cap h_3$ and $h_1 \cap h_3$. We have $h_k \cap h_1 \cap h_2 = \{1 + 2(k-3)\varepsilon, 0, 0\}$, $h_k \cap h_1 \cap h_3 = \{0, 1 + (k-3)\varepsilon, 0\}$, and $h_k \cap h_2 \cap h_3 = \{0, 0, 1 - (k-3)\varepsilon\}$, for $k = 4, 5, \ldots, n-1$, and $h_n \cap h_1 \cap h_2 = \{3, 0, 0\}$, $h_n \cap h_1 \cap h_3 = \{0, 2, 0\}$, and $h_n \cap h_2 \cap h_3 = \{0, 0, 3 + \varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < \frac{1}{2n}$. See Figure 2 for an illustration of an arrangement combinatorially equivalent to $\mathcal{A}^o_{7,3}$ where, for clarity, only the bounded cells belonging to the positive orthant are drawn.

Proposition 4 *For* $n \geq 5$, *the bounded cells of the arrangement* $\mathcal{A}^o_{n,3}$ *consists of* $n-3$ *tetrahedra,* $(n-3)(n-4)-1$ *cells combinatorially equivalent to a prism with a triangular basis,* $\frac{(n-3)(n-4)(n-5)}{6}$ *cells combinatorially equivalent to a cube, and one cell combinatorially equivalent to a shell* S_n *with* n *facets and* $2(n-2)$ *vertices. See Figure 4 for an illustration of* S_7.

PROOF: For $3 \leq k \leq n-1$, let $\mathcal{A}^*_{k,3}$ denote the arrangement formed by the first k planes of $\mathcal{A}^o_{n,3}$. See Figure 3 for an arrangement combinatorially equivalent to $\mathcal{A}^*_{6,3}$. We first show by induction that the bounded cells of arrangement $\mathcal{A}^*_{n-1,3}$ consist of $n-4$ tetrahedra, $(n-4)(n-5)$ combinatorial triangular prisms, and $\binom{n-4}{3}$ combinatorial cubes. We use the following notation to describe the bounded cells of $\mathcal{A}^*_{k-1,3}$: T_+ for a tetrahedron with a facet on h_1 and a vertex above h_1; $P_\triangle$, respectively $P_\diamond$, for a combinatorial triangular prism with a triangular, respectively square, facet on h_1; and C, respectively T and P, for a combinatorial cube, respectively tetrahedron and triangular prism, not touching h_1. The bounded cells of $\mathcal{A}^*_{k-1,3}$ which are to be cut by the addition of h_k are marked with a bar superscript. When the plane h_k is added, the cells $\bar{T}_+$, $\bar{P}_\triangle$, $\bar{P}_\diamond$, and $\bar{C}$ are sliced, respectively, into T and $\bar{P}_\triangle$, P and $\bar{P}_\triangle$, P and $\bar{C}$, and C and $\bar{C}$. In addition, one $\bar{T}_+$ cell and $k-4$ $\bar{P}_\diamond$ cells are created by bounding $k-3$ unbounded cells of

