# Jordan Algebras, Symmetric Cones and General Semidefinite Programming

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# Nonlinear Relaxations

**Question:** Can a combinatorial optimization problem be relaxed to a convex optimization problem?

**Known Answers:**

- Linear Programming
- semidefinite programming

**Glaring Omission:** Lorentz cone programming: Convex quadratic programming with convex quadratic constraints

In this talk we explore generalization of Semidefinite programming

## “Word-by-Word Extensions”

**Question:**

1. How much of analysis, algorithms, proofs, etc. extend from LP to SDP to symmetric cone optimization?
2. Which ones extend from SDP to symmetric cone optimization?

# Some Algebra

- $(\mathcal{A}, *)$ is an Algebra if

  1. $\mathcal{A}$ is a vector space over $\Re$,
  2. "*" distributes over "+": $x * (y + z) = x * y + x * z$, $(y + z) * x = y * x + z * x$

  **Observe:**

  1. If $x * y = z$ then $z_i$ are bilinear in $x$ and $y$:

  $$z_i = x^T A_i y \qquad A_i \in \Re^{n \times n}$$

  2. For every $x \in \mathcal{A}$ there is a matrix $L(x)$:

  $$L(x) y = x * y$$

- $(\mathcal{A}, .)$ is *associative* if $x(yz) = (xy)z$ always
  1. in associative algebras $x \to L(x)$ defines an endomorphism into algebra of $n \times n$ matrices
  2. Thus every associative algebra is isomorphic to some subalgebra of $n \times n$ matrices.
- $(\mathcal{V}, \cdot)$ is *power-associative algebra* if the subalgebra generated by an element $\mathbf{x}$ is associative
  1. We assume identity $\mathbf{e}$: $\mathbf{ex} = \mathbf{xe} = \mathbf{x}$
  2. $x^p$ is unambiguous
- Power associative algebras have many of formal properties of $n \times n$ matrices

In power associative algebra $(\mathcal{V}, \cdot)$:

- for each $\mathbf{x} \in \mathcal{V}$ there is a least integer $r$ such that

$$\{\mathbf{e}, \mathbf{x}, \mathbf{x}^2, \ldots, \mathbf{x}^r\}$$

  is linearly dependent: $\mathbf{x}^r - a_1(\mathbf{x})\mathbf{x}^{r-1} + \cdots + (-1)^r a_r(\mathbf{x})\mathbf{e} = \mathbf{0}$

- *The minimum polynomial* of $\mathbf{x}$:
  $p(\lambda; \mathbf{x}) \stackrel{\text{def}}{=} \lambda^r - a_1(\mathbf{x})\lambda^{r-1} + \cdots + (-1)^r a_r(\mathbf{x})$ and its roots are *eigenvalues* of $\mathbf{x}$ is $\mathcal{V}$.

- $a_i(\mathbf{x})$ are homogeneous of degree $i$: $a_i(t\mathbf{x}) = t^i a_i(\mathbf{x})$

- $$\mathrm{tr}(\mathbf{x}) \stackrel{\text{def}}{=} a_1(\mathbf{x}) \qquad \text{is linear in } \mathbf{x}$$

  $$\det(\mathbf{x}) \stackrel{\text{def}}{=} a_r(\mathbf{x})$$

  $$\mathrm{cork}(\mathbf{x}) \stackrel{\text{def}}{=} \text{multiplicity of zero eigenvalue}$$

  $$\deg(\mathbf{x}) \stackrel{\text{def}}{=} r$$

  $$r \stackrel{\text{def}}{=} \mathrm{rank}(\mathcal{V}) = \max_{\mathbf{x}}\{\mathbf{x} \in \mathcal{V}\}$$

  $$\mathrm{rk}(\mathbf{x}) \stackrel{\text{def}}{=} r - \mathrm{cork}(x)$$

- An element $\mathbf{x}$ is *regular* if $\deg(\mathbf{x}) = \mathrm{rank}(\mathcal{V})$
- The set of regular elements of $\mathcal{V}$ are dense and generic
- **Characteristic polynomial**:
  1. For regular $\mathbf{x}$: $P_{\mathrm{Char}}(\mathbf{x}) \stackrel{\text{def}}{=} P_{\mathrm{Min}}(\mathbf{x})$
  2. Analytically continue characteristic polynomials all $\mathbf{x} \in \mathcal{V}$

## Jordan Algebras

- $(\mathcal{J}, \circ)$ is *Jordan algebra* if

  1. $\mathbf{x} \circ \mathbf{y} = \mathbf{y} \circ \mathbf{x}$
  2. $\mathbf{x} \circ (\mathbf{x}^2 \circ \mathbf{y}) = \mathbf{x}^2 \circ (\mathbf{x} \circ \mathbf{y})$ that is $\mathsf{L}(\mathbf{x})\mathsf{L}(\mathbf{x}^2) = \mathsf{L}(\mathbf{x}^2)\mathsf{L}(\mathbf{x})$.

