A Linear Programming Approach to Semidefinite Programming Problems

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Abstract

A semidefinite programming problem can be regarded as a convex nonsmooth optimization problem, so it can be represented as a semi-infinite linear programming problem. Thus, in principle, it can be solved using a cutting plane approach; we describe such a method. The cutting plane method uses an interior point algorithm to solve the linear programming relaxations approximately, because this resulted in the generation of better constraints than a simplex cutting plane method.

Further, the bundle method of Helmberg and Rendl can be used to generate a set of linear constraints. Typically, these alone are not sufficient to give a good linear programming relaxation. Nonetheless, if they are used in conjunction with some families of problem specific constraints, tight LP relaxations can be obtained.

Why solve an SDP as an LP?

1. Interior Point Methods are limited in the size of problems they can handle.

2. We eventually want to solve an integer program (IP). We can incorporate the SDP via LP approach in a cutting plane LP framework to solving the IP.

3. Since the SDP relaxations are fairly tight, we could also use the above approach in fixing variables in a branch & cut approach.

Contents

1 Introduction

Standard form:

$$\begin{array}{lrcl} \min & C \bullet X & & \\ \text{subject to} & \mathcal{A}(X) & = & b \\ & X & \succeq & 0, \end{array} \qquad (SDP)$$

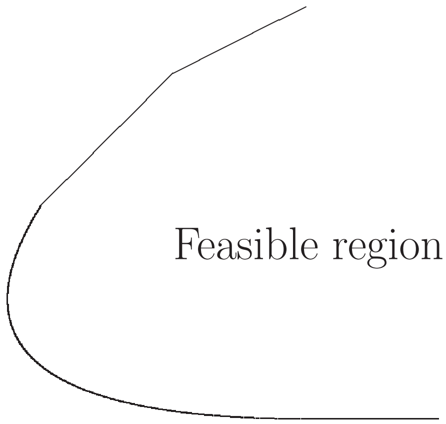
with dual

$$\begin{array}{lrcl} \max & b^T y & & \\ \text{subject to} & \mathcal{A}^T y \ + \ S & = & C \\ & S & \succeq & 0 \end{array} \qquad (SDD)$$

- X and S are square matrices.
- Constraints require that X and S be **positive semidefinite**.
- Inner products are the **Frobenius inner product**.
- $\mathcal{A}X = \begin{bmatrix} A_1 \bullet X \\ \vdots \\ A_k \bullet X \end{bmatrix}$, $\mathcal{A}^T y = \Sigma_{i=1}^k y_j A_j$, where each A_j is a symmetric matrix.
- C is symmetric and b is an k-vector.
- **Primal-dual interior point approaches** are limited in the size of problems they can solve.

2 A semi-infinite linear program

- (SDP) and (SDD) are convex programming problems.
- Only nonlinearity is the positive semidefiniteness (PSD) requirement.
- **Replace PSD constraint by linear constraints.**



$$\begin{aligned} X \succeq 0 \;\; &\Leftrightarrow \;\; d^T X d \geq 0, \forall d \\ &\Leftrightarrow \;\; dd^T \bullet X \geq 0, \forall d \end{aligned}$$

However there are an *infinite* number of these.

Primal formulation:

$$\begin{array}{lrcl} \min & C \bullet X & & \\ \text{subject to} & \mathcal{A}(X) & = & b \\ & d^T X d & \geq & 0 \;\; \forall \text{d with } ||d||^2 = 1. \end{array}$$

- Since X is $n \times n$ and symmetric, we have a linear program in $\frac{n(n+1)}{2}$ variables.
- Need to be **selective** with the vectors d included in the constraints. For example, use a **cutting plane approach** to select vectors d.
- Typically, we will need some vectors d that are **dense**, leading to a dense linear programming problem.

Dual formulation:

$$\begin{array}{lll} \max & b^T y & (LDD) \\ \text{subject to} & d^T(C - \mathcal{A}^T y)d \ \geq \ 0 \ \ \forall \text{d with } ||d||^2 = 1. & \end{array}$$

- If (SDP) has k constraints, then (LDD) has k variables.
- Need to be **selective** with the vectors d included in the constraints. For example, use a **cutting plane approach** to select vectors d.
- Typically, we will need some vectors d that are **dense**, leading to a dense linear programming problem.
- Have far fewer variables than in the primal formulation, so get smaller linear programs (although they are still dense).

3 Regaining a primal solution

- Because the linear programs are smaller, it was more efficient to work with the **dual formulation**.
- We only use a finite number of vectors d, so we are solving a **relaxation** of the dual:

$$\begin{array}{lrcl} \max & b^T y & & \qquad (LDR) \\ \text{subject to} & d_i^T(C - \mathcal{A}^T y)d_i & \geq & 0 \\ & & & \text{for vectors } d_i, i = 1, \ldots, m. \end{array}$$

- The optimal value to (LDR) gives an upper bound on the optimal value of (SDP).

