SDP and LP-relaxations for global optimization

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Consider the **global optimization** problem

$$\mathbb{P} \mapsto p^* \;:=\; \min\{g_0(x) \,|\, g_i(x) \;\geq\; 0,\; i = 1, \ldots m\},$$

where $g_i(x) : \mathbf{R}^n \rightarrow \mathbf{R}$ are all real-valued polynomials. Let

$$\mathbb{K} \;:=\; \{x \in \mathbf{R}^n \,|\, g_i(x) \geq 0,\; i = 1, \ldots m\},$$

be the feasible set. Let

$$1, x_1, x_2, \ldots x_n, x_1^2, x_1 x_2, \ldots, x_n^r,$$

be a basis (of dimension $s(r)$) of the vector space of real-valued polynomials of degree at most r, and in this basis write

$$p(x) \;=\; \sum_{\alpha \leq r} p_\alpha \, x^\alpha \;=\; p_\alpha \left[x_1^{\alpha_1} \ldots x_n^{\alpha_n}\right],$$

with $\alpha = \sum_{i=1}^n \alpha_i$, and $p \in \mathbf{R}^{s(r)}$ its vector of coefficients.

The univariate case

In this case one considers

$$\mathbb{P} \to p^* \;=\; \min_{x \in \mathbb{K}} g_0(x), \;\text{ with } \begin{cases} \mathbb{K} \;\equiv\; \mathbf{R} \\ \mathbb{K} \;\equiv\; \mathbf{R}_+ \\ \mathbb{K} \;\equiv\; [a,b] \end{cases}$$

With $\mathbb{K} \equiv \mathbf{R}$, Shor (1987) was the first to show that $\mathbb{P}$ is a convex problem. Later Nesterov (1997) proposed an LMI formulation for the three cases.

The multivariate case is NP-hard in general.

Two dual points of view

p^* global minimum $\Leftrightarrow g_0(x) - p^* \geq 0 \,\forall x \in \mathbb{K}$, i.e.,

$g_0(x) - p^*$ **is a nonnegative polynomial on** $\mathbb{K}$.

But we also have

$$p^* \;=\; \min_{\mu} \{ \int g_0(x)\, \mu(dx) \,|\, \mu \in \mathcal{P}(\mathbb{K}) \},$$

where $\mathcal{P}(\mathbb{K})$ is the space of probability measures with support contained in $\mathbb{K}$. Indeed,

$$\int g_0(x)\, \mu(dx) \;\geq p^*,\; \forall \mu \in \mathcal{P}(\mathbb{K}), \tag{0.1}$$

and with $\mu := \delta_{x^*}$ at a global minimizer x^*,

$$\int g_0(x)\, \delta_{x^*}(dx) \;=\; p^*.$$

Observe that both properties are valid for global optima only. Moreover, (0.1) is a linear optimization problem.

$$\min_{\mu} \{ \langle g_0, \mu \rangle \,|\, \langle 1\!\!1_{\mathbb{K}}, \mu \rangle \;=\; 1;\; \mu \;\geq\; 0 \}.$$

II. The point of view of moments

The dual linear program of (0.1) is

$$\max_{\gamma}\{\lambda, \gamma \mid \lambda 1\!\mathrm{l}_{\mathbb{K}} + \gamma \leq \; g_0(x), \; \forall x \in \mathbf{R}^n\},$$

or, equivalently, $\max_\gamma\{\gamma \mid g_0(x) - \gamma \; \geq \; 0 \;$ on $\mathbb{K}\}$. Writing

$$\int g_0(x)\, \mu(dx) \; = \; \sum_\alpha (g_0)_\alpha \int x^\alpha \, \mu(dx) \; = \; \sum_\alpha (g_0)_\alpha y_\alpha,$$

with y_α being a moment of order α of μ, (0.1) reads

$$\begin{cases} \min_y \sum_\alpha (g_0)_\alpha y_\alpha \\ \\ y_\alpha \; = \; \int x^\alpha \, \mu(dx) \;\; \forall \alpha \;\; \text{for some probability } \mu \text{ on } \mathbb{K} \end{cases}$$

Hence, translate the condition

there is some probability μ on $\mathbb{K}$ such that

$$y_\alpha \; = \; \int x^\alpha \, \mu(dx), \; \forall \alpha \leq r,$$

into a condition on the vector y. This the $\mathbb{K}$**-moment problem**, which dates back to Hausdorff, Markov, Stieltjes, Hamburger, etc ...