- Jordan algebras are power-associative

## Three Examples

1. Every associative algebra $\mathcal{A}$ induces a Jordan algebra: $\mathbf{x} \circ \mathbf{y} \stackrel{\text{def}}{=} \frac{\mathbf{xy}+\mathbf{yx}}{2}$ for example $n \times n$ matrices $(\mathcal{M}_n, \circ)$
2. Symmetric matrices under $\circ$: $(\mathcal{S}_n, \circ)$ Here $L(X) = \frac{X \otimes I + I \otimes X}{2}$
3. $(\Re^{n+1}, \circ)$ where:

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \circ \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} x_0 y_0 + \cdots + x_n y_n \\ x_0 y_1 + y_0 x_1 \\ \vdots \\ x_0 y_n + y_0 x_n \end{pmatrix}$$

and

$$\mathrm{L}(\mathbf{x}) = \mathrm{Arw}(\mathbf{x}) = \begin{pmatrix} x_0 & \bar{\mathbf{x}}^\top \\ \bar{\mathbf{x}} & x_0 \mathrm{I} \end{pmatrix}$$

where

$$\bar{\mathbf{x}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{x}} = \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

## Euclidean Jordan Algebras

- if $(\mathcal{E}, \circ)$ is Jordan and $\mathrm{tr}(\mathbf{x}^2) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ then $\mathcal{E}$ is a *Euclidean Jordan Algebra*
- **Jordan Frame:**
  1. $\mathbf{c}$ is idempotent if $\mathbf{c}^2 = \mathbf{c}$
  2. An orthogonal system of idempotents $\{\mathbf{c}_1, \ldots, \mathbf{c}_k\}$:

$$\mathbf{c}_i \circ \mathbf{c}_j = \mathbf{0} \quad \text{for } i \neq j \qquad \sum \mathbf{c}_i = \mathbf{e}$$

- An idempotent is *primitive* if it is not sum of two other idempotents
- Definition: A *Jordan frame* is a system of primitive orthogonal idempotents.
- It can be shown that if $\{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$ is a Jordan frame then $r = \mathrm{rank}(\mathcal{J})$

- FACT: Spectral decomposition theorem
  1. All eigenvalues $\lambda_i$ of $\mathbf{x} \in \mathcal{E}$ are real numbers
  2. For each $\mathbf{x} \in \mathcal{E}$ there is a Jordan frame $\{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$ such that

$$\mathbf{x} = \lambda_1 \mathbf{c}_1 + \cdots + \lambda_r \mathbf{c}_r$$

  3. The cone of squares (or the *semidefinite cone:*) $\mathcal{K}(\mathcal{E}) = \{\mathbf{x}^2 : \mathbf{x} \in \mathcal{E}\}$
  4. FACT: A cone is symmetric (homogeneous and self-dual) iff it is the cone of squares of some Euclidean Jordan Algebra

## Examples

- $n \times n$ **Symmetric matrices** $(\mathcal{S}, \circ)$**:** Here concepts of eigenvalues, rank, determinant and norms are the usual ones

$$X = Q \Lambda Q^T = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \cdots + \lambda_n \mathbf{q}_n \mathbf{q}_n^T$$

- If $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ an orthonormal set $\{\mathbf{q}_1 \mathbf{q}_1^T, \ldots, \mathbf{q}_n \mathbf{q}_n^T\}$ is a Jordan frame

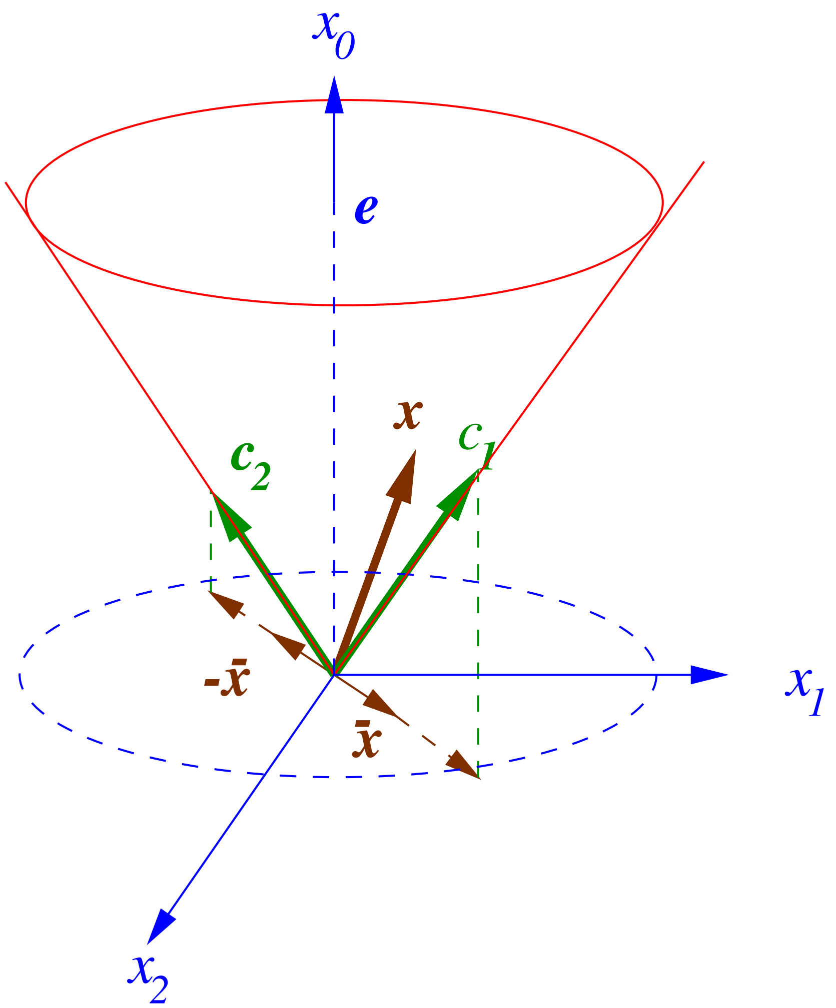
- Cone of squares is the positive semidefinite cone

