On the distribution of values of the Quadratic Assignment Problem

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Preprint available at:

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Introduction

- Quadratic Assignment Problem (QAP).

- Optimization problem on the set S_n of permutations.

- Input two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$.

- Maximize on permutations $\sigma \in S_n$:

$$f(\sigma) = \langle B, \sigma(A) \rangle = \sum_{i,j=1}^{n} b_{\sigma(i)\sigma(j)} a_{ij}$$

- Natural generalization of the TSP.

- Very difficult in practice, even at $n = 30$.

Goals

- Understand the distribution of values of a quadratic objective function.
- Find what fraction of the objective values are "large", and the location of the large values relative to the maximum.
- Get approximation guarantees for simple algorithms.
- Identify special cases.
- Evaluate heuristics.

Definitions

- We have the quadratic objective function:
$$f(\sigma) = \langle B, \sigma(A) \rangle := \sum_{i,j=1}^{n} b_{\sigma(i)\sigma(j)} a_{ij}$$

- Use Hamming distance to measure "nearness": $\mathrm{dist}(\sigma, \tau) := |i : \sigma(i) \neq \tau(i)|$.

- Let $U(\tau; r) = \Big\{ \sigma : \mathrm{dist}(\sigma, \tau) = r \Big\}$, the r-th ring around τ.

- The average value of f is $\overline{f} := \dfrac{1}{n!} \displaystyle\sum_{\sigma \in S_n} f(\sigma)$.

- Let $f_0 := f - \overline{f}$. This has average value 0.

Bullseye Special Case

Suppose that the matrix A is symmetric and has constant row and column sums and a constant diagonal. Let $f := \langle B, \sigma(A) \rangle$ for A and some matrix B and let τ be an optimal permutation: $f(\tau) = \max_{\sigma \in S_n} f(\sigma)$. Let $\overline{f}$ be the average value of f on S_n and let $f_0 = f - \overline{f}$. For $r \geq 0$ let $U(\tau; r) = \left\{ \sigma : \text{dist}(\sigma, \tau) = r \right\}$ be the r-th "ring" around τ and let

$$\alpha(n, r) = \frac{(n-r)^2 - 3(n-r)}{n^2 - 3n}.$$

Then

$$\frac{1}{|U(\tau; r)|} \sum_{\sigma \in U(\tau; r)} f_0(\sigma) \geq \alpha(n, r) f_0(\tau).$$

Distribution of values of the objective function with respect to the Hamming distance

from the maximum point

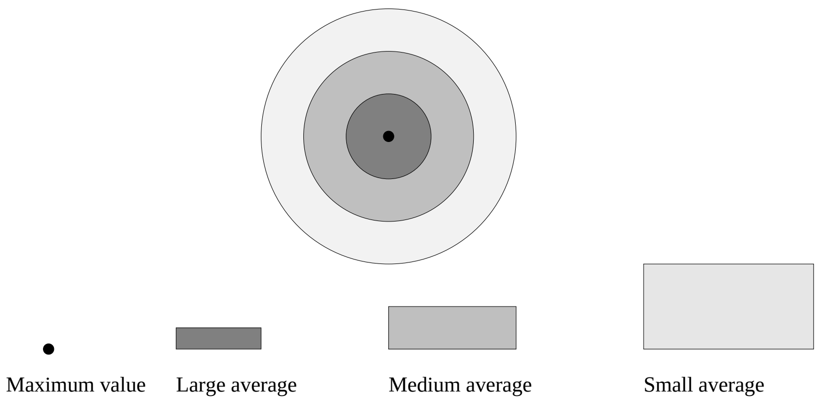

Figure 1

Corollary (Special Case)

1. Fix any $\alpha > 1$. Then there exists a $\beta = \beta(\alpha) > 0$ such that for all sufficiently large $n \geq N(\alpha)$ the probability that a random permutation σ in S_n gives approximation ratio
$$f_0(\sigma) \geq \frac{\alpha}{n^2} f_0(\tau)$$
is at least βn^{-2}. In particular, one can choose $\beta = \exp\left\{c\sqrt{\alpha}\ln\alpha\right\}$ for some absolute constant $c < 0$.