- (LDR) is a relaxation of (SDD). The dual linear program to (LDR) is a **constrained version** of (SDP).
- (LDR) can be rewritten as

$$\begin{array}{lrcl} \max & b^T y & & \\ \text{subject to} & \Sigma_{j=1}^k y_j(d_i^T A_j d_i) & \leq & d_i^T C d_i \\ & & & \text{for vectors } d_i, i = 1, \ldots, m. \end{array} \qquad (LDR)$$

- Writing the dual to (LDR) directly gives

$$\begin{array}{lrcl} \min & \Sigma_{i=1}^m (d_i^T C d_i) x_i & & \\ \text{subject to} & \Sigma_{i=1}^m (d_i^T A_j d_i) x_i & = & b_j \ \text{ for } j = 1, \ldots, k \\ & x & \geq & 0. \end{array}$$

- This can be rewritten as

$$\begin{array}{lrcl} \min & C \bullet (\Sigma_{i=1}^m x_i d_i d_i^T) & & \\ \text{subject to} & \mathcal{A}(\Sigma_{i=1}^m x_i d_i d_i^T) & = & b \\ & x & \geq & 0. \end{array} \qquad (LPR)$$

- Thus, any feasible solution to (LPR) immediately gives a **feasible solution to (SDP)**
  Just set $X = \Sigma_{i=1}^m x_i d_i d_i^T$ which is psd.

4 Helmberg's bundle method

- Need to assume that all feasible solutions X to (SDP) satisfy trace$(X) = a$ for some (problem dependent) scalar a.
- Then (SDD) is equivalent to the **eigenvalue problem**
$$\max_y \; a\lambda_{min}(C - \mathcal{A}^T \mathrm{y}) \; + \; b^T \mathrm{y}$$
- Solve this problem using a **bundle method**.
- This requires in the construction of a matrix P (the current bundle) and the solution of the eigenvalue problem on a **restriction of the feasible region**,
$$\max_y \; a\lambda_{min}(P^t(C - \mathcal{A}^T y)P) + b^T y$$
- This bundle P is refined as the algorithm proceeds.
- At termination, the dual slack matrix $C - \mathcal{A}^T y$ will be **positive semidefinite on the subspace** spanned by the columns of P, but may well not be positive semidefinite over the whole of $I\!R^n$.

SDP with a constant trace

$\Leftrightarrow$

Eigenvalue Optimisation Problem

Primal

$$\begin{array}{llll} \min & C \bullet X & & \\ \text{s.t.} & A_i \bullet X & = & b_i \quad \forall i = 1, \ldots, m \\ & I \bullet X & = & a \\ & X & \succeq & 0 \end{array}$$

Dual

$$\begin{array}{llll} \max & a\lambda + b^T y & & \\ \text{s.t.} & \lambda I + \Sigma_{i=1}^m y_i A_i + S & = & C \\ & S & \succeq & 0 \end{array}$$

- $XS = 0$ (Assume strong duality holds)
- $X = P\Lambda_X P^T$ and $S = P\Lambda_S P^T$
  (Simultaneously Diagonalisable since they commute)
- $XS = 0 \Leftrightarrow \Lambda_X \Lambda_S = 0 \Rightarrow S$ singular
- $\lambda_{min}(C - \lambda I - \Sigma_{i=1}^m y_i A_i) = 0 \Rightarrow$
  $\lambda = \lambda_{min}(C - \Sigma_{i=1}^m y_i A_i)$
- Thus Dual is $\max_y a\lambda_{min}(C - \Sigma_{i=1}^m y_i(A_i - b_i I))$

The restricted eigenvalue problem

The minimum eigenvalue function is the following SDP

$$\begin{aligned}\lambda_{min}(C - \mathcal{A}^T y) \;=\; & \min (C - \mathcal{A}^T y) \bullet W \\ & I \bullet W = 1 \\ & W \succeq 0\end{aligned}$$

Instead choose the following subset of the feasible region above

$$W \;=\; \{\alpha \bar{W} + PVP^T : \alpha + \mathrm{tr}(V) = 1, \alpha \geq 0, V \succeq 0\}$$

- $\bar{W}$ is the aggregrate matrix containing the less important subgradient information.
- P is the bundle containing approximately the eigenvectors corr to almost minimal eigenvalues of $\mathcal{A}^T y - C$.

Thus the basic idea is

1. Consider only a subset of the feasible X matrices and use this to improve the dual solution y.
2. Use this y in turn to improve the feasible solutions in the primal.

One iteration of the bundle method

- Solve the restricted eigenvalue problem using a semidefinite programming solver, to obtain y.
- Calculate the function value and a subgradient of the function for this y.
- The subgradient is found by examining the **minimum eigenvalue and corresponding eigenvector** of $C - \mathcal{A}^T \mathrm{y}$.
- Update the matrix P.

More about P

- The **subgradients** are stored as the **columns** of P.
- The **important subgradients** are kept in P, less important ones are aggregated.
- The number of columns in P is **bounded**, typically by the square root of the number of constraints. Thus, the semidefinite subproblems remain relatively easy to solve.
- P is orthonormalized at each iteration.