- **Example:** $(\Re^{n+1}, \circ)$ Here one can see that for any $(x_0; \bar{\mathbf{x}})$

1. Cone of squares: Lorentz cone: $\mathcal{L} = \{(x_0; \bar{\mathbf{x}}) : x_0 \geq \|\bar{\mathbf{x}}\|\}$
2. Since $\mathbf{x}^2 - 2x_0\mathbf{x} + (x_0^2 - \|\bar{\mathbf{x}}\|^2)\mathbf{e} = \mathbf{0}$, the characteristic polynomial is $\lambda^2 - 2x_0\lambda + (x_0^2 - \|\bar{x}\|^2)$
3. Eigenvalues are $x_0 + \|\bar{x}\|$ and $x_0 - \|\bar{x}\|$
4. $\operatorname{tr}_{\Re^{n+1}}(x) = 2x_0$, $\det_{\Re^{n+1}}(x) = x_0^2 - \|\bar{x}\|^2$, and $\operatorname{rk}(\Re^{n+1}) = 2$, independent of $n$
5. $\operatorname{tr}_{\Re^{n+1}}(x^2) = 2\|x\|^2 \geq 0$ and thus $(\Re^{n+1}, \circ)$ is Euclidean
6. For each $(x_0; \bar{\mathbf{x}}) \in \Re^{n+1}$

$$\mathbf{x} = \frac{1}{2}(x_0 + \|\bar{\mathbf{x}}\|)\begin{pmatrix} 1 \\ \frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|} \end{pmatrix} + \frac{1}{2}(x_0 - \|\bar{\mathbf{x}}\|)\begin{pmatrix} 1 \\ -\frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|} \end{pmatrix}$$

- The Jordan frame $\left\{ \frac{1}{2} \begin{pmatrix} 1 \\ \frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|} \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -\frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|} \end{pmatrix} \right\}$



- Since all $\lambda_i$ are real we can define functions and norms:
  1. $\|\mathbf{x}\|_{F/\mathcal{E}} \stackrel{\text{def}}{=} \sqrt{\lambda_1^2 + \cdots + \lambda_r^2}$
  2. $\|\mathbf{x}\|_2 \stackrel{\text{def}}{=} \max_i |\lambda_i|$
  3. $f(\mathbf{x}) \stackrel{\text{def}}{=} f(\lambda_1)\mathbf{c}_1 + \cdots + f(\lambda_r)\mathbf{c}_r$
  4. $x^{-1} \stackrel{\text{def}}{=} \lambda_1^{-1}\mathbf{c}_1 + \cdots + \lambda_r^{-1}\mathbf{c}_r$
- $\mathbf{x} \succcurlyeq \mathbf{0}$ if $\lambda_i(\mathbf{x}) \geq 0$,
- **Quadratic Representation:** For each $\mathbf{x}, \mathbf{y} \in \mathcal{V}$:

$$Q_{\mathbf{x},\mathbf{y}} \stackrel{\text{def}}{=} L(\mathbf{x})L(\mathbf{y}) - L(\mathbf{y})L(\mathbf{x}) - L(\mathbf{x} \circ \mathbf{y})$$

$$Q_{\mathbf{x}} \stackrel{\text{def}}{=} 2L^2(\mathbf{x}) - L(\mathbf{x}^2)$$

## Examples

- **Symmetric matrices:** Concepts of norms are the usual ones
- $Q_{X,Y}(Z) = \frac{XZY+YZX}{2}$ and $Q_X(Y) = XYX$, $Q = X \otimes X$
- **Lorentz cone algebra:**
- $\|x\|_{F/\mathfrak{R}^{n+1}} = \sqrt{2}\|x\|$ and $\|x\|_2 = \max\Big\{\big|x_0 + \|\bar{x}\|\big|, \big|x_0 - \|\bar{x}\|\big|\Big\}$
- 

$$\mathbf{Q_x} = \begin{pmatrix} \|\mathbf{x}\|^2 & 2x_0\bar{\mathbf{x}}^T \\ 2x_0\bar{\mathbf{x}} & (x_0^2 - \|\bar{\mathbf{x}}\|^2)I + 2\bar{\mathbf{x}}\bar{\mathbf{x}}^T \end{pmatrix}$$

