Chapter 1

STRENGTHENED SEMIDEFINITE PROGRAMMING RELAXATIONS FOR THE MAX-CUT PROBLEM

Miguel F. Anjos
and Henry Wolkowicz
Combinatorics & Optimization,
University of Waterloo,
Ontario,Canada N2L 3G1
anjos@stanfordalumni.org
hwolkowi@orion.math.uwaterloo.ca

Abstract In this paper we summarize recent results on finding tight semidefinite programming relaxations for the Max-Cut problem and hence tight upper bounds on its optimal value. Our results hold for every instance of Max-Cut and in particular we make no assumptions on the edge weights. We present two strengthenings of the well-known semidefinite programming relaxation of Max-Cut studied by Goemans and Williamson. Preliminary numerical results comparing the relaxations on several interesting instances of Max-Cut are also presented.

Keywords: Semidefinite Programming, Discrete Optimization, Lagrangian Relaxation, Max-Cut Problem, Semidefinite Relaxations, Cut Polytope, Metric Polytope.

1. INTRODUCTION

The Max-Cut problem (MC) is a combinatorial optimization problem on undirected graphs with weights on the edges. Given such a graph, the problem consists in finding a partition of the set of vertices into two parts so as to maximize the sum of the weights on the edges that have one end in each part of the partition. It is well known that MC is an NP-hard problem [17], although some special cases can be solved efficiently, for example those where the graph is not contractible to K_5, the complete graph on five vertices [8]. We consider the general case where the graph is complete and we require no restriction on the type of edge weights. So, in particular, negative or zero edge weights are permitted.

Following [19], we can formulate the MC problem as follows. Let the given graph G have vertex set $\{1,\ldots,n\}$ and let it be described by its weighted adjacency matrix $A(G)$. We tacitly assume that the graph in question is complete (if not, missing edges can be given weight 0 to complete the graph). Let L denote the *Laplacian matrix* associated with the graph; hence $L := \mathrm{Diag}\,(A(G)\cdot e) - A(G)$, where the linear operator Diag returns a diagonal matrix with diagonal formed from the vector given as its argument, and e denotes the vector of all ones. Let the vector $v \in \{\pm 1\}^n$ represent any cut in the graph via the interpretation that the sets $\{i : v_i = +1\}$ and $\{i : v_i = -1\}$ form a partition of the vertex set of the graph. Then we can formulate MC as:

$$\text{(MC1)} \qquad \begin{array}{rll} \mu^* = & \max & \frac{1}{4} v^T L v \\ & \text{s.t.} & v_i^2 = 1, i = 1,\ldots,n. \end{array}$$

Define $Q := \frac{1}{4}L$ and consider the change of variable $X := vv^T, v \in \{\pm 1\}^n$. Then $v^T L v = \mathrm{trace}\, LX$ and in fact an equivalent formulation for MC is:

$$\text{(MC2)} \qquad \begin{array}{rll} \mu^* = & \max & \mathrm{trace}\, QX \\ & \text{s.t.} & \mathrm{diag}\,(X) = e \\ & & \mathrm{rank}(X) = 1 \\ & & X \succeq 0, X \in \mathcal{S}^n \end{array}$$

where diag denotes the linear operator that returns a vector with the diagonal elements of its matrix argument, $X \succeq 0$ denotes that X is positive semidefinite and $\mathcal{S}^n$ denotes the space of symmetric $n \times n$ matrices. Note that μ^* denotes the optimal value of MC. Our interest is in finding tight (upper) bounds on μ^* using semidefinite programming.

The best previously known bound was proved by Goemans and Williamson [11]. They considered the relaxation of MC obtained by dropping the constraint $\mathrm{rank}(X) = 1$ from MC2. The resulting problem is a semidefinite programming problem (SDP) and its solution can be approximated in polynomial-time to within any given (fixed) precision using interior-point methods. This follows from the seminal work of Nesterov and Nemirovskii much of which is summarized in [21]. They also implemented the first interior-point method for SDP in [20]. Independently, Alizadeh extended interior-point polynomial-time algorithms from linear programming to SDP and studied applications to discrete optimization [1, 2]. Non-smooth optimization methods for solving semidefinite programming problems have also been proposed (see e.g. [13]).

