LPDASA and LP bounds for QAP *

William W. Hager
Department of Mathematics
University of Florida, Gainesville, FL 32611
hager@math.ufl.edu
http://www.math.ufl.edu/~hager

April 26, 2001

*This work was supported by the National Science Foundation.

OVERVIEW

1. Dual Active Set Algorithm (DASA)
2. Linear Programming DASA (LPDASA)
3. Proximal LPDASA
4. Convergence properties
5. LP lower bounds for QAP
6. Numerical comparisons

1. DASA

Primal Problem:

$$\text{minimize } f(\mathbf{x}) \text{ subject to } h(\mathbf{x}) = 0, \ \mathbf{x} \geq \mathbf{0},$$

Lagrangian:

$$\mathcal{L}(\boldsymbol{\lambda}, \mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}).$$

Dual Function:

$$\mathcal{L}(\boldsymbol{\lambda}) = \inf \mathcal{L}(\boldsymbol{\lambda}, \mathbf{x}) \text{ subject to } \mathbf{x} \geq \mathbf{0}.$$

Dual Problem:

$$\text{maximize } \mathcal{L}(\boldsymbol{\lambda}) \text{ over } \boldsymbol{\lambda}.$$

DASA approach: Solve for $\boldsymbol{\lambda}^* \rightsquigarrow \mathbf{x}^*$

NOTE: Dual problem is unconstrained!

Modified dual functions:

$$\mathcal{L}_B(\boldsymbol{\lambda}) = \inf \mathcal{L}(\boldsymbol{\lambda}, \mathbf{x}) \text{ subject to } \mathbf{x}_B \geq \mathbf{0},$$

$\mathbf{x}_B = [x_i : i \in B]$, $B \subset \{1, 2, \ldots, n\}$

$$\mathcal{L}_B^0(\boldsymbol{\lambda}) = \inf \mathcal{L}(\boldsymbol{\lambda}, \mathbf{x}) \text{ subject to } \mathbf{x}_B = \mathbf{0}$$

Primal minimizer in dual:

$$\mathbf{x}(\boldsymbol{\lambda}) = \arg\min \mathcal{L}(\boldsymbol{\lambda}, \mathbf{x}) \text{ subject to } \mathbf{x} \geq \mathbf{0}$$

Dual Active Set Algorithm

Initialization:

$$j = 0, \quad \boldsymbol{\nu}_0 = \boldsymbol{\lambda}_k, \quad B_0 = \{i : x_i(\boldsymbol{\lambda_k}) = 0\}$$

Subiteration:

$$\begin{aligned} \boldsymbol{\mu}_j &= \arg\max \mathcal{L}_{B_j}^0(\boldsymbol{\lambda}) \text{ (unconstrained max)} \\ \boldsymbol{\nu}_{j+1} &= \arg \max_{\boldsymbol{\mu} \in [\boldsymbol{\nu}_j, \boldsymbol{\mu}_j]} \mathcal{L}_{B_j}(\boldsymbol{\mu}) \text{ (line search)} \end{aligned}$$

Constraint deletion: $B_{j+1} = \{i \in B_j : y_i = 0\}$ where $\mathcal{L}(\boldsymbol{\nu}_{j+1}, \mathbf{y}) = \mathcal{L}_{B_j}(\boldsymbol{\nu}_{j+1})$, $\mathbf{y}_{B_j} \geq \mathbf{0}$.

Stopping criterion: if $\boldsymbol{\nu}_{j+1} = \boldsymbol{\mu_j}$, then $\boldsymbol{\lambda}_{k+1} = \boldsymbol{\nu}_{j+1}$.

Theorem. If f and $\mathbf{h}$ are C^1 and $\mathcal{L}(\boldsymbol{\lambda}, \mathbf{x})$ is strongly convex in $\mathbf{x}$, uniformly in $\boldsymbol{\lambda}$, then the DASA solve the dual problem in a finite number of steps (iteration/subiterations).

Reference: H, The dual active set algorithm in Advances in Optimization and Parallel Computel Computing, P. M. Pardalos, ed., 1992.

Reference: H and D. Hearn, Application of the DASA to quadratic network optimization, Computational Optimization and Applications, 1993.

NOTE: Primal active set schemes typically bind one constraint in each step, or involve a piecewise linear line search. In the DASA, the line search is over a line segment.