## Simple Jordan Algebras and Cones

- If $(\mathcal{E}_1, \circ_1)$ and $\mathcal{E}_2, \circ_2)$ are two Euclidean Algebras then $(\mathcal{E}_1 \oplus \mathcal{E}_2, \circ)$ is also a Euclidean Jordan algebra

$$(\mathbf{x}_1; \mathbf{x}_2) \circ (\mathbf{y}_; \mathbf{y}_2) \stackrel{\text{def}}{=} (\mathbf{x}_1 \circ_1 \mathbf{y}_1; \mathbf{x}_2 \circ_2 \mathbf{y}_2)$$

- An algebra is *simple* if it cannot be decomposed in this way
- There are only 5 Simple Jordan Algebras:
  1. $(\Re^{n+1}, \circ)$ of Lorentz cone
  2. $(\mathcal{S}_n, \circ)$ of $n \times n$ real symmetric matrices
  3. $(\mathcal{H}_n, \circ)$ of $n \times n$ complex Hermitian matrices
  4. $(\mathcal{A}_n, \circ)$ of $n \times n$ quaternion Hermitian matrices
  5. $(\mathcal{O}_n, \circ)$ of $3 \times 3$ octonion Hermitian matrices (Albert)
- All of these except the last one are “representable”

## Representable Jordan algebras

- An Euclidean Jordan Algebra $(\mathcal{E}, \circ)$ is

  **representable:** if it is isomorphic to a subalgebra of $(\mathcal{S}_, \circ)$

  **exceptional** if it is not representable

- Among simple Euclidean Jordan algebras only the *Albert algebra* is exceptional

## Euclidean-Jordan-Associative System

Definition: $(\mathcal{A}, \mathcal{S}, \mathcal{E}, \prime)$ is an EAJ system if

1. $(\mathcal{A}, \cdot)$ is associative
2. $\prime$ is the adjoint operation:
   (a) $\prime : \mathcal{A} \to \mathcal{A}$ is linear and $(\mathbf{x}')' = \mathbf{x}$
   (b) $(xy)' = y'x'$
   (c) $\mathrm{tr}(xx') \geq 0$
3. $\mathcal{S}$ is the set of self-adjoint elements in $\mathcal{A}$ which makes a *Euclidean* Jordan algebra under the $\mathcal{A}$-induced $\circ$
4. $(\mathcal{E}, \circ)$ is a *simple* Jordan subalgebra of $\mathcal{S}$ which generates $\mathcal{A}$.

Note: $\mathcal{E}$ is a Jordan subalgebra of $\mathcal{S}$ but they may have different ranks

In an EAJ system

- $\operatorname{rank}(\mathcal{S})/\operatorname{rank}(\mathcal{E}) = k$ an integer
- $\lambda_i$ is an eigenvalue of $\mathbf{x}$ in $\mathcal{E}$ with multiplicity $t_i$ iff it has multiplicity $kt_i$ in $\mathcal{S}$
- Since $xx' \in \mathcal{S}$ we can define norms and an inner product on $\mathcal{A}$:
  1. $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{A}} \stackrel{\text{def}}{=} \operatorname{tr}_{\mathcal{A}}(\mathbf{xy})$, $\|\mathbf{x}\|_{F/\mathcal{A}} \stackrel{\text{def}}{=} \sqrt{\sum_i \lambda_i(xx')}$,
     $\|\mathbf{x}\|_2 \stackrel{\text{def}}{=} \sqrt{\max_i \lambda_i(\mathbf{xx}')}$
  2. We can define norms in terms of $\mathcal{E}$: $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{E}} \stackrel{\text{def}}{=} \frac{\langle \mathbf{x}, \mathbf{y} \rangle_{\mathcal{A}}}{k}$
     $\|\mathbf{x}\|_{F/\mathcal{E}} \stackrel{\text{def}}{=} \frac{\|\mathbf{x}\|_{F/\mathcal{S}}}{\sqrt{k}}$
- Fact: $\|\mathbf{xy}\|_{F/\mathcal{E}} \leq \|\mathbf{y}\|_2 \, \|\mathbf{x}\|_{F/\mathcal{E}}$
- **Question:** Is it true that $\|\mathbf{xy}\|_{F/\mathcal{E}} \leq \|\mathbf{y}\|_{F/\mathcal{E}} \, \|\mathbf{x}\|_{F/\mathcal{E}}$?

- **Answer:** Only sometimes!

  1. DEFINITION: $\mathbf{x} \in \mathcal{A}$ is *simple* if $\mathbf{x}\mathbf{x}' \in \mathcal{E}$
  2. FACT: if $\mathbf{y} \in \mathcal{A}$ is simple then $\|\mathbf{x}\mathbf{y}\|_{\mathrm{F}/\mathcal{E}} \leq \|\mathbf{y}\|_{\mathrm{F}/\mathcal{E}} \|\mathbf{x}\|_{\mathrm{F}/\mathcal{E}}$
  3. How do we identify simple elements in $\mathcal{A}$?

