A New Class of Facet-Inducing Inequalities for a Polytope Associated with a Constrained Assignment Problem

by

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Abstract

Consider a constrained assignment problem where the side constraint consists of a single equality with 0-1 coefficients. This problem has the following integer programming formulation:

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_{ij} = 1 \qquad \text{for all } j = 1, \ldots, n \tag{2}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \text{for all } i = 1, \ldots, n \tag{3}$$

$$x_{ij} \in \{0, 1\} \qquad \text{for all } i, j = 1, \ldots, n \tag{4}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = r \tag{5}$$

where all c_{ij} are 0 or 1 and r is an integer such that $0 \leq r \leq n$.

Let $I_1 = \{1, 2, \ldots, n_1\}$, $I_2 = \{n_1 + 1, \ldots, n\}$, $J_1 = \{1, 2, \ldots, n_2\}$, $J_2 = \{n_2 + 1, \ldots, n\}$. It was shown in [**3**] that without loss of generality we can assume that $c_{ij} = 1$ if and only if $(i, j) \in (I_1) \times (J_1) \cup I_2 \times J_2$. Furthermore, in this case (5) is equivalent to

$$\sum_{(i,j) \in I_1 \times J_1} x_{ij} = r_1, \tag{6}$$

where $r_1 = (n_1 + n_2 + r - n)/2$.

Define

$P^{n,r_1}_{n_1,n_2}$ = Set of feasible solutions of (2), (3), (6) and $x_{ij} \geq 0$, $i, j = 1, \ldots, n$.

$Q^{n,r_1}_{n_1,n_2}$ = Integer hull of $P^{n,r_1}_{n_1,n_2}$.

The polyhedral structure of $Q^{n,r_1}_{n_1,n_2}$ was investigated in [**3**], where two large classes of facet-inducing inequalities of $Q^{n,r_1}_{n_1,n_2}$ were presented. In this thesis we present a new class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$.

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CHAPTER 1

Introduction

The assignment problem (AP) (also known as the minimum weight bipartite matching problem) is a classical problem with many applications in operations research, economics and computer science. The AP has been studied extensively, and many efficient algorithms have been devised for solving it [**15, 20, 17, 2**]. In particular, the assignment problem can be formulated as a linear programming problem, where the set of feasible solutions is the well-known Birkhoff polytope or the assignment polytope [**24**]. The polyhedral structure, such as the dimension, extreme points, and facets, of this polytope are all well known [**4, 7**].

In many practical applications, one is often faced with problems which can be posed as constrained assignment problems[**8**]. That is, such problems can be formulated as assignment problems together with one or more side constraints. Whereas the assignment problem is a polynomially solvable problem, these constrained assignment problems are in general hard. This thesis is concerned with a constrained assignment problem with one side constraint. This side constraint consists of an equality with 0-1 coefficients. This problem was studied in [**3**] where two large classes of facet-inducing inequalities for the associated polytope $Q^{n,r_1}_{n_1,n_2}$ were presented. Note that this polytope can be thought of as a slice of the Birkhoff polytope. In this thesis we present a new class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$.

The success or failure to obtain a complete description of polytopes associated with combinatorial optimization problems sheds some light on the computational complexity of these problems [**21, 12**]. On one hand, the non-bipartite matching problem (NBMP) is one of the few genuine combinatorial optimization problems where a complete description of its associated polytope is known [**22, 23**]. At the same time, the NBMP is also one of the few genuine combinatorial optimization problems which are solvable by a polynomial time algorithm [**11**]. On the other hand, no complete description of the polytope associated with any NP-hard problem is known.

The constrained assignment problem with 0-1 side constraint that we investigate in this thesis has the following integer programming formulation:

$$\begin{aligned}
&\min && \textstyle\sum_{i=1}^n \sum_{j=1}^n d_{ij}x_{ij} && && (7)\\
&\text{subject to} && \textstyle\sum_{i=1}^n x_{ij} = 1 && \text{for all } j = 1,\dots,n && (8)\\
& && \textstyle\sum_{j=1}^n x_{ij} = 1 && \text{for all } i = 1,\dots,n && (9)\\
& && x_{ij} \in \{0,1\} && \text{for all } i,j = 1,\dots,n && (10)\\
& && \textstyle\sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij} = r && && (11)
\end{aligned}$$

where all c_{ij} are 0 or 1 and r is an integer such that $0 \le r \le n$. An example of such problems arises in the core management of pressurized water nuclear reactors [**6, 14**]. Note that the above problem where c_{ij} are general integers is NP-hard [**10**].

Let $G = (V_1 \cup V_2, E)$, $|V_1| = |V_2| = n$, be a complete bipartite graph where each edge of G is colored either red or blue. Then any feasible solution of (8)-(11) can be interpreted as a perfect matching on G which uses exactly r red edges where an edge (i,j) of G is colored red if and only if $c_{ij} = 1$.

Define:

$P^{n,r} =$ Set of feasible solutions of (8), (9), (11) and $x_{ij} \ge 0$, $i,j = 1,\dots,n$.

$Q^{n,r} =$ integer hull of $P^{n,r}$.

Therefore, the above problem is the problem:

$$\min \sum_{i=1}^n \sum_{j=1}^n d_{ij}x_{ij} : x_{ij} \in Q^{n,r}. \tag{12}$$

Note that $P^{n,r}$ is the intersection of the Birkhoff polytope with a hyper-plane. In general $P^{n,r}$ has fractional extreme points thus $P^{n,r} \ne Q^{n,r}$. In Chapter 4, it is shown that Problem (12) is equivalent to

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_{ij} : x_{ij} \in Q_{n_1,n_2}^{n,r_1}, \tag{13}$$

where Q_{n_1,n_2}^{n,r_1}, which is defined in Chapter 4, is a special case of $Q^{n,r}$. In this thesis we present a new class of facet-inducing inequalities for Q_{n_1,n_2}^{n,r_1}.

1.1. Other Constrained Assignment Problems

Leclerc [**16**] considered the following problem known as the *2-edge restriction matching problem*. Given a bipartite graph $G = (V_1 \cup V_2, E)$, let $W \subset V_1 \cup V_2$. Find a perfect matching M such that $|M \cap \delta(W)| = 2$, where $\delta(W)$ is the set of edges incident with exactly one node in W. He presented an $O(n^{4.5})$ algorithm for this problem where n is the number of nodes of G. Next we show that this 2-edge restriction matching problem is a special case of problem (7)-(11).

Let $W \cap V_1 = I_1$, $W \cap V_2 = J_1$ and let $I_2 = V_1 \setminus I_1$, $J_2 = V_2 \setminus J_1$. Then, the 2-edge restriction matching problem reduces to the problem of finding a feasible solution of the following system:

$$\begin{array}{ll} \sum_{i=1}^{n} x_{ij} = 1 & \text{for all } j = 1, \ldots, n \\ \sum_{j=1}^{n} x_{ij} = 1 & \text{for all } i = 1, \ldots, n \\ x_{ij} \in \{0, 1\} & \text{for all } i, j = 1, \ldots, n \\ \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = 2 & \end{array}$$

where

$$c_{ij} = \begin{cases} 1 & \text{if } (i, j) \in (I_1 \times J_2) \cup (I_2 \times J_1), \\ 0 & \text{if } (i, j) \in (I_1 \times J_1) \cup (I_2 \times J_2). \end{cases}$$

Aboudi and Nemhauser [**1**] studied a constrained assignment problem with m side constraints of the form:

$$x_{2k-1,2k-1} - x_{2k,2k} = 0 \text{ for } k = 1, \ldots, m.$$

They presented a class of facet-inducing inequalities for the associated polytope, and they showed that this class provides a complete description of the associated polytope in the case $m = 1$.

CHAPTER 2

Preliminaries

In this chapter, we present basic definitions and relevant results from polyhedral theory and graph theory that will be needed in this thesis.

2.1. Polyhedral theory

Vectors (or points) $x^1, \ldots, x^k \in \mathbb{R}^n$ are said to be *linearly independent* if $\lambda_1 = \cdots = \lambda_k = 0$ is the unique solution of the system

$$\sum_{i=1}^{k} \lambda_i x^i = 0.$$

Then it easily follows that n is the maximum number of linearly independent points in $\mathbb{R}^n$. On the other hand, vectors (or points) $x^1, \ldots, x^k \in \mathbb{R}^n$ are said to be *affinely independent* if $\lambda_1 = \cdots = \lambda_k = 0$ is the unique solution of the system

$$\sum_{i=1}^{k} \lambda_i x^i = 0,$$

$$\sum_{i=1}^{k} \lambda_i = 0.$$

Note that $n+1$ is the maximum number of affinely independent points in $\mathbb{R}^n$. Clearly, the notions of affine and linear independence are related. Linear independence implies affine independence, but the converse may not be true. The next lemma establishes the exact relation between these two notions.