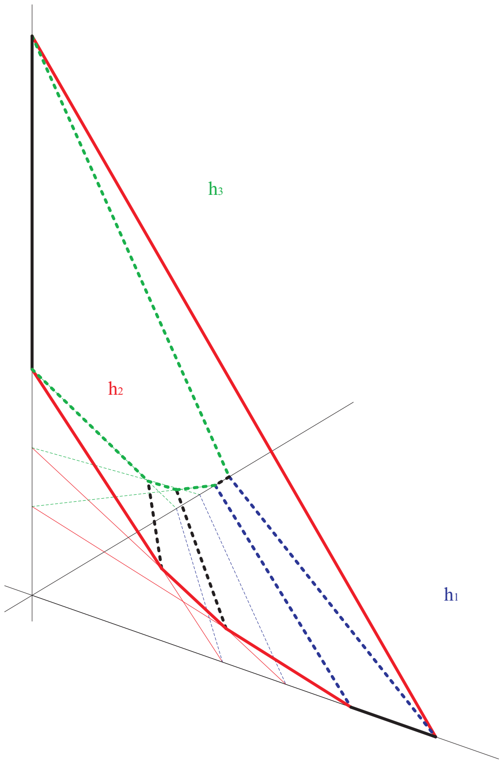

Figure 2: An arrangement combinatorially equivalent to $\mathcal{A}^o_{7,3}$

$\mathcal{A}^*_{k-1,3}$. Let $c(k)$ denotes the number of C cells of $\mathcal{A}^*_{k,3}$, similarly for $\bar{C}$, T, $\bar{T}_+$, P, $\bar{P}_\triangle$ and $\bar{P}_\diamond$. For $\mathcal{A}^*_{4,3}$ we have $\bar{t}_+(4) = 1$ and $t(4) = p(4) = \bar{p}_\triangle(4) = \bar{p}_\diamond(4) = c(4) = \bar{c}(4) = 0$. The addition of h_k removes and adds one $\bar{T}_+$, thus, $\bar{t}_+(k) = 1$. Similarly, all $\bar{P}_\diamond$ are removed and $k-4$ are added, thus, $\bar{p}_\diamond(k) = k-4$. Since $t(k) = t(k-1)+\bar{t}_+(k-1)$ and $\bar{p}_\triangle(k) = \bar{p}_\triangle(k-1)+\bar{t}_+(k-1)$, we have $t(k) = \bar{p}_\triangle(k) = k-4$. Since $p(k) = p(k-1)+\bar{p}_\triangle(k-1)+\bar{p}_\diamond(k-1)$, we have $p(k) = (k-4)(k-5)$. Since $\bar{c}(k) = \bar{c}(k-1) + \bar{p}_\diamond(k-1)$, we have $\bar{c}(k) = \binom{k-4}{2}$. Since $c(k) = c(k-1) + \bar{c}(k-1)$, we have $c(k) = \binom{k-4}{3}$. Therefore the bounded cell of $\mathcal{A}^*_{n-1,3}$ consist of $t(n-1) + \bar{t}_+(n-1) = n-4$ tetrahedra, $p(n-1) + \bar{p}_\triangle(n-1) + \bar{p}_\diamond(n-1) = (n-4)(n-5)$ combinatorial triangular prisms, and $c(n-1) + \bar{c}(n-1) = \binom{n-4}{3}$ combinatorial cubes. Remarking that the addition of h_n is similar to the addition of h_k for $4 \le k \le n-1$, except that one shell S_n is added instead of one triangular prism, we obtain the bounded cells of $\mathcal{A}^o_{n,3}$. $\square$

Corollary 5 *For $n \ge 5$, we have $\delta(\mathcal{A}^o_{n,3}) = 3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)}$.*

PROOF: Since the tetrahedra have diameter 1, the combinatorial triangular prisms have diameter 2, the combinatorial cubes have diameter 3, and the n-shell has diameter $\lfloor \frac{n}{2} \rfloor$, we have $\delta(\mathcal{A}^o_{n,3}) = 3 - 6\frac{2(n-3)+(n-3)(n-4)-1-(\lfloor \frac{n}{2} \rfloor - 3)}{(n-1)(n-2)(n-3)} = 3 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)} = 3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)}$. $\square$

Remark 6 *As there is only one combinatorial type of simple plane arrangement for $n = 5$, we have $\Delta_{\mathcal{A}}(5,3) = \delta(\mathcal{A}^o_{5,3}) = \frac{3}{2}$. We believe that the average diameter of $\mathcal{A}^o_{n,3}$ is not maximal*

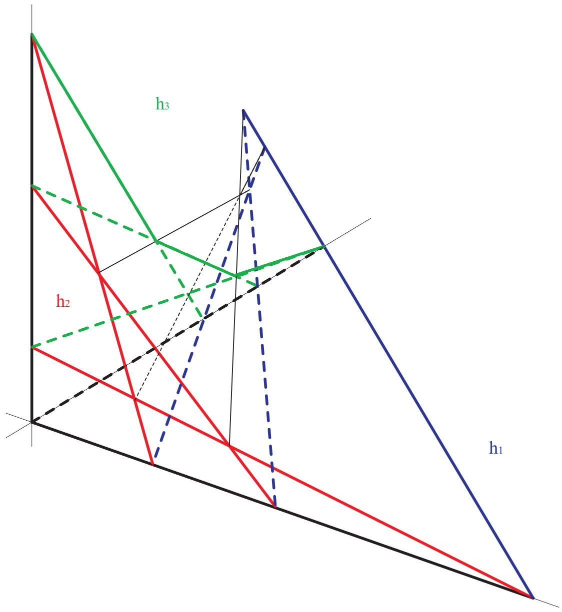

Figure 3: An arrangement combinatorially equivalent to $\mathcal{A}^*_{6,3}$

as some slightly more complicated arrangements ...... The number of external facets of $\mathcal{A}^*_{n,3}$ *is* $2(n+1)^2 + 5$.

A generalization of the arrangements $\mathcal{A}^*_{n,2}$ and $\mathcal{A}^*_{n,3}$ to dimension d yields an arrangement which cells are mainly d-cubes and therefore with an average diameter arbitrarily close to d for n large enough, see [2].

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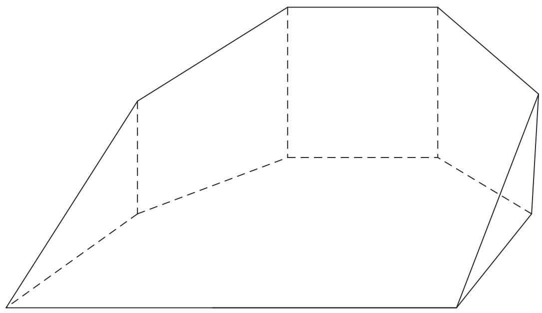

Figure 4: S_7

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