     FACT: If $\alpha \in \Re$, $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in \mathcal{E}$ and $\mathbf{a} \in \mathcal{A}$ is simple then the following are simple:

     (a) $\mathbf{xy}$

     (b) $\mathbf{xa}$

     (c) $\mathbf{u}^{-1}\mathbf{xyu}$

     (d) $\mathbf{uv} + \alpha\mathbf{u}^{-1}\mathbf{v}^{-1}$

     (e) $\mathbf{xy} + \mathbf{u}$

## Peirce Decomposition

Let $(\mathcal{E}, \circ)$ be a Euclidean Jordan algebra of dim $n$ and $\mathrm{rank}(\mathcal{E}) = r$

- for a primitive idempotent $c$ $L(c)$ is symmetric and
  1. has some eigenvalue equal to $0$
  2. has some eigenvalue equal to $1$
  3. has some eigenvalues equal to $1/2$.
- Thus $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \oplus \mathcal{E}_{1/2}$

$$
\begin{aligned}
\mathcal{E}_0 &= \{\mathbf{x} : \mathbf{c} \circ \mathbf{x} = \mathbf{0}\} \\
\mathcal{E}_1 &= \{\mathbf{x} : \mathbf{c} \circ \mathbf{x} = \mathbf{x}\} \\
\mathcal{E}_{\frac{1}{2}} &= \{\mathbf{x} : \mathbf{c} \circ \mathbf{x} = \mathbf{x}/2\}
\end{aligned}
$$

- If $\{\mathbf{c}_1, \dots, \mathbf{c}_r\}$ and $\{\mathbf{d}_1, \dots, \mathbf{d}_r\}$ are Jordan frames then there is $\mathbf{w} \in \mathcal{E}$:
  1. $\mathbf{w}^2 = \mathbf{e}$
  2. $Q_{\mathbf{w}}(\mathbf{c}_i) = \mathbf{d}_i$ for $i = 1, \dots r$

- The Jordan frame $\{\mathbf{c}_1, \ldots, \mathbf{c}_r\}$ induces the following decomposition of $\mathcal{E}$

$$\mathcal{E}_{ii} \stackrel{\text{def}}{=} \mathcal{E}_1(\mathbf{c}_i) \qquad \text{with dimension } 1$$

$$\mathcal{E}_{ij} \stackrel{\text{def}}{=} \mathcal{E}_{\frac{1}{2}}(\mathbf{c}_i) \cap \mathcal{E}_{\frac{1}{2}}(\mathbf{c}_j) \qquad \text{with dimension } \mathbf{d}$$

$$\mathcal{E} = \oplus_{i \leq j} \mathcal{E}_{ij}$$

- Projection of $\mathbf{x}$ to $\mathcal{E}_{i,j}$ are given by
  1. Onto $\mathcal{E}_{ii}$ : $\qquad Q_{\mathbf{c}_i}$
  2. Onto $\mathcal{E}_{ij}$ : $\qquad 4L(\mathbf{c}_i)(L(\mathbf{c}_j))$

One can show

- $\mathcal{E}_{ij} \circ \mathcal{E}_{ij} \subset \mathcal{E}_i + \mathcal{E}_j$
- $\mathcal{E}_{ij} \circ \mathcal{E}_{jk} \subset \mathcal{E}_{ik}$ for $i \neq k$
- $\mathcal{E}_{ij} \circ \mathcal{E}_{kl} = \{\mathbf{0}\}$ for $i, j, k, l$ all distinct

**Conclusion:** Every $\mathbf{x} \in \mathcal{E}$ can be written as

$$\mathbf{x} = \sum_i x_i \mathbf{c}_i + \sum_{i<j} \mathbf{x}_{ij}$$

This is formally similar to an $r \times r$ matrix

# Partial Answer to the Question:

1. **From LP to $\mathcal{K}$-LP:** Primal-only or dual-only algorithms extend "Word-by-Word" to all symmetric cone optimization problems (**Alizadeh-Schmieta** Handbook)
   They include
   - Karmarkar's original algorithms,
   - Ye's primal and dual and projective algorithms
   - Gonzaga's $\mathcal{O}(n^3)$ algorithm etc.

   **Reason:** If a proof *does not* rely on $L(\mathbf{x})L(\mathbf{s}) = L(\mathbf{s})L(\mathbf{x})$ it extends from LP (in LP $L(\mathbf{x}) = \mathrm{Diag}(\mathbf{x})$)

2. Primal-dual algorithms (**Kojima-Mizuno-Yoshise** and **Monteiro-Adler**) don't extend, *however*

3. **From SDP to $\mathcal{K}$-LP:** Those algorithms that need not assume existence of inducing associative algebra (**Faybusovich, Tsuchiya, Schmieta-Alizadeh**):
   (a) Commutative class of Monteiro-Zhang family (includes Nesterov-Todd, $\mathbf{xs}$ and $\mathbf{sx}$ methods (**HRVW/KSH/M**) and
   (b) $\mathbf{x}^{1/2}\mathbf{s}\mathbf{x}^{1/2}$ and variants (**Monteiro-Zanjacomo, Tseng**)
4. **From SDP to *representable* $\mathcal{K}$-LP:** Those algorithms that rely on existence of and inducing associative algebra (**Schmieta Thesis, Schmieta-Alizadeh**):
   The entire Monteiro-Zhang family and most importantly the $\mathbf{xs}+\mathbf{sx}$ method (**AHO**)
5. The last group does not extend to the Albert Algebra of $3 \times 3$ octonions, this algebra is *exceptional* (not representable)