5 Cutting planes from the bundle matrix P

- At termination, the dual slack matrix $C - \mathcal{A}^T y$ will be **positive semidefinite on the subspace** spanned by the columns of P, but may well not be positive semidefinite over the whole of $\mathbb{R}^n$.
- The spectral bundle method works well, so it seems to identify an **important subspace** on which the matrix of dual slacks should be psd.
- Directions not in this subspace seem to not be as important.
- The **columns** of P are good candidates for the **cutting planes** d_i.

Hypothesis 1 *In the region of the optimal solution y^* to (SDD), the set of vectors y such that the matrix of slack variables is positive semidefinite is closely approximated by*

$$\{y : P^T(C - \mathcal{A}^T y)P \succeq 0\}.$$

Denote the columns of P by p_l, $l = 1, \ldots, r$. We have

$$\begin{aligned}&\{y : P^T(C - \mathcal{A}^T y)P \succeq 0\}\\ &\quad \subseteq \{y : p_l^T(\mathcal{A}^T y)p_l \leq p_l^T C p_l, l = 1, \ldots, r\}.\end{aligned}$$

Empirically, these columns provide a **good approximation** to the set of psd slack matrices, if used in conjunction with **additional carefully selected constraints**.

Rationale

- The optimal matrix of dual slacks, S^*, will have a nullspace of dimension greater than one.
- This is the **eigenspace with smallest eigenvalue**.
- If the bundle method is successfully identifying the optimal matrix of dual slacks, it will also find a **basis for this eigenspace**.
- The vectors in this basis will become **columns** of P.
- For matrices S where the columns of P are **eigenvectors**, stating $P^T S P \succeq 0$ is equivalent to stating that $p_l^T S p_l \geq 0$ for $l = 1, \ldots, r$. (Diagonalization!)
- Matrices close to S^* will have eigenvectors close to the eigenvectors of S^*.
- So the weakening of the condition $P^T S P \succeq 0$ may not be too severe.

6 Maxcut problems

Divide the vertices of a graph into two sets so that the number of edges with one endpoint in each set is as large as possible. SDP relaxation:

$$\begin{array}{lrcl} \max & \frac{L}{4} \bullet X & & \\ \text{subject to} & \operatorname{diag}(X) & = & e \\ & X & \succeq & 0, \end{array}$$

with dual

$$\begin{array}{lrcrcl} \min & & e^T y & & & \\ \text{subject to} & -\operatorname{Diag}(y) & + & S & = & -\frac{L}{4} \\ & & & S & \succeq & 0 \end{array}$$

Here $L = \operatorname{Diag}(Ae) - A$ is the Laplacian matrix of the graph, where A is the weighted adjacency matrix. Note that $\operatorname{tr}X = n$, the number of nodes in the graph.

Linear constraints

Require $d^T S d = d^T(\mathrm{Diag}(y) - \frac{L}{4})d \geq 0, \forall d$. Use the following d for the max cut problem.

MC1 Set $d = e_i$, $\forall i = 1 \ldots n$, where e_i is the ith standard basis vector for $\mathbb{R}^n$. This generates the constraint $y_i \geq \frac{L_{ii}}{4}$, $\forall i = 1 \ldots n$.

MC2 Set $d = (e_i \pm e_j)$, $\forall$ edges (i, j) gives the constraint $y_i + y_j \geq \frac{Lii}{4} + \frac{L_{jj}}{4} + |\frac{Lij}{2}|$.

MC3 The columns p_i, $i = 1, \ldots, r$ of the bundle matrix P.

We consider two LP relaxations:

- $LP1$ containing MC1 and MC3.
- $LP2$ containing MC2 in addition to the constraints in $LP1$.

Name	n	m	r (No of hrs)	SDP	LP1	LP2	% Error
toruspm-8-50 [1]	512	1536	16	527.808	525.9138	526.366	0.27
toruspm3-15-50 [1]	3375	10125	21	3.475e+03	3.4341e+03	3.4469e+03	0.81
torusg3-8 [1]	512	1536	13	4.5735e+07	4.5431e+07	4.5540e+07	0.43
torusg3-15 [1]	3375	10125	18	3.13456e+08	3.1004e+08	3.1129e+08	0.69
mcp100 [2]	100	269	10	226.1574	225.7535	225.8586	0.13
mcp124-1 [2]	124	149	20	141.9905	81.0216	141.9459	0.03
mcp124-2 [2]	124	318	11	269.8802	268.9689	269.2138	0.25
mcp124-3 [2]	124	620	11	467.7501	467.3699	467.4437	0.06
mcp124-4 [2]	124	1271	10	864.4119	863.7161	863.8122	0.07
mcp250-1 [2]	250	331	10	317.2643	172.0430	317.2091	0.02
mcp250-2 [2]	250	612	14	531.9301	519.0305	531.3788	0.1
mcp250-3 [2]	250	1283	13	981.1726	980.3248	980.4790	0.07
mcp250-4 [2]	250	2421	13	1681.960	1679.70	1680.0011	0.12
mcp500-1 [2]	500	625	10	598.1485	321.3924	596.6749	0.25
mcp500-2 [2]	500	1223	13	1070.057	1069.90	1069.949	0.01
mcp500-3 [2]	500	2355	14	1847.970	1843.20	1844.1308	0.21
mcp500-4 [2]	500	5120	15	3566.738	3559.80	3560.640	0.17
maxG11 [2]	800	1600	11	629.1648	27.5627	626.9861	0.35
maxG32 [2]	2000	4000	14	1567.640	25.1080	1560.941	0.43
maxG51 [2]	1000	5909	19	4003.809	3988.30	3992.60	0.3
maxG55 [2]	5000	14997	25 (10)	12870	11523.663	12855.031	0.12
maxG60 [2]	7000	17148	21 (1)	15222.27 [6] (15318.404)	8606.6	13837.294	9.1
maxG60 [2]	7000	17148	24 (2)	15222.27 [6] (15298.203)	8615.312	14286.714	6.15
maxG60 [2]	7000	17148	25 (6)	15222.27 [6] (15270.285)	5334.174	14796.327	2.79

