Variational Convexity of Functions and Variational Sufficiency in Optimization

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Joint work with Pham Duy Khanh and Boris Mordukhovich
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Local Convexity Reductions and Variational Convexity
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$-smooth function and $\bar{x} \in \mathbb{R}^n$, the sufficient local optimality condition is

$$\nabla f(\bar{x}) = 0, \quad \text{and} \quad \nabla^2 f(\bar{x}) \text{ is positive definite},$$

which is equivalent to the local strong convexity of $f$ around $\bar{x}$. 
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$\implies$ This reduces to convex optimization.
Local Convexity Reduction in Second-order Sufficient Optimality Conditions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$-smooth function and $\bar{x} \in \mathbb{R}^n$, the sufficient local optimality condition is

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$\implies$ This reduces to convex optimization.

Fundamental question: Do we have such local convexity reduction in nonsmooth optimization, especially in constrained optimization?
Problems with equality constraints:

\[
\text{minimize } f_0(x) \quad \text{subject to } f_i(x) = 0, \ i = 1, 2, \ldots, m
\]

Lagrangian functions: \( L(x, y) = f_0(x) + y_1f_1(x) + \ldots + y_m f_m(x) \).

The local optimality condition of a feasible solution \( \bar{x} \) is

\[
\nabla_x L(\bar{x}, \bar{y}) = 0, \quad \nabla_y L(\bar{x}, \bar{y}) = 0
\]

\[\nabla^2_{xx} L(\bar{x}, \bar{y}) \text{ is positive definite relative to the subspace}
\]

\[
S = \{ \xi \in \mathbb{R}^n | \langle \nabla f_i(\bar{x}), \xi \rangle = 0, \ i = 1, 2, \ldots, m \}.
\]
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\( \implies \) Does this reduce to the local convexity of \( L \) around \( (\bar{x}, \bar{y}) \)?
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\( \implies \) Does this reduce to the local convexity of \( L \) around \( (\bar{x}, \bar{y}) \)?

\( \implies \) The answer is no in general!
**Theorem:**

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a l.s.c., proper function. Then $f$ is convex if and only if $\partial f$ is maximal monotone.

**Question:**

What characterization can be given for the case in which a subgradient mapping is only maximal monotone locally instead of globally?

In the smooth case, we also have the equivalence:

$f$ is convex around $\bar{x} \iff \nabla f$ is maximal monotone around $\bar{x}$.

**Question:** Do we have this equivalence in nonsmooth case?

The answer is no!
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The natural following questions arise:

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- Which property is equivalent to the second-order sufficient optimality condition in NLP, nonsmooth optimization, etc?
- Which property is equivalent to the local maximal monotonicity of subgradient mappings?

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- Which property is equivalent to the second-order sufficient optimality condition in NLP, nonsmooth optimization, etc?
- Which property is equivalent to the local maximal monotonicity of subgradient mappings?

⇒ We need a property more subtle than local convexity.

⇒ This has been answered by Rockafellar\textsuperscript{12}, and this property is called variational convexity.


Tools of Variational Analysis
See\textsuperscript{34} to find more detail.

**Regular normal cone** to $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \Omega$ is

$$\hat{N}_\Omega(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid \limsup_{\begin{subarray}{c} x \to \bar{x} \\ x \in \Omega \end{subarray}} \frac{\langle v, x - \bar{x} \rangle}{\| x - \bar{x} \|} \leq 0 \right\}$$

**Limiting normal cone** to $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \Omega$ is

$$N_\Omega(\bar{x}) := \left\{ v \in \mathbb{R}^n \mid \exists x_k \xrightarrow{\Omega} \bar{x}, \ v_k \to v, \ v_k \in \hat{N}_\Omega(x_k) \right\}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ and $x \in \Omega$

---


Regular coderivative and limiting coderivative of $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x}, \bar{y}) \in \text{gph} F$ are defined, respectively by

$$\hat{D}^* F(\bar{x}, \bar{y})(\nu) := \{ u \in \mathbb{R}^n | (u, -\nu) \in \hat{N}_{\text{gph} F}(\bar{x}, \bar{y}) \}, \ \nu \in \mathbb{R}^m$$

$$D^* F(\bar{x}, \bar{y})(\nu) := \{ u \in \mathbb{R}^n | (u, -\nu) \in N_{\text{gph} F}(\bar{x}, \bar{y}) \}, \ \nu \in \mathbb{R}^m$$