## Generalization of Monteiro-Zhang family

Primal-dual Methods apply Newton's method to the following system:

$$\textbf{Primal}: \qquad \min\{\langle \mathbf{c}, \mathbf{x}\rangle_{\mathcal{E}} : \langle \mathbf{a}_i, \mathbf{x}\rangle_{\mathcal{E}} = b_i, \quad \mathbf{x} \in \mathcal{K}(\mathcal{E})\}$$

$$\textbf{Dual}: \qquad \max\{\mathbf{b}^{\mathsf{T}}\mathbf{y} : \mathcal{A}^{\mathsf{T}}\mathbf{y} + s = \mathbf{c}, \quad \mathbf{s} \in \mathcal{K}(\mathcal{E})\}$$

**Observation:** Polynomiality proofs of primal-dual methods in LP assume $L(\mathbf{x})L(\mathbf{s}) = L(\mathbf{s})L(\mathbf{x})$.

**Monteiro-Zhang Scaling:** Let $\mathbf{p} \succ \mathbf{0}$, and $\mathbf{Q_p} = 2\mathsf{L}^2(\mathbf{p}) - \mathsf{L}(\mathbf{p}^2)$, then

$$\mathbf{x} \to \widetilde{\mathbf{x}} = \mathbf{Q_p x} = \mathbf{pxp}$$

$$\mathbf{s} \to \utilde{\mathbf{s}} \stackrel{\text{def}}{=} \mathbf{Q}_{\mathbf{p}^{-1}} \mathbf{s} = \mathbf{p}^{-1}\mathbf{s}\mathbf{p}^{-1}$$

**Scaled Optimization problem:**

$$\underline{\textbf{Primal:}} \qquad \min \quad \{\langle \utilde{\mathbf{c}}, \widetilde{\mathbf{x}} \rangle_{\mathcal{E}} : \langle \utilde{\mathbf{a}_i}, \widetilde{\mathbf{x}} \rangle_{\mathcal{E}} = b_i, \widetilde{\mathbf{x}} \in \mathcal{K}(\mathcal{E})\}$$

$$\underline{\textbf{Dual:}} \qquad \max \quad \{\mathbf{b}^\mathsf{T}\mathbf{y} : \utilde{A}^\mathsf{T}\mathbf{y} + \utilde{\mathbf{s}} = \utilde{\mathbf{c}}, \utilde{\mathbf{s}} \in \mathcal{K}(\mathcal{E})\}$$

The Newton iteration is applied to

$$\utilde{A}\widetilde{\mathbf{x}} = \mathbf{b}$$

$$\utilde{A}^\mathsf{T}\mathbf{y} + \utilde{\mathbf{s}} = \utilde{\mathbf{c}}$$

$$\widetilde{\mathbf{x}} \circ \utilde{\mathbf{s}} = \mathsf{L}(\widetilde{\mathbf{x}})\mathsf{L}(\utilde{\mathbf{s}})\mathbf{e} = \mu\sigma\mathbf{e}$$

- If $\mathbf{p}$ is chosen such that $\mathsf{L}(\widetilde{\mathbf{x}})$ and $\mathsf{L}(\underset{\sim}{\mathbf{s}})$ commute then we get the Monteiro-Zhang *commutative* class
- Choosing $\mathbf{p} = \mathbf{s}^{1/2}$ yields $\mathbf{xs}$ method, $\mathbf{p} = \mathbf{x}^{-1/2}$ the $\mathbf{sx}$ method, Nesterov-Todd by $\mathbf{p} = \mathbf{w}^{1/2}$ where $\mathbf{w} = \mathbf{Q}_{x^{1/2}} (\mathbf{Q}_{x^{1/2}}\mathbf{s})^{-1/2}$
- The $\mathbf{xs} + \mathbf{sx}$ method is not in commutative class
- Optimization over scaled problem is entirely equivalent to optimization over the original problem