- $\mathbb{K} = \mathbf{R}$ **(truncated) Hamburger problem**
- $\mathbb{K} = (\mathbf{R})^+$ **(truncated) Stieltjes problem**
- $\mathbb{K} = [a, b]$ **(truncated) Hausdorff problem**

Example: Let $H_r(y)$ be the **Hankel** matrix

$$H_r(y) \;:=\; \begin{bmatrix} 1 & y_1 & y_2 & \cdot & y_r \\ y_1 & y_2 & \cdot & \cdot & y_{r+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y_r & y_{r+1} & \cdot & y_{2r-1} & y_{2r} \end{bmatrix}.$$

Then, $H_r(y) \succeq 0$ is a necessary and sufficient condition for the **Hamburger problem**. Similar conditions are valid for the Stieltjes and Hausdorff problems.

In the n-dimensional case, the condition is only necessary....

Particular case of $\mathbb{K} = [0,1]^n$

Hausdorff moment conditions

Ex: $n = 2$; Given a measure $\mu(d(x,z))$ on $\mathbf{R}^2$, let

$$\int x^i(1-x)^m z^j(1-z)^p \, d\mu \;=\; \sum_{k=0}^{m}\sum_{l=0}^{m}\binom{m}{k}\binom{p}{l} x^{i+k}y^{j+l}\, d\mu.$$

Then, given a vector $y \in \mathbf{R}^\infty$, there exists a measure μ on $[0,1]^2$ with

$$\int x^i z^j \, d\mu \;=\; y_{i,j} \qquad \forall i,j = 0,1,\ldots$$

if and only if

$$\sum_{k=0}^{m}\sum_{l=0}^{p}\binom{m}{k}\binom{p}{l} y_{i+k,j+l} \;\geq\; 0,$$

for all $m,p = 1,2,\ldots$ and all $i,j = 0,1,\ldots$

(due to Hausdorff, Bernstein).

Hence, the Hausdorff moment conditions are linear constraints on the y_α's.

... BUT ... notice the **large binomial coefficients** involved ....

II. The point of view of positive polynomials

=Hilbert's 17**th problem** on the representation of positive polynomials. In the one dimensional case,

$$p(x) \geq 0 \Leftrightarrow p(x) = \sum_{k=1}^{s} q_k(x)^2.$$

Not true anymore in $\mathbf{R}^n$, except $(n, 2)$ (quadratic) and $(4, 3)$ (cubic in $\mathbf{R}^4$).

Representation of polynomials,

positive on $\mathbb{K} := \{x \in \mathbf{R}^n \,|\, g_k(x) \geq 0,\ k = 1, \dots m\}$

Theorem : [Schmüedgen, Putinar]

Assume there is a polynomial $u(x) : \mathbf{R}^n \to \mathbf{R}$ *such that*

$$u(x) = q(x) + \sum_{k=1}^{m} g_k(x) v(x),$$

for some polynomials $q(x), v(x)$ *both sums of squares, and such that* $\{u(x) \geq 0\}$ *is compact. Then:*

Every polynomial, $p(x) : \mathbf{R}^n \to \mathbf{R}$*, strictly positive on* $\mathbb{K}$ *has the representation:*

(0.2) $$p(x) = \sum_{j=1}^{r_0} q_j(x)^2 + \sum_{k=1}^{m} g_k(x) \sum_{j=1}^{r_k} t_{kj}(x)^2,$$

for some (finite) families of polynomials $\{q_j(x)\}$, $\{t_{kj}(x)\}$.

For instance, the representation (0.2) holds whenever $\{g_k(x) \geq 0\}$ is compact for some k, when all the $g_k(x)$ are linear and $\mathbb{K}$ is compact, etc... In practice, one may also add the redundant constraint $M - \sum_i x_i^2 \geq 0$ for M large enough. This is also the case when one has integrality constraints $x_i^2 = x$ for all i. $\Rightarrow$ **very general result!**

The case of a convex polytope

$$\mathbb{K} := \{x \in \mathbf{R}^n \,|\, Ax \;\leq\; b\}$$

for some matrix $A \in \mathbf{R}^{m\times n}$.