Subdifferential of $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ at $\bar{x} \in \text{dom} \varphi$ is

$$\partial \varphi(\bar{x}) := \{ \nu \in \mathbb{R}^n | (\nu, -1) \in N_{\text{epi} \varphi}(\bar{x}, \varphi(\bar{x})) \}$$
Combined second-order subdifferential and limiting second-order subdifferential of \( \varphi \) at \( \bar{x} \) relative to \( \bar{v} \in \partial \varphi(\bar{x}) \) are

\[
\tilde{\partial}^2 \varphi(\bar{x}, \bar{x})(u) := (\hat{D}^* \partial \varphi)(\bar{x}, \bar{v})(u), \quad u \in \mathbb{R}^n
\]

\[
\partial^2 \varphi(\bar{x}, \bar{x})(u) := (D^* \partial \varphi)(\bar{x}, \bar{v})(u), \quad u \in \mathbb{R}^n
\]

Note that, we have the inclusion

\[
\tilde{\partial}^2 \varphi(\bar{x}, \bar{x})(u) \subset \partial^2 \varphi(\bar{x}, \bar{x})(u) \quad \text{for all} \quad u \in \mathbb{R}^n.
\]

If \( \varphi \in C^2 \)-smooth around \( \bar{x} \), then

\[
\tilde{\partial}^2 \varphi(\bar{x}, \bar{x})(u) = \partial^2 \varphi(\bar{x}, \bar{v})(u) = \{ \nabla^2 \varphi(\bar{x})u \}, \quad u \in \mathbb{R}^n
\]
**Definition**

\( \varphi: \mathbb{R}^n \to \mathbb{R} \) is prox-regular\(^a\)\(^b\) at \( \bar{x} \in \text{dom} \varphi \) for \( \bar{v} \in \partial \varphi(\bar{x}) \) if \( \varphi \) is lower semicontinuous and there are \( \varepsilon > 0 \) and \( \rho \geq 0 \) such that for all \( x \in B_{\varepsilon}(\bar{x}) \) with \( \varphi(x) \leq \varphi(\bar{x}) + \varepsilon \) we have

\[
\varphi(x) \geq \varphi(u) + \langle \bar{v}, x - u \rangle - \frac{\rho}{2} \| x - u \|^2 \quad \forall (u, v) \in (\text{gph} \partial \varphi) \cap B_{\varepsilon}(\bar{x}, \bar{v})
\]


\( \varphi \) is subdifferentially continuous at \( \bar{x} \) for \( \bar{v} \) if the convergence \( (x_k, v_k) \to (\bar{x}, \bar{v}) \) with \( v_k \in \partial \varphi(x_k) \) yields \( \varphi(x_k) \to \varphi(\bar{x}) \). If both properties hold, \( \varphi \) is continuously prox-regular. This is the major class in second-order variational analysis.
Variationally Convex Functions

Variational Convexity

An l.s.c. function $\varphi : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is called variationally convex at $\bar{x}$ for $\bar{v} \in \partial \varphi(\bar{x})$ if for some convex neighborhood $U \times V$ of $(\bar{x}, \bar{v})$ there exist an l.s.c. convex function $\psi \leq \varphi$ on $U$ and a number $\varepsilon > 0$ such that

$$(U_\varepsilon \times V) \cap \text{gph} \partial \varphi = (U \times V) \cap \text{gph} \partial \psi \quad \text{and} \quad \varphi(x) = \psi(x), \quad (1)$$

at the common elements $(x, \nu)$, where $U_\varepsilon := \{x \in U \mid \varphi(x) < \varphi(\bar{x}) + \varepsilon\}$. We say that $\varphi$ is variationally strongly convex at $\bar{x}$ for $\bar{v}$ with modulus $\sigma > 0$ if (1) holds with $\psi$ being strongly convex on $U$ with this modulus.

Some first-order characterizations of variationally convex functions can be found in \textsuperscript{5}. The characterizations via augmented Lagrangian functions and second subderivative can be found in \textsuperscript{6}.