# Centrality Measure

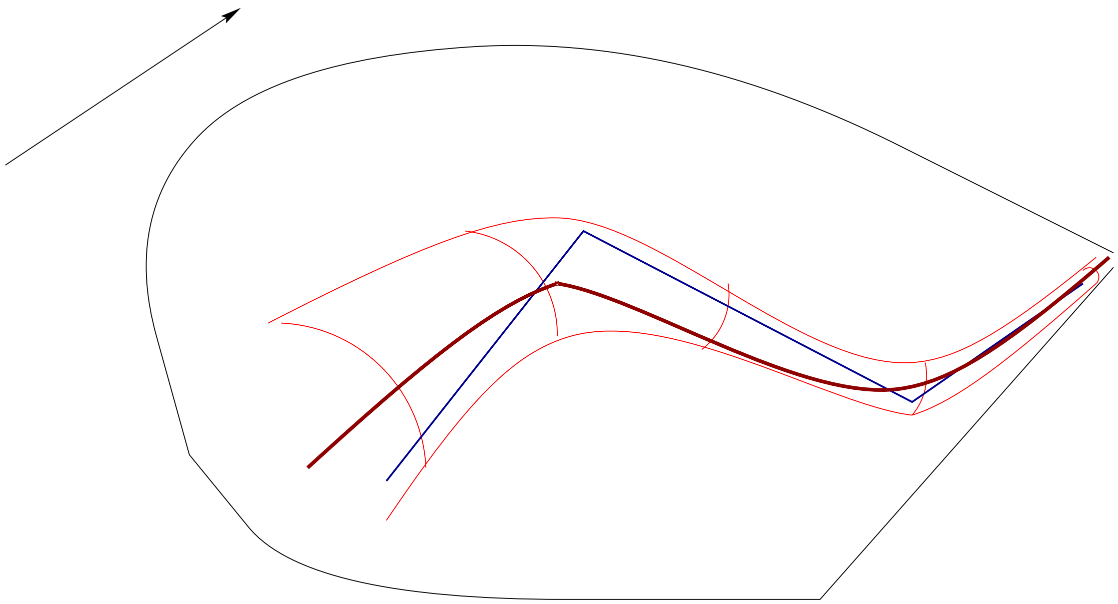
There are several centrality measures for $(\mathbf{x}, \mathbf{s}) \in \mathcal{K} \times \mathcal{K}$

$$
\begin{aligned}
d_F(\mathbf{x}, \mathbf{s}) &\stackrel{\text{def}}{=} \|\mathbf{Q}_{\mathbf{x}^{1/2}} \mathbf{s} - \mu \mathbf{e}\|_{F/\mathcal{E}} \\
&= \sqrt{\frac{1}{k} \sum_{i=1}^{n} (\lambda_i(\mathbf{xs}) - \mu)^2} \\
d_2(\mathbf{x}, \mathbf{s}) &\stackrel{\text{def}}{=} \|\mathbf{Q}_{\mathbf{x}^{1/2}} \mathbf{s} - \mu \mathbf{e}\|_2 = \max_{i=1,\dots,r} |\lambda_i(\mathbf{Q}_{\mathbf{x}^{1/2}} \mathbf{s}) - \mu| \\
&= \max\{\lambda_{\max}(\mathbf{Q}_{\mathbf{x}^{1/2}} \mathbf{s}) - \mu, \mu - \lambda_{\min}(\mathbf{Q}_{\mathbf{x}^{1/2}} \mathbf{s})\} \\
d_{-\infty}(\mathbf{x}, \mathbf{s}) &\stackrel{\text{def}}{=} \mu - \lambda_{\min}(\mathbf{Q}_{\mathbf{x}^{1/2}} \mathbf{s})
\end{aligned}
$$

For a given constant $\gamma \in (0, 1)$, we denote by $\mathcal{N}_\bullet(\gamma)$ the neighborhood of the central path

$$
\mathcal{N}_\bullet(\gamma) = \{(\mathbf{x}, \mathbf{s}, y) \mid d_\bullet(\mathbf{x}, \mathbf{s}) \le \gamma\mu, (\mathbf{x}, \mathbf{s}) \text{ interior feasible}\}
$$

- Let $r = \langle \mathbf{e}, \mathbf{e} \rangle_{\mathcal{E}}$. Then technical lemmas go through when $n$ is replaced by $r$.
- In particular one get $\mathcal{O}(\sqrt{r})$ iteration complexity for short-step path-following method



Here is the what we can prove

- Assume that $(\mathbf{x}, \mathbf{s}, y) \in \mathcal{N}_\bullet(\gamma)$ and let $(\Delta\mathbf{x}, \Delta\mathbf{s}, \Delta\mathbf{y})$ be the solution of the perturbed Newton system.
- By choosing $\gamma, \Gamma$, and $\delta$ appropriately, we can assure that

  $$(\mathbf{x} + \Delta\mathbf{x}, \mathbf{s} + \Delta\mathbf{s}, y + \Delta\mathbf{y}) \in \mathcal{N}_\bullet(\Gamma),$$

  and

  $$\langle \mathbf{x} + \Delta\mathbf{x}, \mathbf{s} + \Delta\mathbf{s} \rangle = (1 - \frac{\delta \kappa_{\mathbf{p}}}{\sqrt{r}}) \langle \mathbf{x}, \mathbf{s} \rangle$$

- **Scaling invariance** $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{N}_\bullet(\gamma)$ iff $(\widetilde{\mathbf{x}}, \mathbf{y}, \underset{\sim}{\mathbf{s}}) \in \mathcal{N}_\bullet(\gamma)$
- Therefore any proof for $\mathbf{xs} + \mathbf{sx}$ method extends to all scalings

- $\kappa_{\mathbf{p}}$ depends on our choice of scaling parameter $\mathbf{p}$
- $\kappa_{\mathbf{p}} \stackrel{\text{def}}{=} \operatorname{cond} \mathsf{L}(\widetilde{\mathbf{x}})\mathsf{L}(\utilde{\mathbf{s}})$ the condition number of $\mathsf{L}(\widetilde{\mathbf{x}})\mathsf{L}(\utilde{\mathbf{s}})$