The columns in the tables represent

n: Problem size i.e. the number of nodes in the graph.

k: Number of SDP constraints.

m: Number of edges in the graph.

r: Bundle size, the number of columns in P.

% Error: $|\frac{SDP-LP}{SDP} \times 100|$.

SDP LP1 LP2: The objective value of the various relaxations.

m1 m2: Number of constraints in LP relaxations.

q: Number of equipartitions required.

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[1] DIMACS
[2] SDPLIB
[3] Rendl
[4] Mitchell
[5] Memory exceeded
[6] Spectral bundle failed to terminate

Notes on maxcut results

- In most cases $LP1$ alone provides a fairly good approximation to the SDP objective value.
- For problems *mcp124-1*, *mcp500-1*, *maxG11*, *maxG32* we need to work with the relaxation $LP2$. These examples underline the importance of the m rank 2 constraints in $LP2$. The LP relaxation $LP2$ provides an **excellent approximation** to the SDP with the $\%error$ well below 1% of the SDP objective value.
- The bundle approach fails to converge for problems *maxG55* and *maxG60*.
- For the *maxG55* problem we run the bundle code for 10 hours. The bundle approach attains an objective value of 12870 (reported SDP objective in SDPLIB is 9999.210). Moreover since the LP relaxations $LP1$ and $LP2$ are larger than 9999.210 it appears that the reported solution in SDPLIB is incorrect.
- For the *maxG60* problem we consider three runs with the bundle code, of durations 1 hour, 2 hours and 6 hours respectively. It is seen that longer the run, the closer the bundle objective to the SDP objective value, and also tighter is the $LP2$ relaxation.

Name	SDP		LP1	LP2
	k	n	m1	m2
toruspm-8-50	512	512	528	2064
toruspm3-15-50	3375	3375	3396	13521
torusg3-8	512	512	525	2061
torusg3-15	3375	3375	3393	13518
mcp100	100	100	110	379
mcp124-1	124	124	144	293
mcp124-2	124	124	135	453
mcp124-3	124	124	135	755
mcp124-4	124	124	134	1405
mcp250-1	250	250	260	591
mcp250-2	250	250	264	876
mcp250-3	250	250	263	1546
mcp250-4	250	250	263	2684
mcp500-1	500	500	510	1135
mcp500-2	500	500	513	1736
mcp500-3	500	500	514	2869
mcp500-4	500	500	515	5635
maxG11	800	800	811	2411
maxG32	2000	2000	2014	6014
maxG51	1000	1000	1019	6928
maxG55	5000	5000	5025	20022
maxG60	7000	7000	7025	24172

Table 1: Sizes of the max cut relaxations

7 Min Bisection

Divide the vertices of a graph into two equal sized sets, so as to minimize the number of cut edges.

SDP formulation:

$$\begin{array}{lrcl} \min & \frac{L}{4} \bullet X & & \\ \text{subject to} & \operatorname{diag}(X) & = & e \\ & (ee^T + I) \bullet X & = & n \\ & X & \succeq & 0, \end{array}$$

with dual

$$\begin{array}{lrcccccc} \min & ny_0 & + & e^T y & & & & \\ \text{subject to} & S & = & \operatorname{Diag}(y) & + & y_0(ee^T + I) & + & \frac{L}{4} \\ & & & & & S & \succeq & 0 \end{array}$$

Note that $(ee^T + I) \bullet X = n$ is equivalent to $ee^T \bullet X = 0$, since $\operatorname{diag}(X) = e$. We used the stated formulation because it resulted in y_0 having a nonzero objective function coefficient.

Linear constraints:

Need $d^T S d = d^T(y_0(ee^T + I) + \mathrm{Diag}(y) + \frac{L}{4})d \geq 0$, $\forall d$.

MB1 Set $d = e_i$, $\forall i = 1 \ldots n$, gives $2y_0 + y_i \geq -\frac{Lii}{4}$, $\forall i = 1 \ldots n$.

MB2 Set $d = e_i - e_j$, $\forall \{i, j\} \in E$, gives $y_i + y_j + 2y_0 \geq \frac{Lij}{2} - \frac{Lii}{4} - \frac{Ljj}{4}$, $\forall \{i, j\} \in E$.

MB3 Set $d = e_i + e_j$, $\forall \{i, j\} \in E$, gives $6y_0 + y_i + y_j \geq -\frac{L_{ij}}{2} - \frac{L_{ii}}{4} - \frac{L_{jj}}{4}$, $\forall \{i, j\} \in E$.