Variational Convexity via Moreau Envelopes
Given an extended-real-valued, proper, l.s.c. function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a positive number $\gamma$, the *Moreau envelope* $e_{\gamma} \varphi$ and the *proximal mapping* $\text{Prox}_{\gamma \varphi}$ are defined by, respectively,

$$e_{\gamma} \varphi(x) := \inf_{y \in \mathbb{R}^n} \left\{ \varphi(y) + \frac{1}{2\gamma} \|y - x\|^2 \right\},$$

$$\text{Prox}_{\gamma \varphi}(x) := \arg\min_{y \in \mathbb{R}^n} \left\{ \varphi(y) + \frac{1}{2\gamma} \|y - x\|^2 \right\}.$$
Theorem 1\[^8\]: Let \( \varphi : \mathbb{R}^n \to \overline{\mathbb{R}} \) be an l.s.c. and prox-bounded function with \( \bar{x} \in \text{dom} \varphi \) and \( \bar{v} \in \partial \varphi(\bar{x}) \). The following assertions are equivalent:

(i) \( \varphi \) is variationally convex at \( \bar{x} \) for \( \bar{v} \).

(ii) \( \varphi \) is prox-regular at \( \bar{x} \) for \( \bar{v} \), and the Moreau envelope \( e_\lambda \varphi \) is locally convex around \( \bar{x} + \lambda \bar{v} \) for small \( \lambda > 0 \).

\[^8\]P. D. Khanh, B. S. Mordukhovich, V. T. Phat, Variational convexity of functions and variational sufficiency in optimization, arXiv: 2208.14399
Theorem 2\textsuperscript{9}: Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an l.s.c. and prox-bounded function with $\bar{x} \in \text{dom} \varphi$ and $\bar{v} \in \partial \varphi(\bar{x})$. The following assertions are equivalent:

(i) $\varphi$ is variationally strongly convex at $\bar{x}$ for $\bar{v}$ with modulus $\sigma > 0$.

(ii) $\varphi$ is prox-regular at $\bar{x}$ for $\bar{v}$ and $e_{\lambda} \varphi$ is locally strongly convex around $\bar{x} + \lambda \bar{v}$ with modulus $\frac{\sigma}{1 + \sigma \lambda}$ for all numbers $\lambda > 0$ sufficiently small.

\textsuperscript{9}P. D. Khanh, B. S. Mordukhovich, V. T. Phat, Variational convexity of functions and variational sufficiency in optimization, arXiv: 2208.14399
Theorem 3\textsuperscript{10}: Let $\varphi : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be an l.s.c. and prox-bounded function with $\bar{x} \in \text{dom } \varphi$ and $\bar{v} \in \partial \varphi(\bar{x})$. The following assertions are equivalent:

(i) $\varphi$ is \textbf{variationally strongly convex} at $\bar{x}$ for $\bar{v}$.

(ii) $\varphi$ is \textbf{prox-regular} at $\bar{x}$ for $\bar{v}$ and $e_{\lambda} \varphi$ is \textbf{locally strongly convex} around $\bar{x} + \lambda \bar{v}$ for all numbers $\lambda > 0$ sufficiently small.

\textsuperscript{10} P. D. Khanh, B. S. Mordukhovich, V. T. Phat, Variational convexity of functions and variational sufficiency in optimization, arXiv: 2208.14399
C coderivative-Based Characterizations of Variational Convexity
Theorem 4: Let $\varphi : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be subdifferentially continuous at $\bar{x} \in \text{dom} \varphi$ and $\bar{v} \in \partial \varphi(\bar{x})$. Then the following assertions are equivalent:

(i) $\varphi$ is variationally convex at $\bar{x}$ for $\bar{v}$.

(ii) $\varphi$ is prox-regular at $\bar{x}$ for $\bar{v}$ and there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{v}$ such that

$$\langle z, w \rangle \geq 0 \text{ whenever } z \in \partial^2 \varphi(x, y)(w), (x, y) \in \text{gph } \partial \varphi \cap (U \times V), w \in \mathbb{R}^n.$$ (4)

(iii) $\varphi$ is prox-regular at $\bar{x}$ for $\bar{v}$ and there exist neighborhoods $U$ of $\bar{x}$, and $V$ of $\bar{v}$ such that

$$\langle z, w \rangle \geq 0 \text{ whenever } z \in \partial^2 \varphi(x, y)(w), (x, y) \in \text{gph } \partial \varphi \cap (U \times V), w \in \mathbb{R}^n.$$ (5)

$^{11}$P. D. Khanh, B. S. Mordukhovich, V. T. Phat, Variational convexity of functions and variational sufficiency in optimization, arXiv: 2208.14399
**Theorem 5**: Let \( \varphi : \mathbb{R}^n \to \overline{\mathbb{R}} \) be subdifferentially continuous at \( \bar{x} \in \text{dom} \varphi \) and \( \bar{v} \in \partial \varphi(\bar{x}) \). Then the following assertions are equivalent:

(i) \( \varphi \) is variationally strongly convex at \( \bar{x} \) for \( \bar{v} \) with modulus \( \sigma > 0 \).