Let us denote the feasible set of MC2, less the rank constraint, by $\mathcal{E}_n$, so

$$\mathcal{E}_n := \{X \in \mathcal{S}^n : \mathrm{diag}\,(X) = e, X \succeq 0\}.$$

With this notation, Goemans and Williamson consider the semidefinite programming problem SDP1:

$$\text{(SDP1)} \qquad \begin{array}{rcll} \nu_1^* & = & \max & \operatorname{trace} QX \\ & & \text{s.t.} & X \in \mathcal{E}_n, \end{array}$$

and give an algorithm that uses the optimal solution of SDP1 to find a cut whose weight is guaranteed to be within 14% of the weight of the maximum cut. Since SDP1 is a relaxation of MC, ν_1^* is a bound on μ^* and the analysis of Goemans and Williamson implies that it is at most 14% above the value of μ^*. More precisely:

$$\mu^* \geq \alpha\, \nu_1^*,$$

where $\alpha := \min_{0 \leq \theta \leq \pi} \frac{2}{\pi} \frac{\theta}{1-\cos\theta} \approx 0.87856$, and since $\frac{1}{\alpha} \leq 1.14$, we have that $\nu_1^* \leq 1.14\, \mu^*$. It is important to note that this result requires the assumption that there are no negative edge weights.

Observe that $\mathcal{E}_n$ is a convex relaxation of the feasible set of problem MC2 since we dropped the rank constraint. Other convex relaxations of the feasible set of MC2 have been proposed in the literature. The smallest convex set containing all the matrices X which are feasible for MC2 is their convex hull. This relaxation is known as the *cut polytope*:

$$C_n := \text{Conv}\{X : X = vv^T, v \in \{\pm 1\}^n\}.$$

Optimizing $\operatorname{trace} Q\, X$ over C_n would yield exactly μ^*, but an efficient description of the cut polytope is not known. Therefore one approach is to seek good relaxations of C_n, that is, to approximate the cut polytope by a larger convex T such that

- $C_n \subseteq T$;
- $T \backslash C_n$ is as "small" as possible; and
- we can optimize $\operatorname{trace} QX$ over T efficiently.

Note that $\mathcal{E}_n$ is such a set.

A well-known relaxation of the cut polytope is the *metric polytope* M_n, defined as the set of all matrices satisfying the triangle inequalities:

$$\begin{aligned} M_n := \{X \in \mathcal{S}^n \,:\, & \operatorname{diag}(X) = e, \text{ and} \\ & X_{ij} + X_{ik} + X_{jk} \geq -1, X_{ij} - X_{ik} - X_{jk} \geq -1, \\ & -X_{ij} + X_{ik} - X_{jk} \geq -1, -X_{ij} - X_{ik} + X_{jk} \geq -1, \\ & \forall\, 1 \leq i < j < k \leq n\}. \end{aligned}$$

The triangle inequalities model the fact that for any three mutually connected vertices in the graph, it is only possible to cut either zero or two of the edges joining them. In fact, the triangle inequalities are sufficient to describe the cut polytope for graphs with less than five vertices, i.e. $C_n = M_n$ for $n \leq 4$;

however, $C_n \subsetneq M_n$ for $n \geq 5$, see for example [10]. Nonetheless, if the graph G is not contractible to K_5, the complete graph on 5 vertices, then Barahona [9] proved that the linear programming problem

$$\begin{array}{ll} \max & \operatorname{trace} Q\,X \\ \text{s.t.} & X \in M_n \end{array}$$

has optimal value equal to μ^*.

A natural idea is to consider optimizing over the set $\mathcal{E}_n \cap M_n$. This was proposed in [22]. Adding triangle inequalities to SDP1 is considered in e.g. [12, 14, 15]. However, it is not the case that adding a certain subset of the triangle inequalities will improve every instance of MC. Furthermore, Karloff [16] has shown that it is impossible to improve the performance guarantee of the approximation algorithm of Goemans and Williamson by adding valid linear inequalities to SDP1.