MB4 Set $d = e$, gives $(n^2 + n)y_0 + \Sigma_{i=1}^n y_i \geq 0$, since $Le = 0$.

MB5 The columns $p_i, i = 1, \ldots, r$ of the bundle matrix P.

Unbounded optimal face

There is a **ray** in the direction $y_0 = 1, y_i = -1, \forall i = 1, \ldots, n$. Therefore, we also investigated imposing an upper bound of u on y_0.

Thus, we are in essence solving the following pair of SDP's

$$
\begin{array}{lrcl}
\min & \begin{bmatrix} \frac{L}{4} & 0 \\ 0 & u \end{bmatrix} & \bullet & \begin{bmatrix} X & 0 \\ 0 & x_s \end{bmatrix} \\
\text{subject to} & \operatorname{diag}(X) & = & e \\
 & (ee^T + I) \bullet X + x_s & = & n \\
 & \begin{bmatrix} X & 0 \\ 0 & x_s \end{bmatrix} & \succeq & 0
\end{array}
$$

with dual

$$
\begin{array}{lrcl}
\max & ny_0 + e^T y & & \\
\text{subject to} & \operatorname{Diag}(y) + y_0(ee^T + I) + S & = & \frac{L}{4} \\
 & S & \succeq & 0 \\
 & y_0 & \leq & u
\end{array}
$$

Here x_s is the primal variable corresponding to the upper bound constraint $y_0 \leq u$. Since $I \bullet X = n$, we have $ee^T \bullet X = -x_s$ in the SDP formulation of min bisection.

Name	m	n	r	SDP	LP	m1	% Error	$\|ee^T \bullet X\|$
bm1 [1]	4711	882	10	23.4434	24.989	10315	6.59	0.062
gpp100 [2]	264	100	10	44.9435	45.1416	639	0.44	0.019
gpp124-1 [2]	149	124	10	7.3431	7.3441	433	0.01	0.001
gpp124-2 [2]	318	124	10	46.8623	47.2748	771	0.55	0.029
gpp124-3 [3]	620	124	11	153.014	153.4017	1376	0.16	0.011
gpp124-4 [2]	1271	124	11	418.99	419.98	2678	0.14	0.039
gpp250-1 [2]	331	250	10	15.445	15.4466	923	0.88	0.002
gpp250-2 [2]	612	250	12	81.869	82.088	1487	0.26	0.019
gpp250-3 [2]	1283	250	13	303.50	305.208	2830	0.56	0.018
gpp250-4 [2]	2421	250	13	747.30	750.58	5106	0.43	0.029
gpp500-1 [2]	625	500	11	25.3	26.250	1762	3.62	0.182
gpp500-2 [2]	1223	500	13	156.06	157.7272	2960	1.06	0.055
gpp500-3 [2]	2355	500	15	513.02	517.9053	5226	0.95	0.015
gpp500-4 [2]	5120	500	15	1567.02	1575.9512	10756	0.57	0.008
biomedP [1]	629839	6514	-	33.6014	MM [5]	-	-	-
industry2 [1]	798219	12637	-	65.6131	MM [5]	-	-	-

The value of the dual variable, corresponding to the upper bound constraint $y_0 \leq 1$ provides an estimate for $|ee^t \bullet X|$. This value is below 0.1 for all, but one, of the reported instances.

8 The q-equipartition problem

Divide the n nodes in a graph into **q equally sized sets**, so as few edges as possible are cut.

Quadratic program formulation in binary variables:

$$\begin{array}{lrcl} \min_V & \frac{1}{2}\mathrm{Trace} V^T L V & & \\ \text{subject to} & V^T e & = & ge \\ & V e & = & e \\ & V_{ij} & = & \{0,1\} \;\; \forall i,j \end{array}$$

- L is the Laplacian matrix of the graph.
- $g = \frac{n}{q}$
- An entry V_{ij} of this matrix is 1 if vertex i belongs to component j in the partition.

An SDP relaxation is the following

$$\begin{array}{lrcl} \min & \frac{1}{2}\text{Trace}LX & & \\ \text{subject to} & \text{diag}(X) & = & e \\ & Xe & = & ge \\ & X & \succeq & 0 \end{array}$$

This can also be written as

$$\begin{array}{lrcll} \min & \frac{L}{2} \bullet X & & & \\ \text{subject to} & e_i e_i^T \bullet X & = & 1 & \forall i = 1, \ldots, n \\ & \frac{1}{2}(ee_i^T + e_i e^T) \bullet X & = & g & \forall i = 1, \ldots, n \\ & X & \succeq & 0 & \end{array}$$

with dual

$$\begin{array}{llcl} \min & \Sigma_{i=1}^n y_i & + & g\,\Sigma_{i=1}^n y_{n+i} \\ \text{subject to} & S & = & \\ & \multicolumn{3}{l}{\Sigma_{i=1}^n e_i e_i^T y_i + \Sigma_{i=1}^n \frac{1}{2}(ee_i^T + e_i e^T) y_{n+i} + \frac{L}{2}} \\ & S & \succeq & 0 \end{array}$$