LEMMA 2.1.1. *$x^1, \ldots, x^k \in \mathbb{R}^n$ are affinely independent iff $x^2 - x^1, \ldots, x^k - x^1$ are linearly independent.*

A set $S \subseteq \mathbb{R}^n$ is said to be *convex* if the line segment joining any two points x^1, x^2 in S is contained entirely in S. i.e. S is convex if $\forall x^1, x^2 \in S$ it follows that

$\lambda x^1 + (1-\lambda)x^2 \in S$ for all $0 \leq \lambda \leq 1$. Given a set $S = \{x^1, x^2, \ldots, x^m\} \subseteq \mathbb{R}^n$, a point $x \in \mathbb{R}^n$ is said to be a *convex combination* of $x^1, x^2, \ldots, x^k$, if there exist nonnegative scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$, $\sum_{i=1}^k \lambda_i = 1$ such that $x = \sum_{i=1}^k \lambda_i x_i$. In particular, x is a convex combination of x^1, x^2 if x lies in the closed line segment joining x^1 and x^2. The *convex hull* of S, denoted by $\text{conv}(S)$, is the set of all points that are convex combinations of points in S. Given a set $S \subseteq \mathbb{R}_\ltimes$, the *integer hull* of S is the convex hull of the integral points in S. The following result, due to *Carathéodory*, is well known [**20**].

THEOREM 2.1.2. *Let $S \subset \mathbb{R}^n$, then every point $x \in conv(S)$ can be represented as a convex combination of $n+1$ points from S. i.e. for every point $x \in conv(S)$, there exist $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \geq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$ such that $x = \sum_{i=1}^{n+1} \lambda_i x^i$.*

A set $H \subseteq \mathbb{R}^n$ of the form $\{x \in \mathbb{R}^n,\ p^T x = \alpha_0,\ p \neq 0,\ \alpha_0 \in \mathbb{R}\}$ is called a *hyperplane*. Every hyperplane H divides the space into two halfspaces:

$$H^+ = \{x \in \mathbb{R}^n : p^T x \geq \alpha_0\}$$
$$H^- = \{x \in \mathbb{R}^n : p^T x \leq \alpha_0\}$$

It is easy to see that both halfspaces H^+, H^- are convex sets. A set $P \subseteq \mathbb{R}^n$ is a *polyhedron* if it is the intersection of a finite number of halfspaces. Equivalently, a polyhedron is the set of points that satisfy a finite number of linear inequalities; Obviously, a polyhedron is a convex set. A polyhedron $P \subseteq \mathbb{R}^n$ is bounded if there exists a positive scalar ω such that $P \subseteq \{x \in \mathbb{R}^n : -\omega \leq x_j \leq \omega \text{ for } j = 1, \ldots, n\}$. Bounded polyhedra are called *polytopes*. We say a polyhedron P is of dimension k, denoted by $dim(P) = k$, if the maximum number of affinely independent points in P is $k+1$. In addition, a polyhedron $P \in \mathbb{R}^n$ is said to be *full-dimensional* if $\dim(P)=n$. If P is not full-dimensional, then at least one of the inequalities $p^i x \leq \alpha_i$ describing P is satisfied as an equality by all points of P. The inequality $p^T x \leq \alpha_0$ is called a *valid inequality* for P if it is satisfied by all points in P. Equivalently, $p^T x \leq \alpha_0$ is a valid inequality of P if and only if P lies in the half-space $\{x \in \mathbb{R}^n : p^T x \leq \alpha_0\}$[**26, 13**].

If $p^T x \leq \alpha_0$ is a valid inequality of P, then $F = \{x \in P : p^T x = \alpha_0\}$ is called a *face* of P, and we say that (p^T, α_0) represents F. Note that F is a polyhedron and P and Φ are faces of P. A face F is said to be *proper* if $F \neq \Phi$ and $F \neq P$. The face F represented by (p^T, α_0) is nonempty if and only if $\max \{p^T x : x \in P\} = \alpha_0$. When F is nonempty, we say that the hyperplane $p^T x = \alpha_0$ supports P. If F is a proper face of P, then $\dim(F) < \dim(P)$. In particular, the dimension of F is k if the maximum number of affinely independent points that lie in F is $k + 1$.

A face F of P is called a *facet* of P if $dim(F) = dim(P) - 1$, and a face F of P is called an *edge* of P if $dim(F) = 1$. Given a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, one is interested in finding out which of the inequalities $a^i x \leq b_i$ are necessary in the description of P and which are redundant.

Facets, which have the highest dimension among all proper faces, are crucial for the description of a polyhedron in the sense that, for each facet F of P, at least one of the inequalities representing F is necessary in the description of P. If P is full-dimensional, then for each facet of P, there exists a unique (up to a multiplication by a scalar) inequality representing it. However, if P is not full-dimensional, then there are more than one inequality representing each facet.

Polyhedra can also be represented in terms of their extreme points. Given a convex set S, $\bar{x} \in S$ is said to be an *extreme point* of S, if it is impossible to represent $\bar{x}$ as a *proper* convex combination of two other points in S. i.e. $\overline{x}$ is an extreme point of S iff whenever $\overline{x} = \lambda x^1 + (1 - \lambda)x^2,\ x^1, x^2 \in S,\ 0 < \lambda < 1$, we must have $x^1 = x^2 = \overline{x}$. A polyhedron P has a finite number of extreme points. Let x^1, x^2 be two distinct extreme points of P, then x^1, x^2 are said to be *adjacent* if the line segment $[x^1, x^2]$ is an edge of P. Note that a face F of P is an *extreme point* if and only if $dim(F) = 0$. The following well-known result shows that a polytope can be expressed as the convex hull of its extreme points.

THEOREM 2.1.3 (Weyl-Minkowski). *Let P be a nonempty polytope, and let $\hat{x}^1, \ldots, \hat{x}^k$ be its extreme points, then*

$$P = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{k} \lambda_i \hat{x}^i, \text{ where } \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0 \text{ for } i = 1, \ldots, k\}$$

THEOREM 2.1.4. *Let P be a polyhedron defined by $P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$. Then dimension $P \leq n - rank(A)$.*

2.2. Graph Theory

In this section, we review basic definitions and results from graph theory.

A *graph G=(V,E)* consists of a finite set V of *vertices* and a collection E of unordered pairs of vertices called *edges*. Two or more edges that join the same pair of distinct vertices are called *parallel* edges. An edge represented by an unordered pair in which the two vertices are the same is known as a *loop*. A *simple graph* is a graph with no parallel edges and loops. The *complete graph* K_n is a graph with n vertices in which there is an edge joining every pair of vertices. A *bipartite graph*, denoted by $G = (V_1 \cup V_2, E)$, is a graph in which the set of vertices can be partitioned into two subsets V_1 and V_2 such that every edge has one end node in V_1 and the other in V_2. The *complete bipartite graph* is the graph $(V_1 \cup V_2, E)$ in which there is an edge between every vertex in V_1 and every vertex in V_2. A *walk* in G is a finite nonempty sequence $W = v_0, e_1, v_1, e_2, v_2, \ldots e_k, v_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. If the edges $e_1, e_2, \ldots, e_k$ of a walk are distinct, W is called a *trail*; in addition, if the vertices $v_0, v_1, \ldots, v_k$ are distinct, W is called a *path*. A graph $G = (V, E)$ is *connected* if for each two vertices v_i, v_j of G, there exists a path from v_i to v_j. A path is *closed* if its origin and terminus are the same. A closed path containing at least one edge is a *cycle*. The length of a path or a cycle is the number of edges in it. The following theorem characterizes bipartite graphs in terms of cycles.

THEOREM 2.2.1. *A graph is bipartite if and only if it has no odd cycle.*

Given a graph $G = (V, E)$, a *matching* M is a subset of edges no two of which are incident with a common vertex. $V(M)$ denotes the set of vertices incident to an edge in a matching M. A matching is said to be *perfect* if $V(M) = V$, that is, every node is matched.

Two of the most studied problems concerning matchings are the *maximum cardinality matching* problem and the *minimum weight matching* problem[**9**]. The maximum cardinality matching problem is concerned with finding a maximum cardinality matching in a given graph. One of its many applications is the problem of assigning students to two-person dormitory rooms. In particular, given a list of pairs of students who would be willing to share a room, this problem asks for an assignment of students to rooms so as to maximize the number of roommates who are acceptable to each other.

The minimum weight matching problem is the problem of finding a perfect matching with minimum weight in an edge-weighted complete bipartite graph. It is also known as the assignment problem since it models the following problem. Given n men, n jobs and a cost d_{ij} of man i performing job j, how should these men be assigned to jobs in order to minimize the total cost. The feasible region of the linear programming formulation of the assignment problem, known as the assignment polytope or the Birkhoff polytope, is the subject of the next chapter.

