CO 227 – Introduction to Optimization
Lecture 1

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Winter 2026

Course information

- Instructor: Professor Henry Wolkowicz
- MC 6312; hwolkowicz@uwaterloo.ca
- Homework (HW15%), Midterm (M30%), Final Exam (F55%)
- HW: best 5 of 6 assignments. (Late assignments are not accepts; no extensions.)
- Final Grade: .15 HW + max(.85 F, .55 F + .30 M).
- Join Piazza! https://piazza.com/uwaterloo.ca/winter2026/co227
- Office hour: Thursday 1:30PM-2:30PM; MC6312 and/or clickable zoom link

Optimization in the Real World

> *"Nothing takes place in the world whose meaning is not that of some maximum or minimum"*
> *Leonard Euler, 1744 book.*

Key idea: Optimization is about making the best use of scarce resources.

- sending a rocket ship to Mars (with limited fuel);
- airline optimal scheduling of planes and flight and ground crews, while maximizing income;
- allocating study and assignment time for different subjects;
- Choosing groceries, healthy diet, within a budget;
- Packing luggage to maximize items within weight/size limits.
- drugs/CT scans/military/ etc...

Example 1: Fruit Stand Problem

Suppose we are selling apples and bananas at a stand. Apples sell for \$2 per kilogram, and bananas sell for \$1.5 per kilogram. Our stand holds up to 75 kilogram of fruits. Also, there are only 4 square metres of shelf space. Each kilogram of apples/bananas takes up roughly 0.08/0.05 square metres of shelf space, respectively. How much of each fruit should we stock to maximize the total sales?

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Objective: "What are we optimizing?"

Constraints: "What limits the variable values?"

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Constraints: Weight constraint: $x_a kg + x_b kg \leq 75kg$

Space constraint: $0.08\frac{m^2}{kg} x_a kg + 0.05\frac{m^2}{kg} x_b kg \leq 4m^2$

Non-negativity: $x_a, x_b \geq 0$

Linear Programming Formulation

Let $x_1 \cong x_a, x_2 \cong x_b$;
use affine functions $f(x) = a^T x - b, x \in \mathbb{R}^n, b \in \mathbb{R}$

$$\begin{array}{crl} \max & 2x_1 + 1.5x_2 & \\ \text{subject to} & x_1 + x_2 & \leq 75 \\ & 0.08x_1 + 0.05x_2 & \leq 4 \\ & x_1, x_2 & \geq 0 \end{array}$$

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- E.g. $(x_1, x_2) = (10, 60)$ is a feasible solution, with its objective value $2x_1 + 1.5x_2 = 20 + 90 = 110$.

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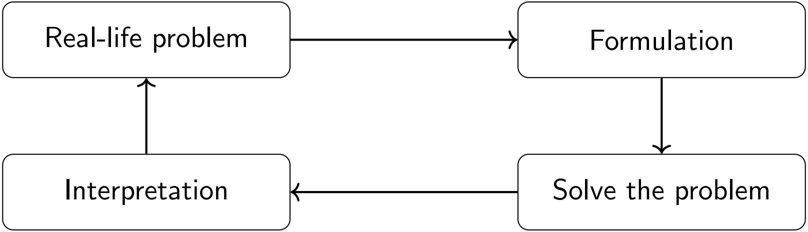
- We can draw the feasible region in $\mathbb{R}^2$.

Optimization Workflow



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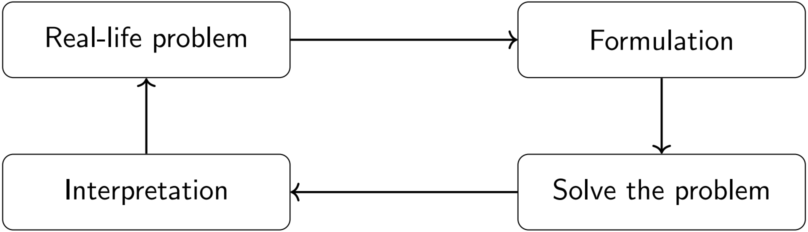

In this course:

- Formulations and models of real world problems using linear functions.
- The simplex algorithm for solving linear programs.
- Theory, geometry, and duality that lies behind the algorithm.
- Strategies in solving integer programs.
- Applications to graph theory problems.

What is a Linear Program?

Definition

Using variables $x_1, \ldots, x_n$, a function is *affine* if it has the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n + b$ for some constants $a_1, \ldots, a_n, b$. If $b = 0$, then this is *linear*.

Example: $7x_1 + 5x_2 - 3x_3 + 9$, $x_1^2 + x_2$, $5x_1x_2$, $5x_1 - 2x_2$

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A **Linear Program (LP)** consists of:

- An objective function $\max f(x)$ or $\min f(x)$, where $f(x)$ is affine
- Finitely many constraints of the form $g(x) \leq b$, $g(x) = b$, $g(x) \geq b$, for some linear function $g(x)$ and constant b
- Also, finitely many variables

Note: Constraints with strict inequalities (e.g. $3x_1 + 2x_2 < 5$) are treated differently.
(E.g., a constraint with $\log(x)$ needs a $x > 0$ constraint. Nonlinear functions such as $\log$ are used at the end of the course.)

LP formulation

A company makes 4 types of products, each requiring time on two different machines and two types of labour. The amount of machine time and labour needed to produce one unit of each product along with its sale price are summarized in the following table.

Product	**M1**	**M2**	**Skilled**	**Unskilled**	**Price**
1	11	4	8	7	300
2	7	6	5	8	260
3	6	5	5	7	220
4	5	4	6	4	180

Each month, the company can use up to 700 hours on machine 1, and 500 hours on machine 2, with no cost. The company can hire up to 600 hours of skilled labour at $8 per hour, and up to 650 hours of unskilled labour at $6 per hour. How should the company operate to maximize their monthly profit?

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