In this paper we summarize recent results on finding tight SDP relaxations of the cut polytope and hence better bounds on the optimal value of MC. Let us note that these results hold for every instance of MC; in particular, we make no assumptions on the edge weights. In the next section we present two strengthenings of SDP1. The first one, denoted SDP2, is proved to give a tighter bound than SDP1 whenever the optimal value of SDP1 is not equal to the optimal value of MC [4]. The second one, denoted SDP3, is a further tightening of SDP2 and its feasible set corresponds, in a way that is described later, to a convex set $F_n \subseteq \mathcal{S}^n$ that is larger than C_n but nonetheless is strictly contained in $\mathcal{E}_n \cap M_n$ [5]. In the last section we report some preliminary numerical results comparing these relaxations on several interesting graphs.

2. THE STRENGTHENED SDP RELAXATIONS

The derivation of SDP2 begins by adding to MC2 the *redundant* quadratic constraints

- $X^2 - nX = 0$ and
- $X \circ X = E$, where $\circ$ denotes the Hadamard (elementwise) product of matrices and E denotes the matrix of all ones.

One motivation for adding these redundant constraints to the formulation MC2 is that we will be using Lagrangian duality to obtain new SDP relaxations. To obtain tighter bounds we therefore want the duality gap incurred in the process to be as small as possible. The interest in these redundant quadratic constraints comes from the results in [6, 7] where the addition of redundant constraints of this type was shown to guarantee strong duality for certain problems where duality gaps can exist.

The validity of the constraint $X^2 - nX = 0$ follows from the observation that $X^2 = vv^T vv^T$ and $v^T v = n$ for all $v \in \{\pm 1\}^n$. Since we can simultaneously diagonalize X and X^2, the eigenvalues of X must satisfy $\lambda^2 - n\lambda = 0$, which implies that the only eigenvalues of X are 0 and n. This shows that the constraint $X \succeq 0$ becomes redundant and may be removed. Moreover, since the diagonal constraint implies that the trace of X is n, we conclude that X must be rank-one and the rank constraint can also be removed. Finally, if $v \in \{\pm 1\}^n$ then all the entries of $X = vv^T$ are ± 1 and clearly $X \circ X = E$ holds.

The resulting problem MC3 is thus another formulation of MC:

$$
\text{(MC3)} \qquad \begin{array}{rll} \mu^* = & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X) = e \\ & & X \circ X = E \\ & & X^2 - nX = 0. \end{array}
$$

Taking the Lagrangian dual of MC3 and then the dual of the dual yields the relaxation SDP2. To express this relaxation let us define $t(i) := \frac{i(i+1)}{2}$ and

$$
T(i,j) := \begin{cases} t(j-1) + i, & \text{if } i \le j \\ t(i-1) + j, & \text{otherwise.} \end{cases}
$$

Further define sMat to be the linear operator that, given a vector $x \in \Re^{t(n)}$, returns a matrix $X \in \mathcal{S}^n$ obtained by filling in columnwise the upper triangular part of X with the $t(n)$ components of x and completing the strictly lower triangle by symmetry. Then SDP2 is:

$$
\text{(SDP2)} \qquad \begin{array}{rll} \nu_2^* = & \max & \operatorname{trace} Q \operatorname{sMat}(x) \\ & \text{s.t.} & \operatorname{diag}(Y) = e \\ & & Y_{0,t(i)} = 1, i = 1, \ldots, n \\ & & Y_{0,T(i,j)} = \frac{1}{n} \sum\limits_{k=1}^{n} Y_{T(i,k),T(k,j)}, \\ & & \qquad \forall\, i,j \text{ s.t. } 1 \le i < j \le n \\ & & Y \succeq 0, \\ & & Y = \begin{pmatrix} 1 & x^T \\ x & \bar{Y} \end{pmatrix} \in \mathcal{S}^{t(n)+1}, \end{array}
$$

where Y is partitioned so that x is its first column less the top left element and $\bar{Y} \in \mathcal{S}^{t(n)}$. Note that $T(i,i) = t(i)$ and that we index the rows and columns of Y by $0, 1, \ldots, t(n)$.