CHAPTER 3

The Assignment Polytope

In this chapter, we review the properties of the assignment polytope, which is also known as the Birkhoff polytope. In particular, we present known results concerning its dimension, facets, and extreme points.

The assignment problem is concerned with finding a minimum weight perfect matching in a bipartite graph. Given a bipartite graph $G = (V_1 \cup V_2, E)$ and $|V_1| = |V_2| = n$, let us associate with each edge $(i, j) \in E$ a weight d_{ij} and a binary variable x_{ij} such that $x_{ij} = 1$ if (i, j) belongs to a matching and $x_{ij} = 0$ otherwise. Then the assignment problem can be formulated as the following integer programming problem:

$$\min \quad \sum_{i,j=1}^{n} d_{ij} x_{ij} \tag{14}$$

$$s.t. \quad \sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i = 1, \ldots, n \tag{15}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for all } j = 1, \ldots, n \tag{16}$$

$$x_{ij} \in \{0, 1\} \text{ for all } i, j = 1, \ldots, n \tag{17}$$

As it will be shown later, condition (17) can be replaced by

$$x_{ij} \geq 0 \text{ for all } i, j = 1, \ldots, n, \tag{18}$$

since the constraint matrix of the assignment problem is *Totally Unimodular(TU)*. A matrix A is said to be TU if the determinant of every square sub-matrix of A is $0, 1$ or -1.

The assignment polytope of order n, denoted by P_n, is the set of all feasible solutions of the assignment problem, i.e. the set of all $x = (x_{ij})$ satisfying (15), (16), and (18). A non-negative $n \times n$ matrix is called a *doubly stochastic* if the sum of the entries in each row and in each column is equal to 1. The simplest example of stochastic matrices are permutation matrices. A *permutation matrix* P is a square matrix with exactly one '1' in each row and in each column (the rest of the entries being zero). Thus by considering the variables x_{ij} as the entries of an $n \times n$ matrix, P_n can be equivalently defined as the set of all doubly stochastic matrices of order n. Furthermore, there is one-to-one correspondence between feasible assignments and permutation matrices of the same order.

Let $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, n\}$. In some cases we will find it convenient to represent variables x_{ij} of a feasible assignment by $(i,j)th$ cells in the two dimensional array $I \times J$ with the values of the variables entered in their associated cells. In other cases a feasible assignment will be represented by a permutation $(\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_n))$, such that $x_{1i_1} = 1, x_{2i_2} = 1, \ldots, x_{ni_n} = 1$, and $x_{ij} = 0$ otherwise. For example, the diagonal assignment is represented by the permutation $(1, 2, \ldots, n)$.

The following result is well known[**7**]. We present a proof for completeness.

THEOREM 3.0.2. *Let P_n be the assignment polytope of order n. Then the dimension of P_n is $(n-1)^2$.*

Proof: Since the rank of the constraint matrix in (15-16) is $2n-1$, then by Theorem 2.1.4, we have that $dimP_n \leq n^2 - (2n-1) = (n-1)^2$. Next, we will show that $dimP_n \geq (n-1)^2$ by exhibiting $(n-1)^2 + 1$ affinely independent assignments in P_n thus proving the theorem.

Represent each assignment either as a permutation $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ or as a permutation matrix.

Step 1: First, let $x^1 = (1, 2, \ldots, n)$. See Table 3.1

Then by switching column 1 and column k in assignment x^1, for $k = 2, \ldots, n$. We obtain assignments: $x^{1,2}, x^{1,3}, \ldots, x^{1,n}$, where $x^{1,k} = (k, 2, 3, \ldots, k-1, 1, k+1, \ldots, n)$ for $k = 2, \ldots, n$. In this step we have a total of $n-1$ new assignment.

Step 2, Now let $x^2 = x^{1,2}$. See table 3.2

By switching column 1 and column k in x^2, for all $k \neq 3$, we obtain the assignments: $x^{2,3}, x^{2,4}, \ldots, x^{2,n}$, where $x^{2,k} = (2, k, 3, 4, \ldots, k-1, 1, k+1, \ldots, n)$ for $k = 3, \ldots, n$.

Thus generating $(n-2)$ new assignment.

1	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1

Table 3.1 x^1

0	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0
0	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	1

Table 3.2 $(x^2 = x^{1,2})$

Step 3: Now let $x^3 = x^{1,3}$. By switching column 1 and column k in x^3 for all $k \neq 3$, we obtain the assignment $x^{3,2}, x^{3,4}, x^{3,5}, \ldots, x^{3,n}$. Thus generating $(n-2)$ new assignments.

By repeating the same process as above on $x^4 = x^{1,4}, \ldots, x^n = x^{1,n}$, we obtain the assignments:

$$\begin{array}{cccccc} x^{4,2}, & x^{4,3}, & x^{4,5}, & x^{4,6}, & \dots, & x^{4,n} \\ x^{5,2}, & x^{5,3}, & x^{5,4}, & x^{5,6}, & \dots, & x^{5,n} \\ x^{n-1,2}, & x^{n-1,3}, & x^{n-1,4}, & \dots, & x^{n-1,n-2}, & x^{n-1,n} \\ x^{n,2}, & x^{n,3}, & x^{n,4}, & \dots, & x^{n,n-2}, & x^{n,n-1} \end{array}$$

Therefore, the total number of assignment generated is $1+(n-1)+(n-2)(n-1) = 1+(n-1)^2$.

Next we show that these assignments are affinely independent. Let $x^{1,1}$ denote the diagonal assignment x^1. Arrange all these $(n-1)^2+1$ assignments in the order they were generated. Thus for all these assignments we have: the ij^{th} component of assignment $x^{i,j}$ is equal to 1, while the ij^{th} component of all assignments generated before $x^{i,j}$ is equal to 0. Therefore, all these assignments are affinely independent, and the results follows.■

The following is an immediate corollary to the proof of the previous theorem.

Corollary 3.0.3. *Let P_n be the assignment polytope of order n, then $x_{ij} \geq 0$ is a facet-inducing inequality of P_n for all $i, j = 1, \dots, n$.*

From the definition of doubly stochastic matrices it immediately follows that a convex combination of permutation matrices is a doubly stochastic matrix. The converse, namely that every doubly stochastic matrix can be expressed as a convex combination of permutation matrices was independently proven by Birkhoff and Von Neumann[**25**].

Theorem 3.0.4. *(Birkhoff-Von Neumann theorem) Let A be a doubly stochastic matrix of order n, then A can be written as a convex combination of permutation matrices of order n.*

Proof: Since $A = (a_{ij})$ is a doubly stochastic matrix, all entries are non-negative. Let P^1 be a permutation matrix such that $\lambda_1 = \min\{a_{ij} : {P^1}_{ij} = 1\}$ is positive. Then $R^1 = A - \lambda_1 P^1$ is non-negative and has equal row and column sums. Furthermore, the number of zero entries of R^1 is at least one more than those of A. Repeating this argument on R^1 and noting that A has at most n^2 non-zero entries, after a finite, say k, steps we have

$$A = \lambda_1 P_1 +, \ldots, +\lambda_k P_k$$

where each P_i is a permutation matrix, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. ■

Following is an example of the decomposition process used in the above proof of Birkhoff-Von Neumann theorem.

Given the stochastic matrix

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

The minimum positive entry in A is $\frac{1}{6}$, so let λ_1 be $\frac{1}{6}$, and

$$P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then $R_1 = A - \lambda_1 P_1$ is nonnegative and has equal row and column sums.

$$R_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{5}{6} \end{pmatrix}$$

Now the minimum positive entry in R_1 is $\frac{1}{3}$, so let $\lambda_2 = \frac{1}{3}$, and

$$P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_2 = R_1 - \lambda_2 P_2$ is again nonnegative and has equal row and column sums.

$$R_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

The minimum positive entry in R_2 is $\frac{1}{2}$, so let $\lambda_3 = \frac{1}{2}$, and

$$P_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

.

After this decomposition, $A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3$, where each P_i is a permutation matrix, and each λ_i, for i=1,2,3 and $\lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1$.

Because of the Birkhoff-Von Neumann Theorem, the extreme points of the assignment polytope P_n are exactly the $n \times n$ permutation matrices [**5**]. Another way to arrive at this result is by using the notion of total unimodularity[**9**]. It is easy to prove that the constraint matrix of the assignment problem is TU. Therefore, all extreme points of the assignment polytope are integral. Because of this, condition(17) can be replaced by condition(18) in the integer programming formulation of the assignment problem.

Adjacency on the assignment polytope is characterized in the following theorem. [**18**]

THEOREM 3.0.5. *Let M_1 and M_2 be two distinct assignments. Then M_1 and M_2 are adjacent on the assignment polytope iff $(M_1 \setminus M_2) \cup (M_2 \setminus M_1)$ forms one cycle.*

Related to the notion of adjacency of extreme points is the notion of diameter of a polytope. The *distance* between a pair of extreme points in a polytope is the number of edges in a shortest path connecting these extreme points. The *diameter* of a polytope is the greatest distance between any pair of extreme points in the polytope. The following theorem establishes the diameter of the assignment polytope. [**4**]

THEOREM 3.0.6. *The assignment polytope has diameter 2.*

This theorem implies that any two distinct feasible assignments are either adjacent on the Birkhoff Polytope or are both adjacent to some feasible assignment.