Linear constraints:

Use the following d:

kEQ1 Set $d = e_i$, $\forall i = 1, \ldots, n$, gives $y_i + y_{n+i} \geq -\frac{L_{ii}}{2}$, $\forall i$.

kEQ2 Set $d = e$, gives $\Sigma_{i=1}^{n} y_i + n \, \Sigma_{i=1}^{n} y_{n+i} \geq 0$.

kEQ3 Set $d = (e_i + e_j)$, $\forall \{i, j\} \in E$, gives $y_i + y_j + 2y_{n+i} + 2y_{n+j} \geq -\frac{L_{ii}}{2} - \frac{L_{jj}}{2} - L_{ij}$.

kEQ4 Set $d = (e_i - e_j)$, $\forall \{i, j\} \in E$, gives $y_i + y_j \geq -\frac{L_{ii}}{2} - \frac{L_{jj}}{2} + L_{ij}$.

kEQ5 The vectors $p_i, \forall i = 1, \ldots, r$ in the bundle P.

The constraints in kEQ2 ensure that the y_{n+i}, $\forall i = 1, \ldots, n$ don't take arbitrarily large negative values, making the LP unbounded.

Name	m	n	q	r (No of Hrs)	SDP	LP	m1	% Error
nfl1 [4]	496	32	8	9	7.3372e+05	7.3202e+05	1034	0.23
nfl2 [4]	496	32	4	9	6.2890e+05	6.2890e+05	1034	0
A100 [3]	4070	100	2	10	8.61803e+05	8.61807e+05	8251	4.6e-04
gpp100 [2]	364	100	2	10	44.9435	44.9449	839	0.0031
gpp124-1 [2]	149	124	2	10	7.3431	43.0370	433	486.09
gpp124-1 [2]	149	124	2	31	7.3431	7.3443	454	0.016
gpp124-2 [2]	318	124	2	8	46.8623	46.864	769	0.0036
gpp250-1 [2]	331	250	2	40 (3)	15.445 [6] (15.414)	18.757	953	
gpp250-2 [2]	612	250	2	11	81.869	81.872	1486	0.0037

Notes on the q-equipartition results

- The LP relaxation of the bisection problem, i.e. q-equipartition with $q = 2$, does not have an unbounded optimal face, unlike the earlier LP relaxation of min bisection.

- The bipartition formulation of min bisection gives an SDP relaxation which is relatively harder to solve than the original min bisection formulation using the bundle approach, since we are dealing with more constraints.

- Need a larger bundle for *gpp124-1*, *gpp250-1*. Can then get tighter LP approximations to *gpp124-1*. For instance, with the default bundle size we are only able to get an LP approximation of 43.0370 (SDP objective value is 7.3431), but with the larger bundle size our LP relaxation is 7.3443 for a $\%error$ of 0.016. However this results in bundle sizes r much larger than $\sqrt{k}$.

9 The Lovasz theta function

The SDP representing the Lovasz theta function is

$$\begin{array}{lrcl} \min & C \bullet X & & \\ \text{subject to} & I \bullet X & = & 1, \\ & A_k \bullet X & = & 0, \;\; k = 1, \ldots, n \\ & X & \succeq & 0, \end{array}$$

with dual

$$\begin{array}{lccccccc} \min & y_0 & & & & & & \\ \text{subject to} & -y_0 I & + & -\Sigma_{k=1}^{m} y_k A_k & + & S & = & C \\ & & & & & S & \succeq & 0 \end{array}$$

Here C is the matrix of all minus ones, $A_k = e_i e_j^T + e_j e_i^T$, where the kth edge of the graph (with m edges) is from vertex i to vertex j. We have $\mathrm{tr}(X) = 1$, a constraint in the original SDP.

The Lovasz theta number is an upper bound on the maximum clique number and a lower bound on the colouring number of the complement of the graph.

Linear constraints:

Require $d^T S d = d^T(y_0 I + \Sigma_{k=1}^m y_k A_k + C)d \geq 0, \forall d$.

Use the following d for the Lovasz theta problem.

LT1 Set $d = e_i$, $\forall i = 1 \ldots n$, gives $y_0 \geq 1$. This constraint implies that a lower bound on the Lovasz theta number is one.

LT2 Set $d = e_i + e_j$, $\forall i, j = 1 \ldots n$, gives $y_0 + y_k \geq 2$, $\forall k = 1, \ldots, m$.

LT3 Set $d = e_i - e_j$, $\forall i, j = 1 \ldots n$, gives $y_0 - y_k \geq 0$, $\forall k = 1, \ldots, m$.

LT4 Set $d = e$, gives $ny_0 + 2\,\Sigma_{i=1}^n y_i \geq n^2$.

LT5 The columns $p_i, \forall i = 1, \ldots, r$ of the bundle matrix P.

The constraints in LT2 and LT3 are required to force y_0 to move as y_k moves. Otherwise the LP is trivial to solve: the optimal solution is $y_0 = 1$ for an objective value of 1, adjusting the y_k , $k = 1, \ldots, m$ so as to satisfy the remaining constraints.