2. Fix any $\alpha > 0$. Then there exists a $\beta = \beta(\alpha) < 1$ such that for all sufficiently large $n \geq N(\alpha)$ the probability that a random permutation σ in S_n satisfies the inequality
$$f_0(\sigma) \geq n^{-\alpha} f_0(\tau)$$
is at least $\exp\{-n^{\beta}\}$. In particular, one can choose $\beta = 1 - \frac{\alpha}{2}$.

Spike Distribution

There is a QAP $f := \langle B, \sigma(A) \rangle$ with A and B *symmetric*, such that:

- The average value of f on S_n is $\overline{f} = 0$.

- The maximum value of f on S_n is 1 and attained, in particular, on the identity permutation e.

- On the k-th ring $U(e,k)$ for $3 \leq k \leq n-4$, the function average is negative. That is:

$$\frac{1}{|U(e,k)|} \sum_{\sigma \in U(e,k)} f(\sigma) \leq 0$$

Distribution of values of the objective function with respect to the Hamming distance

from the maximum point

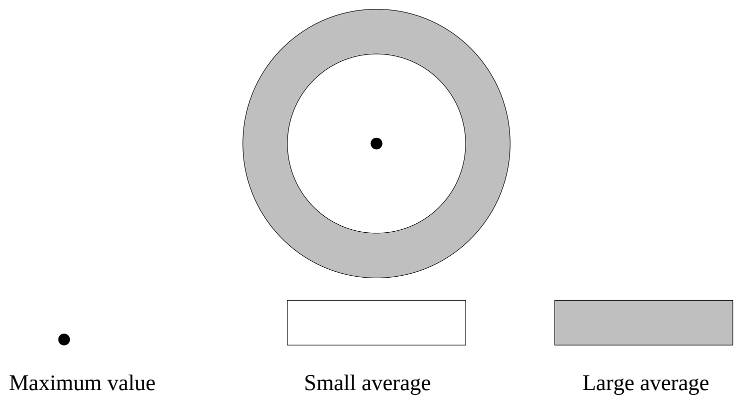

Figure 2

Result for General QAP's

1. Let us fix any $\alpha > 0$. Then there exists a $\beta = \beta(\alpha) > 0$ such that for all sufficiently large $n \geq N(\alpha)$ the probability that a random permutation σ in S_n has approximation ratio
$$f_0(\sigma) \geq \frac{\alpha}{n^2} f_0(\tau)$$
is at least βn^{-2}. In particular, one can choose $\beta = \exp\Big\{c\alpha \ln \alpha\Big\}$ for some absolute constant c.

2. Let us fix any $\alpha > 0$. Then there exists a $\beta = \beta(\alpha) < 1$ such that for all sufficiently large $n \geq N(\alpha)$ the probability that a random permutation σ in S_n satisfies the inequality
$$f_0(\sigma) \geq n^{-1-\alpha} f_0(\tau)$$
is at least $\exp\{-n^{\beta}\}$. In particular, one can choose $\beta = 1 - \frac{\alpha}{2}$

Methods

- Shift f so its maximum is at e.
- Project f to a simpler "class function" g.
- Describe the distribution of g as a mixture of a few "extreme" distributions.
- Identify permutations that perform well on various components, then interpolate.
- Apply results from g to f.

Class Functions

- We average the objective function on representation classes. Define g, the *central projection* of f:

$$g(\sigma) = \frac{1}{n!} \sum_{\tau \in S_n} f(\tau^{-1} \sigma \tau).$$

- Then g is a class function from representation theory, constant on equivalence classes of S_n.

- If f is maximized at the identity, then so is g.

- Note that g has the same average as f on rings $U(e, r)$.

- All such g lie in a 4-dimensional vector space.

- Can reduce to a 3-d problem in general case, and down to 2 or 1 dimensions in special cases.

The Central Cone

- The set of g maximized at e form a convex cone K.

- K is generated by 4 extreme rays for n even, 5 for n odd.

- Two look like "bullseyes", one is a "damped oscillator", and one is a "spike".

- From symmetric matrices, we get both bullseyes and spikes.

- From matrices with constant row and column sums, and constant diagonals, we get bullseyes and oscillators.

- If symmetric as well, we get a unique bullseye distribution.

Estimates

- On bullseyes, permutations near e have large values.

- On oscillators, many permutations have large values.

- On spikes, permutations with no fixed points have large values.

- Show any mixture must have (reasonably) large value on some large class of permutations.

- Use a Markov-type estimate to move from central projection back to f.