The following is an example of the characterization of adjacency on the assignment polytope.

Let M_1, M_2 be the sets of edges corresponding to the assignment (2,1,3,4) and (1,2,3,4). $(M_1 \setminus M_2) \cup (M_2 \setminus M_1)$ forms one cycle, thus M_1 and M_2 are adjacent. Now let M_3 be the set of edges of assignment (1,2,4,3), then $(M_1 \setminus M_3) \cup (M_3 \setminus M_1)$ forms two cycles. Hence, M_1 and M_3 are not adjacent.

The existence of many efficient algorithms for solving the assignment problem is due, in part, to the simplicity of its polytope. In the following chapters, this motivates our polyhedral investigation of the polytope $Q^{n,r_1}_{n_1,n_2}$ obtained by intersecting the Birkhoff polytope with the hyperplane: $\sum_{(i,j)\in I_1\times J_1} x_{ij} = r_1$.

CHAPTER 4

Known Facets of $Q^{n,r_1}_{n_1,n_2}$

Recall that our problem is

$$\begin{array}{rlll} \min & \sum_{i=1}^n \sum_{j=1}^n d_{ij}x_{ij} & & (19) \\ \text{subject to} & \sum_{i=1}^n x_{ij} = 1 & \text{for all } j = 1,\dots,n & (20) \\ & \sum_{j=1}^n x_{ij} = 1 & \text{for all } i = 1,\dots,n & (21) \\ & x_{ij} \in \{0,1\} & \text{for all } i,j = 1,\dots,n & (22) \\ & \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij} = r & & (23) \end{array}$$

where all c_{ij} are 0 or 1, and r is an integer such that $0 \le r \le n$.

Let $G = (V_1 \cup V_2, E)$, $|V_1| = |V_2| = n$ be a colored complete bipartite graph, where edges are colored either red or blue. Then any feasible solution to (20)-(23) can be interpreted as a perfect matching on G which uses exactly r red edges, where an edge (i,j) is colored *red* if and only if $c_{ij} = 1$. Let us represent each edge (i,j) by a cell (i,j) in a two dimensional array $I \times J$ where $I = J = \{1, 2. \dots, n\}$.

We say that problem (19)-(23) belongs to a special case called the *partitioned case* if there exist partitions $I = I_1 \cup I_2$ and $J = J_1 \cup J_2$ such that cell (i,j) is red if and only if $(i,j) \in (I_1 \times J_1) \cup (I_2 \times J_2)$. In this partitioned case, the cells of the $I \times J$ array are partitioned into 4 blocks: $B_1 = I_1 \times J_1$, $B_2 = I_1 \times J_2$, $B_3 = I_2 \times J_2$, and $B_4 = I_2 \times J_1$. Let $|I_1| = n_1$ and $|J_1| = n_2$. Then it was shown in [**19**] that in the partitioned case, constraint (23) is equivalent to

$$\sum_{(i,j)\in B_1} x_{ij} = r_1, \qquad (24)$$

where $r_1 = (n_1 + n_2 + r - n)/2$.

It is not difficult to show that (24) is also equivalent to either one of the following constraints:

$$\sum_{(i,j)\in B_2} x_{ij} = r_2, \tag{25}$$

where $r_2 = n_1 - r_1$,

$$\sum_{(i,j)\in B_3} x_{ij} = r_3, \tag{26}$$

where $r_3 = n - n_2 - r_2$,

$$\sum_{(i,j)\in B_4} x_{ij} = r_4, \tag{27}$$

where $r_4 = n_2 - r_1$.

The following theorem was proved in [**3**].

THEOREM 4.0.7 (Alfakih et al [**3**]). *The problem of solving (19)-(23) polynomially reduces to a problem of the same type belonging to the partitioned case.*

Therefore, without loss of generality we assume that our problem belongs to the partitioned case.

Define:

$$\begin{aligned} P^{n,r_1}_{n_1,n_2} &= \text{ Set of feasible solutions of (20), (21), (24) and } x_{ij} \geq 0,\ i,j = 1,\dots,n. \\ Q^{n,r_1}_{n_1,n_2} &= \text{ integer hull of } P^{n,r_1}_{n_1,n_2}. \end{aligned}$$

THEOREM 4.0.8 (Alfakih at al [**3**]). *Suppose $r_i \geq 1$ for $i = 1,\dots,4$ and $Q^{n,r_1}_{n_1,n_2} \neq \emptyset$. Then dimension $Q^{n,r_1}_{n_1,n_2}$ = dimension $P^{n,r_1}_{n_1,n_2} = n^2 - 2n$.*

Two large classes of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$ were presented in [**3**]. Before we present these two classes we remark that the facet-inducing inequalities for the assignment polytope $x_{ij} \geq 0$ are also facet-inducing for $Q^{n,r_1}_{n_1,n_2}$. These facets are called the trivial facets of $Q^{n,r_1}_{n_1,n_2}$ [**3**].

4.1. First Class of Facet-Inducing Inequalities for $Q^{n,r_1}_{n_1,n_2}$

Facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$ of the first class are characterized by a primary defining cell (p, q), a non-empty subset of row indices K_R, and a non-empty subset of column indices K_C.

The defining cell (p, q) for the first class can be any cell in the array. Suppose it is in block B_1. Then the *defining subset of row indices* K_R must be a non-empty proper subset of I_2, and the *defining subset of column indices* K_C must be a non-empty proper subset of J_2, and together they have to satisfy $|K_R| + |K_C| = 1 + r_3$.

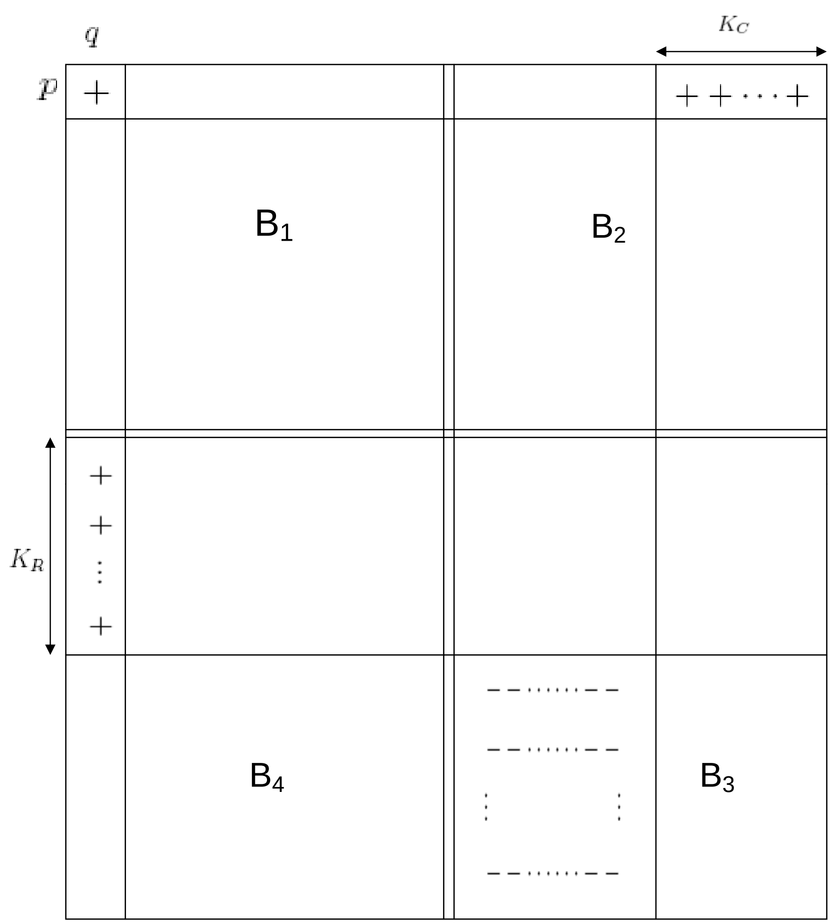

Figure 4.1 Facets of the First class

THEOREM 4.1.1 (Alfakih *et al* [**3**]). *Let (p,q) be the defining cell and K_R and K_C be the defining subsets of row and column indices selected as discussed above. Then*

$$x_{pq} + \sum_{j\in K_C} x_{pj} + \sum_{i\in K_R} x_{iq} - \sum_{i\in I_2\backslash K_R,\ j\in J_2\backslash K_C} x_{ij} \leq 1$$

is a facet-inducing inequality for $Q^{n,r_1}_{n_1,n_2}$.