The following results for SDP2 are proved in [4].

Lemma 1 *[4, Lemma 3.2] Suppose that* $Y = \begin{pmatrix} 1 & x^T \\ x & \bar{Y} \end{pmatrix}$ *is feasible for SDP2. Then*

$$
\operatorname{sMat}(x) \succeq 0
$$

and so $\mathrm{sMat}\,(x)$ *is feasible for SDP1.* ∎

Lemma 2 *[4, Theorem 3.1] The optimal values satisfy*

$$\nu_2^* \leq \nu_1^*.$$

∎

It turns out that SDP2 can be further strengthened by the following observation. Consider again the rank-one matrices $X = vv^T$, $v \in \{\pm 1\}^n$. We know that these matrices X have all their entries equal to ± 1. Hence if x satisfies $X = \mathrm{sMat}\,(x)$, then the matrix

$$Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix}$$

is feasible for SDP2 and has all its entries equal to ± 1.

From SDP2, consider the constraints

$$Y_{0,T(i,j)} = \frac{1}{n} \sum_{k=1}^{n} Y_{T(i,k),T(k,j)}, \ \ \forall\, 1 \leq i < j \leq n.$$

For all the above rank-one matrices Y, the entry $Y_{0,T(i,j)}$ is equal to 1 in magnitude. The corresponding constraint says that it must be equal to the average of n entries in the block $\bar{Y}$. But each of these n entries has magnitude at most 1, so for equality to hold, they must all have magnitude equal to 1, and in fact they must all equal $Y_{0,T(i,j)}$.

We can state this observation in a different way. If Y is rank-one, then the block $\bar{Y} = xx^T$, and therefore $Y_{T(i,k),T(k,j)} = x_{T(i,k)} x_{T(k,j)} = v_i v_k \cdot v_k v_j$. But if $v_k^2 = 1$, then $Y_{T(i,k),T(k,j)} = v_i v_j = X_{ij} = Y_{0,T(i,j)}$.

This discussion leads us to define the relaxation SDP3:

$$(\mathrm{SDP3}) \qquad \begin{array}{rll} \nu_3^* = & \max & \mathrm{trace}\, Q\, \mathrm{sMat}\,(Z_{1:t(n),0}) \\ & \text{s.t.} & \mathrm{diag}\,(Z) = e \\ & & Z_{0,t(i)} = 1,\, i = 1, \ldots, n \\ & & Z_{0,T(i,j)} = Z_{T(i,k),T(k,j)}, \\ & & \quad k = 1, \ldots, n,\ \forall\, 1 \leq i < j \leq n \\ & & Z \succeq 0,\, Z \in \mathcal{S}^{t(n)+1}. \end{array}$$

The linear operator sMat allows us to map the feasible set of SDP3 into $\mathcal{S}^n$ by defining

$$F_n := \{X \in \mathcal{S}^n \ : X = \mathrm{sMat}\,(Z_{1:t(n),0}),\, Z \text{ feasible for SDP3}\}.$$

Since the feasible set of SDP3 is convex and compact, and since F_n is the image of that feasible set under a linear transformation, it follows that F_n is also convex and compact. Furthermore

Lemma 3 *[5, Lemma 4.1]* $C_n \subseteq F_n$. ■

Clearly, every Z feasible for SDP3 is feasible for SDP2. Therefore, by Lemma 1 above, we have the inclusion

Corollary 1 *[5, Corollary 4.1]* $F_n \subseteq \mathcal{E}_n$. ■

It turns out that the matrices in F_n also satisfy the triangle inequalities.

Theorem 1 *[5, Theorem 4.1]* $F_n \subseteq M_n$. ■

By the result of Barahona in [9, Theorem 6.4], this guarantees that the optimal value of SDP3 is exactly equal to the optimal value of MC if the graph G is not contractible to K_5.