Name	m	n	r (No of Hrs)	SDP	LP	m1	% Error
hamming-9-8 [1]	2304	512	23	224	221.6829	4633	1.0344
hamming-7-5-6 [1]	1792	128	12	42.67	42.5297	3598	0.3288
hamming-10-2 [1]	23040	1024	-	102.4	MM [5]	-	-
hamming-11-2 [1]	56320	2048	-	170.67	MM [5]	-	-
hamming-8-3-4 [1]	16128	256	43	25.6	25.32	32301	1.0938
hamming-9-5-6 [1]	53760	512	-	85.33	MM [5]	-	-
theta1 [2]	103	50	10	23	22.9503	218	0.2161
theta2 [2]	497	100	20	32.87	32.8378	1016	0.098
theta3 [2]	1105	150	25	42.166	40.1389	2237	4.8074
theta3 [2]	1105	150	47	42.166	42.107	2259	0.14
theta4 [2]	1948	200	25(6)	50.321 [6] (50.35)	38.1189	3923	24.2485
theta4 [2]	1948	200	51(6)	50.321 [6] (50.331)	49.6258	3949	1.3815
theta5 [2]	3027	250	25(6)	57.232 [6] (57.282)	33.3018	6081	41.8126
theta5 [2]	3027	250	51(6)	57.232 [6] (57.576)	51.8651	6107	9.3774
theta5 [2]	3027	250	51(10)	57.232 [6] (57.391)	54.6415	6107	4.7908
theta6 [2]	4374	300	25(6)	63.477 [6] (63.564)	30.061	8775	52.6427
theta6 [2]	4374	300	59(6)	63.477 [6] (65.05)	50.536	8809	20.3869
theta6 [2]	4374	300	58(10)	63.477 [6] (64.316)	54.569	8808	14.0334
theta6 [2]	4374	300	58(15)	63.477 [6] (63.912)	57.27	8808	9.778

Notes on the Lovasz theta experiments

- Since the number of constraints in the Lovasz theta problem is $O(m)$ a **larger bundle** is required. Thus the bundle method is time consuming on these problems.

- For instance *theta3* the bundle approach requires about 4 hours to converge. It fails to converge for *theta4*, *theta5*, *theta6*, so we used the bundle from a 6 hour run. The objective value attained by the bundle approach is reported in parentheses next to the SDP objective value.

- To improve on the LP relaxations for the SDPLIB problems *theta3*, *theta4*, *theta5* and *theta6*, we choose larger bundle sizes, and compute the LP relaxations using the bundle available after runs of various length. This considerably strengthens the LP relaxation in all cases.

- For the DIMACS *Hamming* instances, the bundle approach finds the optimal bundle in a few iterations. We are unable to solve to solve *hamming-10-2*, *hamming-11-2* and *hamming9-5-6*, where we run out of memory.

- When the bundle approach converges, our LP solution is an excellent approximation to the SDP objective value with the $\%error$ well under 1%.
- It must be emphasised here that all the SDPLIB Lovasz theta problems can be solved in under an hour using any interior point package such as SDPT3.

10 Box constrained QP's

Consider the the following box constrained QP

$$\begin{array}{lrcl} \max & & x^T Q x & \\ \text{subject to} & -e \leq & x & \leq e \end{array}$$

The SDP relaxation of this problem is

$$\begin{array}{lrcl} \max & Q \bullet X & & \\ \text{subject to} & \operatorname{diag}(X) & \leq & e, \\ & X & \succeq & 0, \end{array}$$

with dual

$$\begin{array}{lrcl} \min & e^t y & & \\ \text{subject to} & \operatorname{Diag}(y) & \geq & Q \\ & y & \geq & 0 \end{array}$$

Linear constraints:

We use the following d.

QPB1 Set $d = e_i$, $\forall i = 1, \ldots, n$ gives $y_i \geq Q_{ii}$ $\forall i = 1, \ldots, n$. Since Q is typically a p.s.d. matrix, this can replace the nonnegativity requirement on y_i.

QPB2 The columns $p_i, \forall i = 1, \ldots, r$ in the bundle P.

Name	k	n	r	SDP	LP	m1	% Error
qpG11 [2]	800	1600	11	1.181e+04	1.1764e+04	811	0.39
gpG51 [2]	1000	2000	10	2448.659	2434.502	1010	0.57

This LP provides an excellent approximation to the SDP objective value and has $O(n + \sqrt{m})$ constraints.

11 A Cutting Plane Approach

Given a relaxation, it can be tightened by adding more cutting planes.

1. Solve an LP relaxation of (SDD) to a tolerance τ on the relative duality gap.
2. Use a **Lanczos scheme** to approximately find the (up to) 20 most negative eigenvalues of the matrix S of dual slacks, and the corresponding eigenvectors d_j.
3. If τ is small enough and if the most negative eigenvalue of S is sufficiently close to zero, STOP.
4. Add **constraints** corresponding to these **eigenvectors** to the LP relaxation, namely $d_j^T(\mathcal{A}^T y)d_j \leq d_j^T C d_j$.
5. Reduce τ and return to Step 1.