Note that all coefficients in this facet-inducing inequality are $-1, 1$ *or* 0. This inequality is shown in Figure4.1 where a $+(-)$ sign in cell (i,j) means that the coefficient of x_{ij} in the inequality is $+1(-1)$.

4.2. Second Class of Facet-Inducing Inequalities for $Q^{n,r_1}_{n_1,n_2}$

Facet-inducing inequalities of the second class are characterized by two defining cells called the *primary* and the *secondary* defining cells, and by two defining subsets of row indices, and two defining subsets of column indices.

The primary defining cell, (p,q) can be any cell in the array. Suppose it is contained in block B_1, then the second class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$ exists only if the numbers r_2 and r_4 are both ≥ 2. If this condition is satisfied, the secondary defining cell (m,l) can be any cell in block B_2 or B_4 such that $l \neq q$.

Suppose that $(m,l) \in B_4$. The defining subsets of column indices $K_C, \widetilde{K}_C$ can be any nonempty disjoint proper subsets of J_2. The defining subset K_R can be any nonempty subset of $I_2\backslash\{m\}$, and the defining subset $\widetilde{K}_R$ can be any nonempty subset of I_1. These defining subsets also must satisfy $|K_C| + |K_R| = 1 + r_3$, and $|\widetilde{K}_C| + |\widetilde{K}_R| = r_4$.

With those assumptions mentioned above we have

THEOREM 4.2.1 (Alfakih *et al* [**3**]). *Let (p,q), (m,l), K_R, $\widetilde{K}_R$, K_C and $\tilde{K}_C$ be as discussed above. Then,*

$$x_{pq} + \sum_{j\in K_C} x_{pj} + \sum_{i\in K_R} x_{iq} - \sum_{i\in I_2\backslash(K_R\cup\{m\})\ j\in J_2\backslash K_C} x_{ij} - \sum_{j\in J_2\backslash(K_C\cup\widetilde{K}_C)} x_{mj}$$
$$- \sum_{i\in I_1\backslash(\widetilde{K}_R\cup\{p\},\ j\in J_2\backslash(K_C\cup\widetilde{K}_C)} x_{ij} - \sum_{i\in I\backslash(K_R\cup\widetilde{K}_R\cup\{p,m\})} x_{il} \leq 1$$

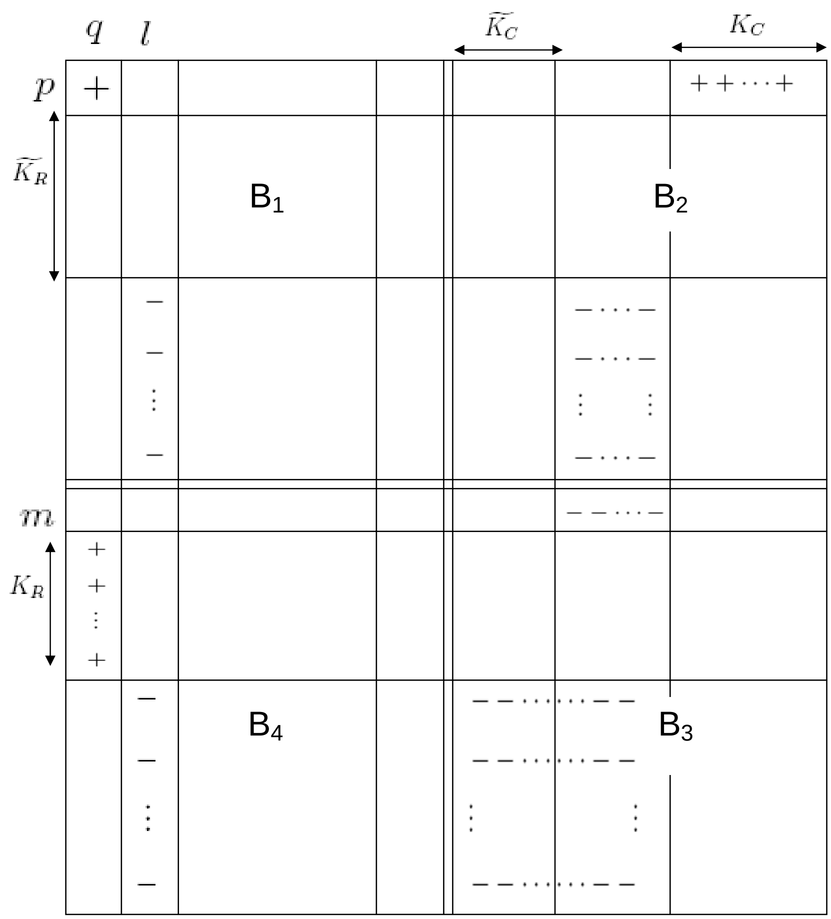

Figure 4.2 Facets of the Second class

is a facet-inducing inequality for $Q^{n,r_1}_{n_1,n_2}$.

This inequality is shown in Figure 4.2. As was pointed out in [**3**], these two classes of facets do not present a complete description of $Q^{n,r_1}_{n_1,n_2}$. In the next chapter we present a new class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$.

CHAPTER 5

New Class of Facet-Inducing Inequalities for $Q^{n,r_1}_{n_1,n_2}$

In this chapter, we present a new, i.e. a third, class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$. Facet-inducing inequalities in this new class are characterized by a *primary defining cell* (p, q), three *secondary defining cells* (l, q) , (m, q) and (m', q); and by four nonempty disjoint defining subsets of columns $\widetilde{K}_C$, $\widetilde{\widetilde{K}}_C$, $\bar{K}_C$, $\bar{\bar{K}}_C$, and by one nonempty defining subset of rows $\widetilde{K}_R$.

The primary defining cell, (p, q), can be any cell in the array. Suppose it is in block B_1, this new class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$ only exists if

$$r_2 \geq 2 \text{ and } r_4 \geq 3, \tag{28}$$

or

$$r_2 \geq 3 \text{ and } r_4 \geq 2. \tag{29}$$

If (28) holds, then the three secondary defining cells can be in Block B_4. On the other hand, if (29) holds, then the three secondary defining cells can be in Block B_2.

Suppose that (28) holds and that the secondary defining cells are in B_4. Then $\widetilde{K}_R \subset I_1 \backslash \{p\}$, $\bar{K}_C, \bar{\bar{K}}_C \subset J_1 \backslash \{q\}$ and $\widetilde{K}_C, \widetilde{\widetilde{K}}_C \subset J_2$ (see Figure 5.1). We require that these defining subsets of rows and columns satisfy

$$|\widetilde{\widetilde{K}}_C| + |\widetilde{K}_C| = r_2, \tag{30}$$

$$|\bar{\bar{K}}_C| + |\bar{K}_C| = r_1 + 1, \tag{31}$$

$$|\widetilde{K}_R| - |\widetilde{K}_C| = |\bar{K}_C| - 1. \tag{32}$$

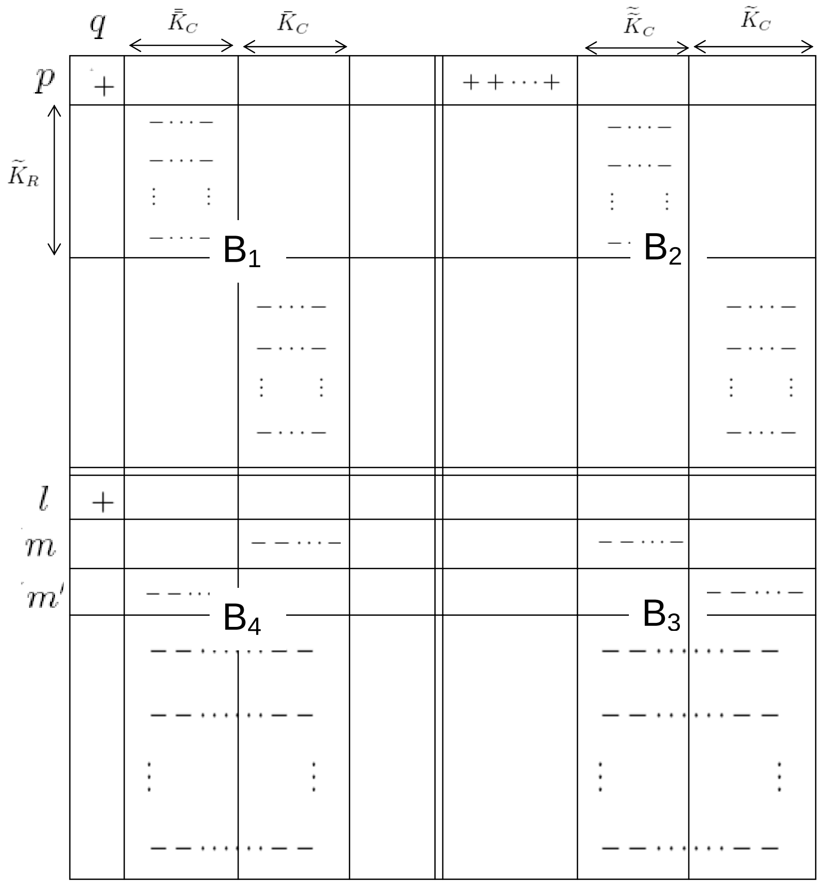

Figure 5.1 Facets of the third class

LEMMA 5.0.2. *Let* $\widetilde{K}_C$, $\widetilde{\widetilde{K}}_C$, $\bar{K}_C$, $\bar{\bar{K}}_C$, $\widetilde{K}_R$ *be as discussed above and assume* $r_4 \geq 3$ *and* $r_2 \geq 2$. *Then*