In fact, appropriate examples are given in [5] to prove the following stronger result:

Theorem 2 *[5, Corollary 6.1]* $C_n \subsetneq F_n \subsetneq \mathcal{E}_n \cap M_n$ *for* $n = 5$, *and hence for* $n \geq 5$. ■

The examples are:

Example 1 *Consider the matrix*

$$X = \begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}.$$

It is known that $X \notin C_5$ *[18]. However, there exists a* 16×16 *matrix* P^* *which is feasible for SDP3 and such that* $\mathrm{sMat}\,(P^*_{1:15,0}) = X$. *Hence* $X \in F_5$.

Example 2 *Consider the matrix*

$$X = \begin{pmatrix} 1 & -0.65 & -0.65 & -0.65 & 0.93 \\ -0.65 & 1 & 0.3 & 0.3 & -0.65 \\ -0.65 & 0.3 & 1 & 0.3 & -0.65 \\ -0.65 & 0.3 & 0.3 & 1 & -0.65 \\ 0.93 & -0.65 & -0.65 & -0.65 & 1 \end{pmatrix}.$$

It is easy to check that $X \in \mathcal{E}_5 \cap M_5$. *However, there is no matrix P feasible for SDP3 such that* $\mathrm{sMat}\,(P_{1:15,0}) = X$. *Hence* $X \notin F_5$.

3. NUMERICAL RESULTS

The relaxations SDP1, SDP2 and SDP3 were compared for several interesting problems using the software package SDPPACK (version 0.9 Beta) [3]. It turns out that the dimension of the variables Y and Z of SDP2 and SDP3 respectively can be slightly reduced by the fact that their feasible sets as described above have no strictly feasible point and thus can be projected onto an appropriate face of the positive semidefinite cone. More details can be found in [4]. For completeness we also solved the linear relaxation over the metric polytope:

$$\begin{array}{ll} \max & \operatorname{trace} Q\,X \\ \text{s.t.} & X \in M_n. \end{array}$$

This relaxation is easily formulated as an LP and we solved it using the Matlab solver LINPROG. The results are summarized in the following table. The value ρ equals the value of the optimal cut divided by the bound, and R.E. denotes the relative error with respect to the optimal cut.

We report numerical results for the following problems:

1. The first line of results corresponds to solving the three SDP relaxations for a 5-cycle with unit edge-weights.

2. The second line corresponds to the complete graph on 5 vertices with unit edge-weights on all edges except one, which is assigned weight zero.

3. The third line corresponds to the complete graph on 5 vertices with unit edge-weights. In this example, none of the four SDP relaxations attains the MC optimal value, and in fact they are not distinguishable numerically. Only the polyhedral relaxation M_n gives a noticeably weaker bound.

4. The fourth line corresponds to the graph defined by the weighted adjacency matrix

$$A(G) = \begin{pmatrix} 0 & 1.52 & 1.52 & 1.52 & 0.16 \\ 1.52 & 0 & 1.60 & 1.60 & 1.52 \\ 1.52 & 1.60 & 0 & 1.60 & 1.52 \\ 1.52 & 1.60 & 1.60 & 0 & 1.52 \\ 0.16 & 1.52 & 1.52 & 1.52 & 0 \end{pmatrix}.$$

   This problem is interesting because it shows a significant difference between SDP3 and all the other relaxations; in this case, SDP3 is the only relaxation that attains the MC optimal value.