Theorem : [Handelman]

Every polynomial, $p(x) : \mathbf{R}^n \to \mathbf{R}$*, strictly positive on* $\mathbb{K}$ *has the representation:*

$$p(x) \;=\; \sum_{|\alpha|\leq s} c_\alpha\, (Ax)_1^{\alpha_1} (Ax)_2^{\alpha-2} \cdots (Ax)_m^{\alpha_m} \tag{0.3}$$

for some integer s *and nonnegative coefficients* $\{c_\alpha\}$.

No true anymore when $\mathbb{K}$ is not a polytope.

Notice the "exponential" number of terms, in contrast to the "linear" Schmüdgen-Putinar representation in terms of squares.

SDP-relaxations

Initiated by **Shor, Lovász and Schrijver, Anjos and Wolkowicz**.

Given a vector $(1, y_1, \dots) \in \mathbf{R}^{s(2r)}$, let $M_r(y)$ be the **moment matrix**. For instance, in $\mathbf{R}^2$ and with $r = 2$,

$$M_2(y) = \begin{bmatrix} 1 & | & y_{1,0} & y_{0,1} & | & y_{2,0} & y_{1,1} & y_{0,2} \\ - & - & - & - & - & - & - & - \\ y_{1,0} & | & y_{2,0} & y_{1,1} & | & y_{3,0} & y_{2,1} & y_{1,2} \\ y_{0,1} & | & y_{1,1} & y_{0,2} & | & y_{2,1} & y_{1,2} & y_{0,3} \\ - & - & - & - & - & - & - & - \\ y_{2,0} & | & y_{3,0} & y_{2,1} & | & y_{4,0} & y_{3,1} & y_{2,2} \\ y_{1,1} & | & y_{2,1} & y_{1,2} & | & y_{3,1} & y_{2,2} & y_{1,3} \\ y_{0,2} & | & y_{1,2} & y_{0,3} & | & y_{2,2} & y_{1,3} & y_{0,4} \end{bmatrix}$$

Given a polynomial $\theta(x) : \mathbf{R}^n \to \mathbf{R}$ of degree w, with coefficient vector $\theta \in \mathbf{R}^{s(w)}$, let $M_r(\theta y)$ be the **localizing matrix**

$$M_r(\theta y) := \sum_\alpha \theta_\alpha y_{\{\alpha(i,j)+\alpha\}}.$$

For instance, with $x \mapsto \theta(x) = 1 - x_1^2 - x_2^2$, $M_2(\theta y) =$

$$\begin{bmatrix} 1 - y_{20} - y_{02}, & y_{10} - y_{30} - y_{12}, & y_{01} - y_{21} - y_{03} \\ y_{10} - y_{30} - y_{12}, & y_{20} - y_{40} - y_{22}, & y_{11} - y_{21} - y_{12} \\ y_{01} - y_{21} - y_{03}, & y_{11} - y_{21} - y_{12}, & y_{02} - y_{22} - y_{04} \end{bmatrix}.$$

If $(1, y)$ is the vector of moments up to order $2r$ of some probability measure μ on the Borel sets of $\mathbf{R}^n$, then for every polynomial $q(x) : \mathbf{R}^n \to \mathbf{R}$ of degree at most r,

$$\langle q, M_r(y) q \rangle \; = \; \int q(x)^2 \, \mu(dx),$$

so that $M_r(y) \succeq 0$. Similarly,

$$\langle q, M_r(\theta y) q \rangle \; = \; \int \theta(x) q(x)^2 \, \mu(dx),$$

and thus $M_r(\theta y) \succeq 0$ whenever μ is supported on $\{\theta(x) \geq 0\}$.

The theory of moments identifies those vectors y with $M_r(y) \succeq 0$ that are the moments of some measure μ.

The $\mathbb{K}$-moment problem identifies those vectors y with $M_r(y) \succeq 0$ that are moment of a measure μ with support contained in $\mathbb{K}$.