$$\begin{aligned} x_{pq} &+ \sum_{j\in J_2\setminus(\widetilde{\widetilde{K}}_C\cup\widetilde{K}_C)} x_{pj} + x_{lq} - \sum_{i\in\widetilde{K}_R,\, j\in\bar{\bar{K}}_C} x_{ij} - \sum_{i\in\widetilde{K}_R,\, j\in\widetilde{\widetilde{K}}_C} x_{ij} \\ &- \sum_{i\in I_1\setminus\widetilde{K}_R\cup\{p\},\, j\in\bar{K}_C} x_{ij} - \sum_{i\in I_1\setminus\widetilde{K}_R\cup\{p\},\, j\in\widetilde{K}_C} x_{ij} - \sum_{j\in\bar{K}_C} x_{mj} - \sum_{j\in\widetilde{\widetilde{K}}_C} x_{mj} \\ &- \sum_{j\in\bar{\bar{K}}_C} x_{m'j} - \sum_{j\in\widetilde{K}_C} x_{m'j} - \sum_{i\in I_2\setminus\{l,m,m'\},\, j\in\bar{\bar{K}}_C\cup\bar{K}_C\cup\widetilde{\widetilde{K}}_C\cup\widetilde{K}_C} x_{ij} \leq 1 \end{aligned} \tag{33}$$

is a valid inequality for $Q^{n,r_1}_{n_1,n_2}$.

Proof: For any assignment $x \in Q^{n,r_1}_{n_1,n_2}$, the sum

$$x_{pq} + x_{lq} + \sum_{j \in J_2 \backslash (\widetilde{\widetilde{K}}_C \cup \widetilde{K}_C)} x_{pj}, \tag{34}$$

is equal to $0, 1$, or 2. If (34) is equal to either 0 or 1 the lemma trivially holds. Therefore, assume that it is equal to 2. This holds when $x_{lq} = 1$ and $x_{pj_0} = 1$ for some $j_0 \in J_2 \backslash (\widetilde{\widetilde{K}}_C \cup \widetilde{K}_C)$.

For ease of notation let $B_C = J_2 \backslash (\widetilde{\widetilde{K}}_C \cup \widetilde{K}_C)$, $A_C = J_1 \backslash (\{q\} \cup \bar{\bar{K}}_C \cup \bar{K}_C)$, $A_R = I_1 \backslash (\{p\} \cup \widetilde{K}_R)$, and $B_R = I_2 \backslash \{l, m, m'\}$(see Figure 5.2). Thus it follows from (30)-(32) that

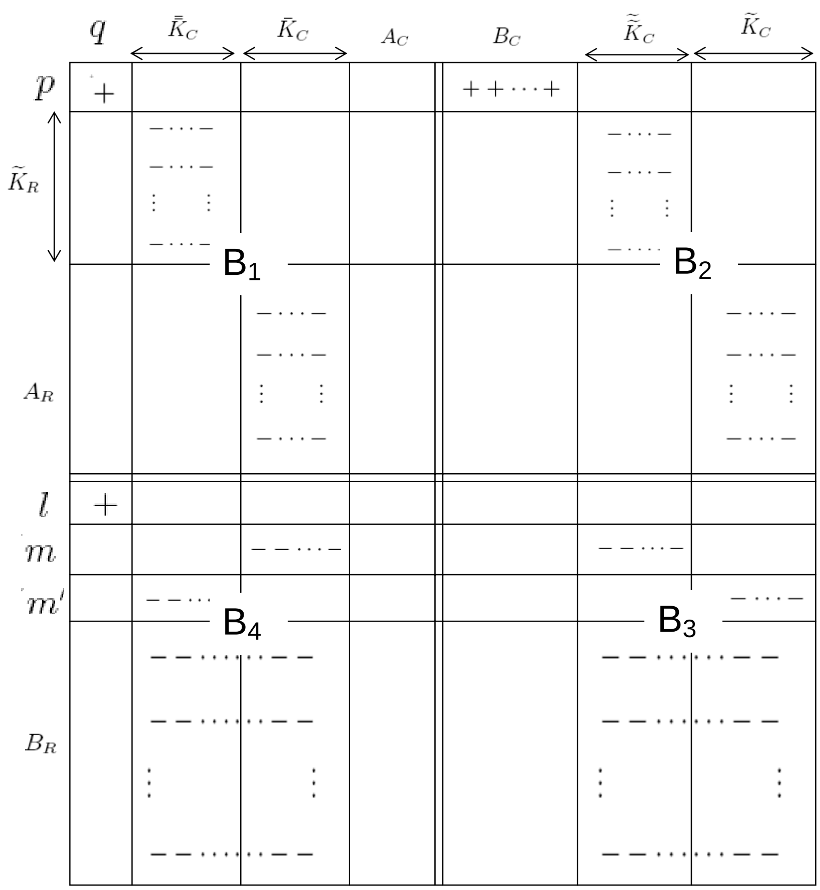

Figure 5.2

$$|B_C| = r_3, \tag{35}$$

$$|A_C| = r_4 - 2, \tag{36}$$

$$|A_R| - |\tilde{\tilde{K}}_C| = |\bar{\bar{K}}_C| - 1. \tag{37}$$

We say a sub-block $(X \times Y)$ has k *allocations* if there exists an assignment $x \in Q^{n,r_1}_{n_1,n_2}$ such that $\sum_{(i,j)\in(X\times Y)} x_{ij} = k$.

Recall that in any assignment $x \in Q^{n,r_1}_{n_1,n_2}$, blocks B_1, B_2, B_3 and B_4 must have allocations r_1, r_2, r_3 and r_4 respectively. Four cases will be considered (see Figure 5.2):

Case 1: $\displaystyle\sum_{j\in\tilde{K}_C} x_{mj} = 0$ and $\displaystyle\sum_{j\in\tilde{\tilde{K}}_C} x_{m'j} = 0$.

Recall that $x_{lq} = 1$. Since $x_{pj_0} = 1$ for some $j_0 \in B_C$, sub-block $(I_2 \times B_C)$ can have at most $r_3 - 1$ allocations. But for any assignment x in $Q^{n,r_1}_{n_1,n_2}$, Block B_3 should have r_3 allocations. Therefore, at least some cell in Block B_3 with a negative sign must have an allocation and the result follows.

Case 2: $x_{m'j_1} = 1$ for some $j_1 \in \tilde{\tilde{K}}_C$ and $\displaystyle\sum_{j\in\tilde{K}_C} x_{mj} = 0$.

If some cell (i, j) with a negative sign in Block B_3 has an allocation then we are done. So assume that sub-block $(I_2 \times B_C)$ has $r_3 - 1$ allocations. i.e., $x_{ij} = 1$ for all $j \in B_C$. Thus sub-block $((\tilde{K}_R \cup A_R) \times B_C)$ has no allocations. Now we have two subcases.

Subcase 2a: $x_{mj_2} = 1$ for some $j_2 \in B_C$.

Then the sub-block $(\{m, m'\} \times J_1\backslash\{q\})$ can not have any allocations. Therefore sub-block $(B_R \times J_1\backslash\{q\})$ must have $r_4 - 1$ allocations in any feasible assignment x. However, sub-block $(B_R \times A_C)$ can have at most $r_4 - 2$. Thus one cell with a negative sign in sub-block $(B_R \times (\bar{\bar{K}}_C \cup \bar{K}_C))$ must have an allocation and the result follows.

Subcase 2b: $\sum_{j\in B_C} x_{mj} = 0$.