Table 1.1 Numerical comparison of relaxations for selected test problems

Graph	μ^*	*SDP1 bound*	*SDP2 bound*	M_n *bound*	$\mathcal{E}_n \cap M_n$ *bound*	*SDP3 bound*
C_5	4	4.5225 ρ = 0.8845 R.E.: 13.06%	4.2889 ρ = 0.9326 R.E.: 7.22%	4.0000 ρ = 1.0000 R.E.: 0%	4.0000 ρ = 1.0000 R.E.: 0%	4.0000 ρ = 1.0000 R.E.: 0%
$K_5 \backslash e$	6	6.2500 ρ = 0.9600 R.E.: 4.17%	6.1160 ρ = 0.9810 R.E.: 1.93%	6.0000 ρ = 1.0000 R.E.: 0%	6.0000 ρ = 1.0000 R.E.: 0%	6.0000 ρ = 1.0000 R.E.: 0%
K_5	6	6.2500 ρ = 0.9600 R.E.: 4.17%	6.2500 ρ = 0.9600 R.E.: 4.17%	6.6667 ρ = 0.9000 R.E.: 11.11%	6.2500 ρ = 0.9600 R.E.: 4.17%	6.2500 ρ = 0.9600 R.E.: 4.17%
Given by $A(G)$	9.28	9.6040 ρ = 0.9663 R.E.: 3.49%	9.4056 ρ = 0.9866 R.E.: 1.35%	9.3867 ρ = 0.9886 R.E.: 1.15%	9.2961 ρ = 0.9983 R.E.: 0.17%	9.2800 ρ = 1.0000 R.E.: 0%
AW_9^2	12	13.5 ρ = 0.8889 R.E.: 12.50%	12.9827 ρ = 0.9243 R.E.: 8.19%	12.8571 ρ = 0.9333 R.E.: 7.14%	12.6114 ρ = 0.9515 R.E.: 5.10%	12.4967 ρ = 0.9603 R.E.: 4.14%
Pet.	12	12.5 ρ = 0.9600 R.E.: 4.17%	12.3781 ρ = 0.9695 R.E.: 3.15%	12.0000 ρ = 1.0000 R.E.: 0%	12.0000 ρ = 1.0000 R.E.: 0%	12.0000 ρ = 1.0000 R.E.: 0%
Rand. gen.	88	90.3919 ρ = 0.9735 R.E.: 2.72%	89.5733 ρ = 0.9824 R.E.: 1.79%	89.3333 ρ = 0.9851 R.E.: 1.52%	88.0029 ρ = 1.0000 R.E.: $3.3E-5$	88.0000 ρ = 1.0000 R.E.: $9.9E-7$

5. The fifth line corresponds to the graph in the figure with unit edge weights. This graph is the antiweb AW_9^2 and it is the hardest example[1] that the authors know for the relaxation $\mathcal{E}_n \cap M_n$. It is interesting that SDP3 performs better on this example than on the K_5 with unit edge weights.

6. The last two lines correspond to slightly larger graphs. The graph on 10 vertices is the Petersen graph with unit edge-weights. The graph on 12 vertices is a randomly generated graph that gives slightly different results for each relaxation (the exact description of the graph can be found in [5]).

In all cases a relative error equal to zero means that the relative error was below 10^{-11}, the value of the smallest default stopping criteria used by SDPPACK.

[1]Thanks to Franz Rendl for suggesting this interesting example.

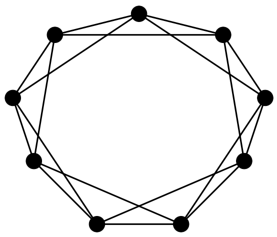

Figure 1.1 Antiweb AW_9^2

Numerical tests have so far been done only for small problems. Because SDP2 has $O(n^2)$ equality constraints and SDP3 has $O(n^3)$ equality constraints, solving them using an interior-point method becomes very time-consuming and requires large amounts of memory, even for moderate values of n. However, the constraints of both problems, and especially those of SDP3, are sparse and have a very special structure so it should be possible to develop a specialized algorithm that exploits the structure of the problem.

References

[1] F. ALIZADEH. *Combinatorial optimization with interior point methods and semidefinite matrices*. PhD thesis, University of Minnesota, 1991.

[2] F. ALIZADEH. Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM J. Optim.*, 5:13–51, 1995.

[3] F. ALIZADEH, J.-P. HAEBERLY, M.V. NAYAKKANKUPPAM, M.L. OVERTON, and S. SCHMIETA. SDPpack user's guide – version 0.9 Beta. Technical Report TR1997–737, Courant Institute of Mathematical Sciences, NYU, New York, NY, June 1997.