Dual theory in algebraic geometry of representation of polynomials, positive on a semi-algebraic set $\mathbb{K}$

$\mathbb{Q}$ Introduce the family $\{\mathbb{Q}_i\}$ of convex LMI problems

$$\mathbb{Q}_i \begin{cases} \min\limits_{y} \sum\limits_{\alpha} (g_0)_\alpha y_\alpha \\ M_i(y) \qquad\qquad \succeq 0 \\ M_{i-v_k}(g_k y) \qquad \succeq 0, \quad k = 1, \ldots m. \end{cases}$$

and the family $\{(\mathbb{Q}_i)^*\}$ of their dual

$$\mathbb{Q}_i^* \begin{cases} \max\limits_{X, Z_1, \ldots Z_m \succeq 0} -X(1,1) - \sum\limits_{k=1}^{m} g_k(0) Z_k(1,1) \\ \text{s.t.} \;\; \langle X, B_\alpha \rangle + \sum\limits_{k=1}^{m} \langle Z_k, C_\alpha^k \rangle = (g_0)_\alpha, \; \forall \alpha. \end{cases}$$

where we write

$$\begin{aligned} M_i(y) &= \sum_{\alpha} y_\alpha B_\alpha \\ M_{i-v_k}(g_k y) &= \sum_{\alpha} y_\alpha C_\alpha^k, \; k = 1, \ldots m, \end{aligned}$$

Interpretation :

From the dual, $\max \mathbb{Q}_i^* \leq p^*$ and

$$p(x) - \max \mathbb{Q}_i^* = \sum_{j=1}^{s} q_j(x)^2 + \sum_{k=1}^{m} g_k(x) \left[\sum_{l=1}^{s_k} q_{kl}(x)^2 \right],$$

with degree$(q_i) \leq i$ and degree$(q_{kl}) \leq i - v_k$.

SDP-relaxation $\longleftrightarrow$ Schmüdgen-Putinar representation of $p(x) - p^*$.

Theorem 1. *Assume that there exists a polynomial* $u(x) : \mathbf{R}^n \to \mathbf{R}$ *of the form (0.2) with* $\{u(x) \geq 0\}$ *compact. Then:*

$$\inf \mathbb{Q}_i \uparrow p^* = \min \mathbb{P}. \tag{0.4}$$

In addition, if $p(x) - p^*$ *has the representation (0.2) for polynomials* $\{q_j(x)\}$ *et* $\{t_{kj}(x)\}$*, of degree at most* N*, then*

$$p^* = \min \mathbb{Q}_i, \qquad \forall i \geq N, \tag{0.5}$$

and for every optimal solution x^* *of* $\mathbb{P}$*, the vector*

$$y^* := (x_1^*, \ldots, x_n^*, \ldots (x_1^*)^{2i}, \ldots (x_n^*)^{2i}) \tag{0.6}$$

is an optimal solution of $\mathbb{Q}_i$*.*

Karush-Kuhn-Tucker Global optimality conditions

Proposition 2. *Assume that there exists a polynomial* $u(x) : \mathbf{R}^n \to \mathbf{R}$ *of the form (0.2) with* $\{u(x) \geq 0\}$ *compact, and let* x^* *be an optimal solution of* $\mathbb{P}$. *If* $p(x) - p^*$ *has the representation (0.2), then*

$$g_k(x^*) \left[\sum_{j=1}^{r_k} t_{kj}(x^*)^2\right] = 0, \; k = 1, \ldots m \tag{0.7}$$

$$\nabla g_0(x^*) = \sum_{k=1}^{m} \nabla g_k(x^*) \left[\sum_{j=1}^{r_k} t_{kj}(x^*)^2\right] \tag{0.8}$$

Thus, we may interpret the representation (0.2) of $p(x) - p^*$ as a **global optimality condition** à la Karush-Kuhn-Tucker, with **polynomials multipliers** $\sum_{j=1}^{r_k} t_{kj}(x)^2$ in lieu of the usual scalars λ_k^*, $k = 1, \ldots m$.

Moreover, if (x^*, λ^*) is KKT optimal point of $\mathbb{P}$, and the gradients $\{\nabla g_k(x^*)\}$ are linearly independent, then

$$\sum_{j=1}^{r_k} t_{kj}(x)^2 = \lambda_k^*, \; \forall k = 1, \ldots m.$$

SDP-relaxations for 0-1 Programming

Integrality constraints $x \in \{0,1\}^n$ are taken into account via $g_k(x) := x_k^2 - x_k = 0$ for all $k = 1, \dots n$, which yields significant simplifications in the moment and localizing matrices $M_i(y) \succeq 0$ and $M_i(g_j y) \succeq 0$.

For instance, an entry like y_{321} which corresponds to the monomial $x_1^3 x_2^2 x_3^1$ is replaced with y_{111}.