The sub-block $((\widetilde{K}_R \cup A_R) \times ((\widetilde{\widetilde{K}}_C \backslash \{j_1\}) \cup \widetilde{K}_C))$ must have $r_2 - 1$ allocations. Now if any cell with a negative sign in sub-blocks $(\widetilde{K}_R \times (\widetilde{\widetilde{K}}_C \backslash \{j_1\}))$ or $(A_R \times \widetilde{K}_C)$ has an allocation, we are done. So assume that sub-blocks $(\widetilde{K}_R \times \widetilde{\widetilde{K}}_C)$ and $(A_R \times \widetilde{K}_C)$ have no allocations. Now from (30) it follows that sub-blocks $(\widetilde{K}_R \times \widetilde{K}_C)$ and $(A_R \times \widetilde{\widetilde{K}}_C)$ have $r_2 - 1$ allocations. Hence, all columns in $\widetilde{K}_C$ and $\widetilde{\widetilde{K}}_C$ have allocations. Recall that $j_1 \in \widetilde{\widetilde{K}}_C$ has an allocation in cell (m', j_1).
Now in Block B_1 if any cell in sub-block $(\widetilde{K}_R \times \bar{\bar{K}}_C)$ or sub-block $(A_R \times \bar{K}_C)$ has an allocation then we are done. Therefore assume that sub-blocks $(\widetilde{K}_R \times \bar{\bar{K}}_C)$ and $(A_R \times \bar{K}_C)$ have no allocations. This implies that sub-blocks $(\widetilde{K}_R \times (\bar{K}_C \cup A_C))$ and $(A_R \times (\bar{\bar{K}}_C \cup A_C))$ must have r_1 allocations since the column q already has an allocation in the cell $(l, q) \in B_4$. Now we have to consider two subcases depending on whether or not a column in A_C has an allocation in Block B_1.

Subcase i: $x_{ij_3} = 1$ for some $i \in \widetilde{K}_R \cup A_R$ and some $j_3 \in A_C$.

In this case, Sub-block $((\{m\} \cup B_R) \times (J_1 \backslash \{q\}))$ must have $r_4 - 1$ allocations. However, it follows from (36) that $((\{m\} \cup B_R) \times A_C)$ can have at most $r_4 - 3$ allocations. Hence, Sub-block $((\{m\} \cup B_R) \times (\bar{\bar{K}}_C \cup \bar{K}_C))$ must have at least 2 allocations. This implies that sub-block $(B_R \times (\bar{\bar{K}}_C \cup \bar{K}_C))$ must have at least 1 allocation and the result follows.

Subcase ii: Sub-block $((\widetilde{K}_R \cup A_R) \times A_C)$ has no allocations.

In this case, Sub-blocks $(\widetilde{K}_R, \bar{K}_C)$ and $(A_R \times (\bar{\bar{K}}_C))$ must have r_1 allocations. Now it follows from (32) and (37) that $|\widetilde{K}_R| - |\widetilde{K}_C| = |\bar{K}_C| - 1$ and $|A_R| - |\widetilde{\widetilde{K}}_C \backslash \{j_1\}| = |\bar{\bar{K}}_C|$. Hence, Sub-block $(\widetilde{K}_R \times \bar{K}_C)$ has $|\bar{K}_C| - 1$ allocations and Sub-block $(A_R \times \widetilde{\widetilde{K}}_C)$ has $|\bar{\bar{K}}_C|$ allocations. Therefore, the Sub-block $((\{m\} \cup B_R) \times (\bar{K}_C \cup A_C))$ must have $r_4 - 1$ allocations. However, from (36) we have

$|A_C| = r_4 - 2$. Thus the Sub-block $((\{m\} \cup B_R) \times \bar{K}_C)$ must have an allocation and the result follows.

Case 3: $x_{mj_4} = 1$ for some $j_4 \in \widetilde{K}_C$ and $\sum_{j \in \widetilde{\widetilde{K}}_C} x_{m'j} = 0$.

This case is similar to Case 2.

Case 4: $x_{mj_5} = 1$ for some $j_5 \in \widetilde{K}_C$ and $x_{m'j_6} = 1$ for some $j_6 \in \widetilde{\widetilde{K}}_C$. This case is similar to Case 2a.

THEOREM 5.0.3. *The valid inequalities of the form(33) are facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$.*

The following lemma is crucial for the proof of the above theorem. It allows us to lift a facet-inducing inequality for $Q^{n,r_1}_{n_1,n_2}$ into another facet-inducing inequality for $Q^{n+1,r_1}_{n_1,n_2}$, $Q^{n+1,r_1}_{n_1+1,n_2}$, $Q^{n+1,r_1+1}_{n_1+1,n_2+1}$, and $Q^{n+1,r_1}_{n_1,n_2+1}$.

LEMMA 5.0.4 (Alfakih et al [**3**]). *Let $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_{ij} \le a_0$ be a non trivial facet-inducing inequality for $Q^{n,r_1}_{n_1,n_2}$ and let $A^* = (a^*_{ij})$ be the $(n+1)\times(n+1)$ matrix derived from $A = (a_{ij})$ such that:*

$$A^* = \begin{pmatrix} A & A_{.j_0} \\ A_{i_o.} & 0 \end{pmatrix}$$

for any $i_0 \in \{n_1+1, \ldots, n\}$ and any $j_0 \in \{n_2+1, \ldots, n\}$ satisfying $a_{i_0j_0} = 0$, where $A_{.j_0}$ and $A_{i_0.}$ denote, respectively, the j_0th column and the i_0th row of A. Then $\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a^*_{ij}x^*_{ij} \le a_0$ is a facet-inducing inequality for $Q^{n+1,r_1}_{n_1,n_2}$ provided that it is a valid inequality for it.

Proof of Theorem 5.0.3:

Consider the problem where $n = 7$, $n_1 = 3$, $n_2 = 4$, $r_1 = 1$. Then $r_2 = 2$, $r_3 = 1$ and $r_4 = 3$ (see Figure 5.3).

Let $(p, q) = (1, 1)$, $l = 4$, $m = 5$ and $m' = 6$. Further, let $B_C = 5$, $\bar{\bar{K}}_C = \{2\}$, $\bar{K}_C = \{3\}$, $\widetilde{\widetilde{K}}_C = \{6\}$, $\widetilde{K}_C = \{7\}$ and $\widetilde{K}_R = \{2\}$. Then

$$x_{11}+x_{15}+x_{41}-x_{22}-x_{26}-x_{33}-x_{37}-x_{53}-x_{56}-x_{62}-x_{67}-x_{72}-x_{73}-x_{76}-x_{77} \le 1 \quad (38)$$

	q	$\bar{\bar{K}}_C$	$\bar{K}_C$			$\tilde{\tilde{K}}_C$	$\tilde{K}_C$
p	+				+		
$\tilde{K}_R$		-				-	
			-				-
l	+						
m			-			-	
m'		-					-
		-	-			-	-

Figure 5.3 Facets for the $Q^{7,1}_{3,4}$

is a facet-inducing inequality of $Q^{7,1}_{3,4}$, since it is a valid inequality of $Q^{7,1}_{3,4}$ by Lemma 5.0.2 and since the following 35 feasible assignments, represented as permutations, are affinely independent and satisfy (38) as an equality. Recall that dim $Q^{7,1}_{3,4} = 35$.

$x^1 = (1,5,6,2,7,3,4)$, $x^2 = (5,1,6,2,7,3,4)$, $x^3 = (5,1,6,7,2,3,4)$
$x^4 = (1,7,6,5,2,3,4)$, $x^5 = (1,7,6,2,5,3,4)$, $x^6 = (2,7,6,1,5,3,4)$
$x^7 = (5,4,6,1,2,3,7)$, $x^8 = (5,4,6,1,7,3,2)$, $x^9 = (5,4,6,2,7,3,1)$
$x^{10} = (5,2,6,1,7,3,4)$, $x^{11} = (5,3,6,1,7,2,4)$, $x^{12} = (5,3,6,1,7,4,2)$
$x^{13} = (5,3,6,7,2,1,4)$, $x^{14} = (5,3,6,1,2,7,4)$, $x^{15} = (7,4,6,1,2,3,5)$
$x^{16} = (4,7,6,1,2,3,5)$, $x^{17} = (3,7,6,1,2,4,5)$, $x^{18} = (3,7,6,1,2,5,4)$
$x^{19} = (1,7,6,3,2,5,4)$, $x^{20} = (1,7,6,4,2,3,5)$, $x^{21} = (1,7,6,2,4,3,5)$

$x^{22} = (5,7,2,6,4,3,1)$, $x^{23} = (5,7,2,6,1,3,4)$, $x^{24} = (5,7,1,6,2,3,4)$
$x^{25} = (5,7,1,3,2,6,4)$, $x^{26} = (5,7,4,3,2,6,1)$, $x^{27} = (5,7,3,1,2,6,4)$
$x^{28} = (5,7,4,1,2,6,3)$, $x^{29} = (5,7,4,1,2,3,6)$, $x^{30} = (6,7,4,1,2,3,5)$
$x^{31} = (5,6,2,1,7,3,4)$, $x^{32} = (5,7,2,1,6,3,4)$, $x^{33} = (5,7,2,1,3,6,4)$
$x^{34} = (5,3,7,1,2,6,4)$, $x^{35} = (7,3,5,1,2,6,4)$,

Next assume that $n \geq 7$ and that the assertion is true for assignments of order n. Using the lifting procedure mentioned above, we will show that it is true for assignments of order $n+1$. Symbols with $*$ refer to assignments of order $n+1$. Without loss of generality assume that the primary defining cell is $(p,q) = (1,1)$ and the three secondary defining cells are $(l,q) = (n_1+1,1)$, $(m,q) = (n_1+2,1)$ and $(m',q) = (n_1+3,1)$. Let $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_{ij} \leq 1$ be a facet-inducing inequality of the form (33) (see Figure 5.2) for $Q^{n,r_1}_{n_1,n_2}$. We will refer to this inequality as Vineq(n). Consider the problem of order $n+1$ and its corresponding array $I^* \times J^*$. Thus the $(n+1)\times(n+1)$ array $I^* \times J^*$ is obtained from the $(n \times n)$ array $I \times J$ by adding one row and one column. The new row can be added either on the top or the bottom of $I \times J$, and the new column can be added either to the left or the right of $I \times J$. Thus four cases have to be considered.