2. LPDASA

Linear Program:

$$\text{minimize} \quad \mathbf{c}^{\mathsf{T}}\mathbf{x} \text{ subject to } \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}.$$

Dual Function:

$$\mathcal{L}(\boldsymbol{\lambda}, \mathbf{x}) = \mathbf{c}^{\mathsf{T}}\mathbf{x} + \boldsymbol{\lambda}^{\mathsf{T}}(\mathbf{b} - \mathbf{A}\mathbf{x}).$$

Problem: $\mathcal{L}$ not strongly convex in $\mathbf{x}$

LPDASA

Strict ascent: Choose $\mathbf{y} \in \mathbf{x}(\boldsymbol{\lambda}_k)$,

$\mathbf{z} = \arg\min \quad \|\mathbf{Ax}-\mathbf{b}\| \quad$ subject to $\quad \mathbf{x}_B = \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0}$,

where $B = \{i : (\mathbf{c} - \mathbf{A}^\mathsf{T}\boldsymbol{\lambda}_k)_i \neq 0\}$. Defining $\boldsymbol{\mu}(t) = \boldsymbol{\lambda}_k + t(\mathbf{b} - \mathbf{Az})$,

$$\boldsymbol{\nu}_0 = \boldsymbol{\mu}(\bar{t}), \bar{t} = \arg\max_{t \geq 0} \mathcal{L}(\boldsymbol{\mu}(t))$$

Initialization:

$$j = 0, \quad B_0 = \{i : (\mathbf{c} - \mathbf{A}^\mathsf{T}\boldsymbol{\nu}_0)_i \neq 0\}$$

Subiteration: $\mathbf{p}$ = projection of $\mathbf{b} - \mathbf{A}_{B_j}\mathbf{z}_{B_j}$ in null space of $\mathbf{A}^\mathsf{T}_{F_j}$, $F_j = B_j^c$, $\boldsymbol{\mu}(t) = \boldsymbol{\nu}_u + t\mathbf{p}$,

$\boldsymbol{\nu}_{j+1} = \boldsymbol{\mu}(\bar{t}), \bar{t} = \arg\max_{t \geq 0} \mathcal{L}^0_{B_j}(\boldsymbol{\mu}(t))$ (line search)

Constraint deletion:

$$B_{j+1} = \{i \in B_j : (\mathbf{c} - \mathbf{A}^\mathsf{T}\boldsymbol{\nu}_{j+1})_i \neq 0\}$$

Stopping criteria: If $\mathbf{Az} = \mathbf{b}$ in strict ascent, done. If $\bar{t} = +\infty$, problem infeasible. If $\mathbf{p} = \mathbf{0}$, then $\boldsymbol{\lambda}_{k+1} = \boldsymbol{\nu}_j$.

Reference: H, The LP dual active set algorithm (High Performance Algorithms and Software in Nonlinear Optimization, R. De Leone, A. Murli, P. M. Pardalos, and G. Toraldo, eds., Kluwer, Dordrecht, 1998.

Theorem. The LPDASA solves an LP in a finite number of steps.

3. Proximal LPDASA

Lagrangian $\rightsquigarrow$ Proximal Lagrangian (shift $\mathbf{y}$):

$$\mathcal{M}(\boldsymbol{\lambda}, \mathbf{x}) = \mathcal{L}(\boldsymbol{\lambda}, \mathbf{x}) + \frac{\epsilon}{2}\|\mathbf{x} - \mathbf{y}\|^2$$

Dual Function $\rightsquigarrow$ Proximal Function (shift $\boldsymbol{\mu}$):

$$\mathcal{M}(\boldsymbol{\lambda}) = -\frac{\delta}{2}\|\boldsymbol{\lambda} - \boldsymbol{\mu}\|^2 + \inf\{\mathcal{M}(\boldsymbol{\lambda}, \mathbf{x}) : \mathbf{x} \geq \mathbf{0}\}$$

Proximal Problem:

$$\text{maximize } \mathcal{M}(\boldsymbol{\lambda}) \text{ over } \boldsymbol{\lambda}.$$

NOTE: Proximal Lagrangian strongly convex in $\mathbf{x}$ so DASA converges to a solution of the Proximal Problem in finite number of steps. Proximal Problem is strongly concave so it has a unique solution.

Update: new $\boldsymbol{\mu}$ and $\mathbf{y}$ are solutions $\boldsymbol{\lambda}$ and $\mathbf{x}$ of Proximal Problem

Reference: Rockafellar, SIAM J. Control, Math. of OR, 1976 (general setting)

Reference: Mangasarian, SIAM J. Numer. Anal., 1981 (estimate distance from proximal solution to LP solution set)

Subiteration (unconstrained max) is solution $\boldsymbol{\lambda}$ to

$$(\mathbf{A}_F \mathbf{A}_F^\mathsf{T} + \sigma \mathbf{I})\boldsymbol{\lambda} = \sigma \boldsymbol{\mu} + \mathbf{A}_F \mathbf{c}_F + \epsilon(\mathbf{b} - \mathbf{A}_F \mathbf{y}_F)$$

$F = B_j^c$

NOTE: exact solution is not needed; can use any iterative method which ascends the dual function $\mathcal{L}_{B_j}^0$