More details of the cutting plane scheme

- Use an interior point code to solve the LP relaxations. This results in a test point that is more centered, leading to better constraints and faster convergence — see the results.

- Choose an initial point that corresponds to a feasible solution in (SDP) and (SDD). This leads to a feasible initial point for the primal and dual linear programs, and the algorithm then maintains feasibility.

- This dual feasible point can be found by exploiting the fact that the trace of X is constant.

- A primal feasible point can often be found by exploiting the structure of the problem.

Starting the cutting plane method from scratch

Parameters:

- Computational results reported for Maxcut.
- Initial relaxation: $y_i \geq \frac{L_{ii}}{4}$.
- Initial tolerance $\tau = 1.0$.
- Reduce τ by a factor of 0.95 at each outer iteration.
- Run until a specified time (a few hours).
- Run both simplex and interior point cutting plane schemes. A strictly feasible starting point was chosen for the interior point solver. LIPSOL used as the interior point solver and CPLEX6.5 as the simplex solver.

Name	SDP	Bundle LP	Interior LDR	Interior LPR	Eig	Iter
mcp100 [2]	226.16	225.85	226.13	226.11	-1.1e-3	200
mcp124-1 [2]	141.99	141.95	141.94	141.88	-1.6e-3	185
mcp124-2 [2]	269.88	269.21	269.76	269.72	-2.8e-3	200
mcp124-3 [2]	467.75	467.44	467.40	467.35	-1.2e-2	200
mcp124-4 [2]	864.41	863.81	862.69	862.67	-3.7e-2	200
mcp250-1 [2]	317.26	317.21	316.95	316.91	-1.1e-2	200
mcp250-2 [2]	531.93	531.38	529.77	529.73	-2.7e-2	200
mcp250-3 [2]	981.17	980.48	976.98	976.92	-5.3e-2	200
mcp250-4 [2]	1681.96	1680.00	1675.23	1675.14	-8.5e-2	200
mcp500-1 [2]	598.15	596.67	594.28	586.68	-3.5e-2	100
mcp500-2 [2]	1070.06	1069.95	1059.51	1049.32	-5.3e-2	100
mcp500-3 [2]	1847.97	1844.13	1830.11	1823.00	-1.2e-1	100
mcp500-4 [2]	3566.74	3560.64	3534.11	3532.74	-1.8e-1	140
maxG11 [1]	629.16	626.99	627.31	619.98	-1.8e-2	100
toruspm-8-50 [1]	527.81	525.91	516.90	508.87	-6e-2	100

Table 2: Strengths of various LP relaxations

Name	m_{sb}	m_{icp}
mcp100 [2]	379	869
mcp124-1 [2]	293	718
mcp124-2 [2]	453	1026
mcp124-3 [2]	755	1221
mcp124-4 [2]	1405	1332
mcp250-1 [2]	591	1224
mcp250-2 [2]	876	1911
mcp250-3 [2]	1546	2312
mcp250-4 [2]	2684	2465
mcp500-1 [2]	1135	1626
mcp500-2 [2]	1736	1929
mcp500-3 [2]	2869	2253
mcp500-4 [2]	5635	3195
maxG11 [1]	2411	2610
toruspm-8-50 [1]	2064	2197

Table 3: Sizes of the various LP relaxations

Name	SDP	Interior	Simplex	Iter
mcp100 [2]	226.16	226.06	224.68	200
mcp124-2 [2]	269.88	265.34	263.76	100
mcp124-3 [2]	467.75	461.65	459.15	100
mcp124-4 [2]	864.41	862.17	857.66	200
mcp250-4 [2]	1681.96	1672.18	1643.11	100
mcp500-4 [2]	3566.74	3499.36	3430.75	60

Table 4: Strengths of the interior and simplex CP approaches after an equivalent no of iterations

Notes on the cutting plane results

- The cutting plane method gets bounds comparable to those found using the bundle LP method, although it needs more constraints.
- No constraints are dropped.
- The interior point code has a more **central iterate**, so it is closer to the cone of semidefinite matrices. Therefore, it adds fewer constraints and these constraints are stronger, than those found using the simplex approach.
- At the end of the interior point runs, the **most negative eigenvalue** is typically on the order of **-0.1**.
- The iteration counts are upper bounds; the two cutting plane methods may take fewer iterations in the specified time.

12 Conclusions

- We provide empirical evidence that only **a few linear constraints**, polynomial in the problem size n, are typically required.
- The LP approach is very successful in solving the max cut and the box constrained QP problems.
- Bounding the min bisection problem works well, with the value of $|ee^T \bullet X|$ very small.
- The $\{0, 1\}$ formulation of the min bisection problem which is a q-equipartition problem with $q = 2$ is not unbounded. However, the resulting SDP has more constraints and is more time consuming.
- The spectral bundle approach is time consuming on Lovasz theta problems. Larger bundle sizes are needed. Traditional interior point methods usually outperform the spectral bundle method.
- In certain instances, we need bundle sizes $> \sqrt{k}$ to tighten our LP relaxations.
- The main computational task in the LP approach is in computing the optimal bundle P. Solving the resulting LP relaxations is relatively trivial.

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