Case 1: The added row and the added column are $n+1$ and $n+1$ respectively. This corresponds to the polytope $Q^{n+1,r_1}_{n_1,n_2}$ where $r_3^* = r_3 + 1$. Let i_0 be any row of B_R and j_0 be any column of B_C(See Figure5.2). Note that $A_{i_0j_0} = 0$. Hence

$$\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} a^*_{ij}x_{ij} \leq 1, \tag{39}$$

where $A^* = (a^*_{ij})$ as defined in Lemma (5.0.4) is a facet-inducing inequality with the same defining cells and with the same defining subsets.

Case 2: The added row and the added column are 0 and $n+1$ respectively. This corresponds to the polytope $Q^{n+1,r_1}_{n_1+1,n_2}$ where $r_2^* = r_2 + 1$. In this case Vineq(n) can be lifted in two ways.

(1) Select i_0 to be any row in $\widetilde{K}_R$, and j_0 to be any column in $\widetilde{K}_C$. Using the same argument as in Case 1, it follows that (39) is a facet-inducing inequality for $Q^{n+1,r_1}_{n_1+1,n_2}$ with the same defining cells and with defining subsets $\widetilde{K}^*_R = \widetilde{K}_R \cup \{0\}$ and $\widetilde{K}^*_C = \widetilde{K}_C \cup \{n+1\}$. The other row and column defining subsets are the same.

(2) Select i_0 to be any row in A_R, and j_0 to be any column in $\widetilde{\widetilde{K}}_C$. Using the same argument as in Case 1, it follows that (39) is facet-inducing for $Q^{n+1,r_1}_{n_1+1,n_2}$ with the same defining cells and with defining subsets $A^*_R = A_R \cup \{0\}$ and $\widetilde{\widetilde{K}}^*_C = \widetilde{\widetilde{K}}_C \cup \{n+1\}$. The other row and column subsets are the same.

Case 3: The added row and the added column are 0 and 0 respectively. This corresponds to the polytope $Q^{n+1,r^*_1}_{n_1+1,n_2+1}$ where $r^*_1 = r_1 + 1$. Then Vineq(n) can be lifted in two ways.

(1) Select i_0 to be any row in $\widetilde{K}_R$, select j_0 to be any column in $\bar{K}_C$. Using the same argument as in Case 1, it follows that (39) is a facet-inducing inequality for $Q^{n+1,r^*_1}_{n_1+1,n_2+1}$ with the same defining cells and with defining subsets $\widetilde{K}^*_R = \widetilde{K}_R \cup \{0\}$ and $\bar{K}^*_C = \bar{K}_C \cup \{0\}$. The other row and column defining subsets are the same.

(2) Select i_0 to be any row in A_R, select j_0 to be any column in $\bar{\bar{K}}_C$. Using the same argument as in Case 1, it follows that (39) is a facet-inducing inequality for $Q^{n+1,r^*_1}_{n_1+1,n_2+1}$ with the same defining cells and with defining subsets $A^*_R = A_R \cup \{0\}$ and $\bar{\bar{K}}^*_C = \bar{\bar{K}}_C \cup \{0\}$. The other row and column defining subsets are the same.

Case 4: The added row and the added column are $n+1$ and 0 respectively. This corresponds to the polytope $Q^{n+1,r_1}_{n_1,n_2+1}$ where $r^*_4 = r_4 + 1$. Select i_0 to be any row in B_R and select j_0 be any column in A_C. Using the same argument as in Case 1, it follows that (39) is a facet-inducing inequality for $Q^{n+1,r_1}_{n_1,n_2+1}$ with defining cells and with defining subsets $B^*_R = B_R \cup \{n+1\}$ and $A^*_C = A_C \cup \{0\}$. The other row and column indices are the same.

To complete the proof we need to show that every valid inequality of form (33) for the problem of order $n+1$ can be obtained by lifting some valid inequality of order n. To this end, consider the valid inequality of form (33) for the problem of order $n+1$ with defining cells $(p,q) = (1,1), (l,q) = (n_1+1,1)$, $(m,q) = (n_1+2,1)$ and $(m',q) = (n_1+3,1)$. Since $n+1 \geq 8$, one of the following must hold: $r_1^* \geq 2$, $r_2^* \geq 3$, $r_3^* \geq 2$,or $r_4^* \geq 4$. This is the case since $r_1 \geq 1$ or $r_2 \geq 2$ or $r_3 \geq 1$ or $r_4 \geq 3$.

Now suppose that $r_3^* \geq 2$. Since $|{B_C}^*| = r_3^*$ we have $|{B_C}^*| \geq 2$. Let j_0 be any column of B_C^*. Since $|B_R^*| = n - n_1 - 3 = r_3^* + r_4^* - 3$, and since $|r_4^*| \geq 3$ it follows that $|B_R^*| \geq 2$. Let i_0 be any row of B_R^* and let j_0 be any row of B_C^*. Consider the problem of order n associated with array $I^* \setminus \{i_0\} \times J^* \setminus \{j_0\}$. Then the inequality obtained by deleting i_0 from B_R^* and j_0 from B_C^* is of the form (33) with the same defining cells and with defining subsets $B_R = B_R^* \setminus \{i_0\}$ and $B_C = B_C^* \setminus \{j_0\}$. So it is a valid inequality for the problem of order n. Furthermore, the valid inequality of the problem of order $n+1$ can be lifted from this valid inequality as in Case 1.

Other cases are similar and the result follows.

CHAPTER 6

Conclusion

Given the following linear integer programming problem:

$$\min \quad \sum_{i=1}^n \sum_{j=1}^n d_{ij}x_{ij} \tag{40}$$

$$\text{subject to} \quad \sum_{i=1}^n x_{ij} = 1 \qquad \text{for all } j = 1, \ldots, n \tag{41}$$

$$\sum_{j=1}^n x_{ij} = 1 \qquad \text{for all } i = 1, \ldots, n \tag{42}$$

$$x_{ij} \in \{0, 1\} \qquad \text{for all } i, j = 1, \ldots, n \tag{43}$$

$$\sum_{(i,j)\in I_1\times J_1} x_{ij} = r_1 \tag{44}$$

where $I_1 = \{1, 2, \ldots, n_1\}$ and $J_1 = \{1, 2, \ldots, n_2\}$. Let

$P^{n,r_1}_{n_1,n_2} =$ Set of feasible solutions of (41), (42), (44) and $x_{ij} \geq 0$ for all $i, j = 1, \ldots, n$.

$Q^{n,r_1}_{n_1,n_2} =$ integer hull of $P^{n,r_1}_{n_1,n_2}$.

In this thesis we presented a new class of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$. Two classes of facet-inducing inequalities for $Q^{n,r_1}_{n_1,n_2}$ we presented in [**3**]. Unfortunately, these 3 classes do not provide a complete description of $Q^{n,r_1}_{n_1,n_2}$ for general n, since the following fractional point $\hat{x} = (\hat{x}_{ij})$ defined by

$\hat{x}_{11} = \hat{x}_{15} = \hat{x}_{24} = \hat{x}_{27} = \hat{x}_{35} = \hat{x}_{38} = \hat{x}_{43} = \hat{x}_{44} = \hat{x}_{56} = \hat{x}_{57} =$

$\hat{x}_{62} = \hat{x}_{66} = \hat{x}_{73} = \hat{x}_{78} = \hat{x}_{81} = \hat{x}_{82} = \frac{1}{2}, \hat{x}_{ij} = 0, otherwise,$

is an extreme point of $P^{n,r_1}_{n_1,n_2}$ which satisfies all the inequalities of these three classes. This is yet another proof that whereas the assignment, or the Birkhoff, polytope P_n can be easily described as the intersection of n^2 inequalities of the form $x_{ij} \geq 0$ for all $i, j = 1, \ldots, n$, the polytope obtained by intersecting P_n with a simple hyperplane of the form (44) is not likely to have a simple description in terms of facets-inducing inequalities. Nonetheless, these 3 known classes of facet-inducing inequalities can be used in a branch-and-cut algorithm for solving Problem (40)-(44).

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