EXAMPLE: LEMMA 3.6 (**Monteiro**): For every $\alpha \in \Re$, we have

$$\left\| \mathbf{x}^{-1/2} \left[ \mathbf{x}(\alpha)\mathbf{s}(\alpha) - \mu(\alpha)\mathbf{e} \right] \mathbf{x}^{1/2} \right\|_F \leq (1-\alpha)d(\mathbf{x},\mathbf{s}) + \alpha^2 \delta_x \delta_y + \alpha \left\| \mathbf{w_x} \right\|_F ,$$

where

$$\delta_x = \left\| \mathbf{x}^{-1/2} \Delta \mathbf{x} \mathbf{s}^{1/2} \right\|_F , \qquad \delta_s = \left\| \mathbf{s}^{-1/2} \Delta \mathbf{s} \mathbf{x}^{1/2} \right\|_F .$$

and

$$\mathbf{w_x} = \mathbf{x}^{-1/2} \left[ \Delta\mathbf{x}\mathbf{s} + \mathbf{x}\Delta\mathbf{s} + \mathbf{x}\mathbf{s} - \sigma\mu\mathbf{e} \right] \mathbf{x}^{1/2},$$

**Proof:**

$$\mathbf{x}^{-1/2}\left[\mathbf{x}(\alpha)\ \mathbf{s}(\alpha)-\mu(\alpha)\mathbf{e}\right]\mathbf{x}^{1/2}$$

$$= \mathbf{x}^{-1/2}\left[\mathbf{x}\mathbf{s}+\alpha(\mathbf{x}\Delta\mathbf{s}+\Delta\mathbf{x}\mathbf{s})+\alpha^2\Delta\mathbf{x}\Delta\mathbf{s}-\mu(\alpha)\mathbf{e}\right]\mathbf{x}^{1/2}$$

$$= (1-\alpha)(\mathbf{x}^{1/2}\mathbf{s}\mathbf{x}^{1/2}-\mu\mathbf{e})$$

$$+\ \alpha\mathbf{x}^{-1/2}\left[\mathbf{x}\mathbf{s}+\mathbf{x}\Delta\mathbf{s}+\Delta\mathbf{x}\mathbf{s}-\sigma\mu\mathbf{e}\right]\mathbf{x}^{1/2}$$

$$+\ \alpha^2\mathbf{x}^{-1/2}\Delta\mathbf{x}\Delta\mathbf{s}\mathbf{x}^{1/2}$$

$$\Rightarrow$$

$$\left\|\mathbf{x}^{-1/2}\left[\mathbf{x}(\alpha)\ \mathbf{s}(\alpha)-\mu(\alpha)\mathbf{e}\right]\mathbf{x}^{1/2}\right\|_F$$

$$= (1-\alpha)d(\mathbf{x},\mathbf{s})+\|w_x\|_F+\alpha^2\delta_x\delta_s$$

## Clifford algebras

**Definition 1** *1. The Clifford algebra $\mathcal{C}_n$ is a $2^n$-dimensional algebra which is generated by the set of elements $\{\mathbf{e}, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ having the property*

$$\mathbf{e}^2 = \mathbf{e}, \qquad \mathbf{e}\mathbf{e}_i = \mathbf{e}_i\mathbf{e} = \mathbf{e}_i, \qquad \mathbf{e}_i^2 = \mathbf{e},$$

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad \textit{for } i \neq j,\ i, j \in \{1, \dots, n\}.$$

*2. Its basis consists of the $2^n$ products*

$$\mathbf{e}_I = \prod_{i \in I} \mathbf{e}_i \quad I \subseteq N = \{1, \dots, n\}$$

- Elements of $\mathcal{C}_n$ and their product:

$$x = \sum_{I \subseteq N} x_I e_I$$

$$y = \sum_{I \subseteq N} y_I e_I$$

$$xy = \left(\sum_{I \subseteq N} x_I e_I\right)\left(\sum_{I \subseteq N} y_I e_I\right) = \sum_{I \subseteq N}\left(\sum_{J} \operatorname{sgn}(I, J) x_J y_{I \Delta J}\right) e_I$$

- A base element $\mathbf{e}_I$ is called *simple* if $|I| \leq 1$.
- It is called *even* iff $\mathbf{e}_I^2 = \mathbf{e}$ and *odd* iff $\mathbf{e}_I^2 = -\mathbf{e}$
- $\mathbf{x}$ is called *symmetric* if $x_I \neq 0 \Rightarrow \mathbf{e}_I$ even
- Similarly $\mathbf{x}$ is called *simple* if $x_I \neq 0 \Rightarrow \mathbf{e}_I$ simple

Now We have our EAJ system:

- $\mathcal{C}$ is the associative algebra
- $\mathbf{x}'$ is defined as

$$x'_I \stackrel{\text{def}}{=} \begin{cases} -x_I & \text{if } \mathbf{e}_I \text{ odd} \\ x_I & \text{if } \mathbf{e}_I \text{ even} \end{cases}$$

- $S_{2^n}$ is the set of all symmetric elements in $\mathcal{C}_n$
- $\mathfrak{R}^{n+1}$ is the set of all simple elements in $S_{2^n}$
- The EAJ system: $(\mathcal{C}_n, S_{2^n}, \mathfrak{R}^{n+1}, \prime)$.