Theorem 2: *For the problem*

$$\mathbb{P} \to \min_{x \in \{0,1\}^n} \{g_0(x) \,|\, g_i(x) \ \geq\ 0,\ i = 1, \dots m\},$$

with $deg(g_k) = 2v_k - 1$ *if odd and* $2v_k$ *if even, we have*

$$\min \mathbb{Q}_{n+v} \ =\ p^*,$$

that is, all the relaxations $\mathbb{Q}_i$ *are exact whenever* $i \geq n + v$*, and where* $v := \max_k v_k$*.*

There are at most 2^n variables. This extends the Lovász & Schrijver *lift and project* procedure for 0-1 linear programs.

Exact LMI relaxations for MAX-CUT

$$\min \; q_{110}y_{110} + q_{101}y_{101} + q_{011}y_{011}$$

$$\mathbb{Q}_2 \quad \begin{bmatrix} 1 & y_{100} & y_{010} & y_{001} & y_{110} & y_{101} & y_{011} \\ & 1 & y_{110} & y_{101} & y_{010} & y_{001} & y_{111} \\ & & 1 & y_{011} & y_{100} & y_{111} & y_{101} \\ & & & 1 & y_{111} & y_{100} & y_{010} \\ & & & & 1 & y_{011} & y_{101} \\ & & & & & 1 & y_{110} \\ & & & & & & 1 \end{bmatrix} \succeq 0$$

and

$$\min \; q_{110}y_{110} + q_{101}y_{101} + q_{011}y_{011}$$

$$\mathbb{Q}_3 \quad \begin{bmatrix} 1 & y_{100} & y_{010} & y_{001} & y_{110} & y_{101} & y_{011} & y_{111} \\ & 1 & y_{110} & y_{101} & y_{010} & y_{001} & y_{111} & y_{011} \\ & & 1 & y_{011} & y_{100} & y_{111} & y_{101} & y_{101} \\ & & & 1 & y_{111} & y_{100} & y_{010} & y_{110} \\ & & & & 1 & y_{011} & y_{101} & y_{001} \\ & & & & & 1 & y_{110} & y_{010} \\ & & & & & & 1 & y_{100} \\ & & & & & & & 1 \end{bmatrix} \succeq 0$$

$\mathbb{Q}_3$ is exact.

LP-Relaxations

Basic idea: [Shor, Sherali and Adams]

(1) add **redundant** constraints of the form

$$g_1(x)^{\alpha_1} g_2(x)^{\alpha_2} \cdots g_m(x)^{\alpha_m} \geq 0,$$

with $|\alpha| \leq \delta$, fixed.

(2) **linearize** all the terms

$$y \mapsto y_\beta := x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n},$$

to replace the nonlinear constraint

$$g_1(x)^{\alpha_1} \cdots g_m(x)^{\alpha_m} \geq 0,$$

by the linear constraint

$$\sum_\beta c_\beta y_\beta \geq b$$

The LP-Relaxation of order δ is the LP pogram

$$\mathbb{P}_\delta \to \min_{y_\beta} \{ c'_\beta y_\beta \mid A_\delta y \geq b_\delta \},$$

with dual

$$\mathbb{P}^*_\delta \to \max_{\lambda \geq 0} \{ b'_\delta \lambda \mid A'_\delta \lambda = c_\delta \}$$

Interpretation

Let $\rho_\delta := \min \mathbb{P}_\delta = \max \mathbb{P}^*_\delta$. Then : $\rho_\delta \leq p^*$ and :

$$g_0(x) - \rho_\delta = \sum_{|\alpha|\leq\delta} \lambda_\alpha g_1(x)^{\alpha_1} g_2(x)^{\alpha_2} \cdots g_m(x)^{\alpha_m}, \; x \in \mathbf{R}^n$$

The dual aims at representing $g_0(x) - p^*$ as in **Handelman**'s representation, which in principle is valid only when the g_k are linear and define a polytope $\mathbb{K}$.

Corollary : *Let $\mathbb{K}$ be a convex polytope, i.e., the g_k's are linear : Then*

$$\rho_\delta \uparrow p^* \qquad \textit{as } |\delta| \to \infty,$$

that is, the LP-relaxations converge.

Remarks: (i) ill-conditioned when δ large.

Example : $K = [0,1]^n$. From the constraints $x \geq 0$ and $(1 - x_i) \geq 0$, the linear constraints coming from the linearization of $\prod_{i \in I} x_i^{\alpha_i} \prod_{j \in J} (1 - x_j)^{\alpha_j}$ contains **large binomial coefficients**.