[4] M.F. ANJOS and H. WOLKOWICZ. A strengthened SDP relaxation via a second lifting for the Max-Cut problem. Research Report CORR 99-55, University of Waterloo, Waterloo, Ontario, 1999. 28 pages.

[5] M.F. ANJOS and H. WOLKOWICZ. A tight semidefinite relaxation of the cut polytope. Research Report CORR 2000-19, University of Waterloo, Waterloo, Ontario, 2000. 24 pages.

[6] K.M. ANSTREICHER, X. CHEN, H. WOLKOWICZ, and Y. YUAN. Strong duality for a trust-region type relaxation of the quadratic assignment problem. *Linear Algebra Appl.*, 301(1-3):121–136, 1999.

[7] K.M. ANSTREICHER and H. WOLKOWICZ. On Lagrangian relaxation of quadratic matrix constraints. *SIAM J. Matrix Anal. Appl.*, 22(1):41–55, 2000.

[8] F. BARAHONA. The max-cut problem on graphs not contractible to K_5. *Oper. Res. Lett.*, 2(3):107–111, 1983.

[9] F. BARAHONA. On cuts and matchings in planar graphs. *Math. Programming*, 60(1, Ser. A):53–68, 1993.

[10] M.M. DEZA and M. LAURENT. *Geometry of cuts and metrics*. Springer-Verlag, Berlin, 1997.

[11] M.X. GOEMANS and D.P. WILLIAMSON. .878-approximation algorithms for MAX CUT and MAX 2SAT. In *ACM Symposium on Theory of Computing (STOC)*, 1994.

[12] C. HELMBERG. *An interior point method for semidefinite programming and max-cut bounds*. PhD thesis, Graz University of Technology, Austria, 1994.

[13] C. HELMBERG and F. OUSTRY. Bundle methods to minimize the maximum eigenvalue function. In H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors, *HANDBOOK OF SEMIDEFINITE PROGRAMMING: Theory, Algorithms, and Applications*. Kluwer Academic Publishers, Boston, MA, 2000.

[14] C. HELMBERG and F. RENDL. A spectral bundle method for semidefinite programming. *SIAM J. Optim.*, 10(3):673 – 696, 2000.

[15] C. HELMBERG, F. RENDL, R. J. VANDERBEI, and H. WOLKOWICZ. An interior-point method for semidefinite programming. *SIAM J. Optim.*, 6(2):342–361, 1996.

[16] H. KARLOFF. How good is the Goemans-Williamson MAX CUT algorithm? *SIAM J. on Computing*, 29(1):336–350, 1999.

[17] R. M. KARP. Reducibility among combinatorial problems. In R. E. Miller and J.W. Thatcher, editors, *Complexity of Computer Computation*, pages 85–103. Plenum Press, New York, 1972.

[18] M. LAURENT, S. POLJAK, and F. RENDL. Connections between semidefinite relaxations of the max-cut and stable set problems. *Math. Programming*, 77:225–246, 1997.

[19] B. MOHAR and S. POLJAK. Eigenvalues in combinatorial optimization. In *Combinatorial Graph-Theoretical Problems in Linear Algebra*, IMA Vol. 50. Springer-Verlag, 1993.

[20] Y.E. NESTEROV and A.S. NEMIROVSKII. *Optimization over positive semidefinite matrices: Mathematical background and user's manual*. USSR Acad. Sci. Centr. Econ. & Math. Inst., 32 Krasikova St., Moscow 117418 USSR, 1990.

[21] Y.E. NESTEROV and A.S. NEMIROVSKII. *Interior Point Polynomial Algorithms in Convex Programming*. SIAM Publications. SIAM, Philadelphia, USA, 1994.

[22] S. POLJAK and F. RENDL. Nonpolyhedral relaxations of graph-bisection problems. *SIAM J. Optim.*, 5(3), 1995. 467-487.