They are in fact the **Hausdorff moment conditions** on y to be the vector of moments of a probability measure on $[0,1]^n$.

Let $I(x)$ be the set of active constraints at a feasible point x, i.e., $i \in I(x) \Rightarrow g_i(x) = 0$.

(ii) No relaxation can be exact if a global minimum is in the interior of $\mathbb{K}$ or if there is a nonoptimal feasible point x with $I(x) = I(x^*)$.

LP-relaxations for 0-1 programming

$$\mathbb{K} \;=\; \{g_k(x) \;\geq\; 0\} \bigcap \{0,1\}^n.$$

With $I, J \subset \{1, 2, \ldots n\}$, $I \cap J = \emptyset$ and $|I \cup J| \;=\; \delta$,

linearize

$$\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j) \;=\; \sum_{\beta} c_\beta x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n},$$

with

$$y \;\mapsto\; y_\beta \;:=\; x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n},$$

to obtain the linear constraint $\sum_\beta c_\beta y_\beta \geq 0$.

Do the same thing in the redundant constraints

$$\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j) g_k(x) \;\geq\; 0,\; k = 1, \ldots m,$$

in replacing every occurrence of x_i^k by x_i, to obtain linear constraints of the form $\sum_\beta c_\beta y_\beta \;\geq\; 0$ and a resulting LP-relaxation

$$\mathbb{P}_\delta^* \!\rightarrow \min_{y_\beta} \{ \sum_\beta c'_\beta y_\beta \,|\, A_\delta y_\beta \;\geq\; b_\delta \}.$$

Theorem : [Sherali and Adams] $\min P_n^* \;=\; p^*$.

Convex hull characterization

Still with $\mathbb{K} := \{x \in \mathbf{R}^n \,|\, g_k(x) \geq 0,\ k = 1, \ldots m\}$, consider the constraint set $\mathbb{K} \cap \{0,1\}^n$.

Let $\mathbb{Q}_i(\mathbb{K})$ (resp. $\mathbb{P}_i(\mathbb{K})$) be the projection of the feasible set of the SDP-relaxation $\mathbb{Q}_i$ (resp. the LP-relaxation $\mathbb{P}_i$) onto the space spanned by the n first variables. Then :

$$\mathbb{Q}_{i+v-1}(K) \subseteq \mathbb{P}_i(\mathbb{K}), \qquad \forall i = 1, \ldots,$$

(Laurent (2001)) and

$$\mathbb{Q}_{n+v}(\mathbb{K}) = \mathbb{P}_n(\mathbb{K}) = \textbf{convex-hull}\,(\mathbb{K} \cap \{0,1\}^n)\,.$$

If $\mathbb{K}$ is a polytope, let $N_i^+(\mathbb{K})$ be the domain of the Lovász-Schrijver *lift and project* procedure. Then :

$$\mathbb{Q}_i(\mathbb{K}) \subseteq \mathbb{P}_i(\mathbb{K}) \subseteq N_i^+(\mathbb{K}), \qquad \forall i = 1, \ldots,$$

$\Rightarrow$ the SDP-relaxations $\{\mathbb{Q}_i\}$ are stronger than the Sherali-Adams and Lovász-Schrijver relaxations.

Conclusion

SDP-relaxations $\mathbb{Q}_i$	**LP-relaxations** $\mathbb{P}_i$
LMI constraints	**Linear constraints**
$p(x) - \min \mathbb{Q}_i = \Sigma^2 + \sum_{k=1}^{m} g_k(x)\Sigma^2$	$p(x) - \min \mathbb{P}_i = \sum_\alpha g_1^{\alpha_1} \cdots g_m^{\alpha_m}$
Schmüdgen-Putinar	**Handelman** (polytope)
$0-1 \rightarrow \min \mathbb{Q}_{n+v} = p^*$	$0-1 \rightarrow \min \mathbb{P}_n = p^*$
$\mathbb{K}$ **compact**. $\min \mathbb{Q}_i \uparrow p^*$	$\mathbb{K}$ **polytope**. $\min \mathbb{P}_i \uparrow p^*$
$n = 1$; a single relaxation	$n = 1$; $\min \mathbb{P}_i \uparrow p^*$
SDP-packages : limited	LP-packages : unlimited
	ill-conditionning; binomial coeff.