(ii) \( \varphi \) is prox-regular at \( \bar{x} \) for \( \bar{v} \) and there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{v} \) such that

\[
\langle z, w \rangle \geq \sigma \|w\|^2 \quad \text{whenever} \quad z \in \partial^2 \varphi(x, y)(w), \ (x, y) \in \text{gph} \partial \varphi \cap (U \times V), \ w \in \mathbb{R}^n.
\] (6)

(iii) \( \varphi \) is prox-regular at \( \bar{x} \) for \( \bar{v} \) and there are neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{v} \) such that

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\langle z, w \rangle \geq \sigma \|w\|^2 \quad \text{whenever} \quad z \in \partial^2 \varphi(x, y)(w), \ (x, y) \in \text{gph} \partial \varphi \cap (U \times V), \ w \in \mathbb{R}^n.
\] (7)

Furthermore, the strong variational convexity in (i) with some modulus \( \sigma > 0 \) is equivalent to the prox regularity of \( \varphi \) at \( \bar{x} \) for \( \bar{v} \) together with the fulfillment of the pointbased condition

\[
\langle z, w \rangle > 0 \quad \text{whenever} \quad z \in \partial^2 \varphi(\bar{x}, \bar{v})(w), \ w \neq 0.
\] (8)

---

Variational Sufficiency in Composite Optimization
Let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a lower semicontinuous function. Suppose that $\bar{x} \in \text{dom} \varphi$ is a stationary point, i.e., $0 \in \partial \varphi(\bar{x})$. We have the following implications:

- Variational convexity of $\varphi$ at $\bar{x}$ $\Rightarrow \bar{x}$ is a local minimizer.
- Variational strong convexity of $\varphi$ at $\bar{x}$ $\Rightarrow \bar{x}$ is a tilt-stable local minimizer.
Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function. Suppose that $\bar{x} \in \text{dom} \varphi$ is a stationary point, i.e., $0 \in \partial \varphi(\bar{x})$. We have the following implications:

- variational convexity of $\varphi$ at $\bar{x} \implies \bar{x}$ is a local minimizer.

- variational strong convexity of $\varphi$ at $\bar{x}$ with modulus $\sigma > 0 \implies \bar{x}$ is a tilt-stable local minimizer.
Variational Sufficiency in Optimization

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a lower semicontinuous function. Suppose that $\bar{x} \in \text{dom } \varphi$ is a stationary point, i.e., $0 \in \partial \varphi(\bar{x})$. We have the following implications:

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- variational convexity of $\varphi$ at $\bar{x}$ $\Longrightarrow$ $\bar{x}$ is a local minimizer.
- variational strong convexity of $\varphi$ at $\bar{x}$ $\Longrightarrow$ $\bar{x}$ is a tilt-stable local minimizer.

**Definition**

Let $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$, and consider the unconstrained optimization problem:

$$\begin{align*}
\text{minimize} \quad & \varphi(x) \\
\text{subject to} \quad & x \in \mathbb{R}^n.
\end{align*}$$

(9)

It is said that the variational sufficient condition for local optimality in (9) holds at $\bar{x}$ if $\varphi$ is variationally convex at $\bar{x}$ for $0 \in \partial \varphi(\bar{x})$. If $\varphi$ is variationally strongly convex at $\bar{x}$ for $0$ with modulus $\sigma > 0$, then we say that the strong variational sufficient condition for local optimality at $\bar{x}$ holds with modulus $\sigma$. 
Here we consider the class of composite optimization problems given by:

$$\min \varphi(x) := \varphi_0(x) + \psi(g(x)) \quad \text{subject to} \quad x \in \mathbb{R}^n, \quad (10)$$

where $\psi : \mathbb{R}^m \rightarrow \overline{\mathbb{R}}$ is an extended-real-valued l.s.c. function, $\varphi_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is a $C^2$-smooth function, and $g$ is a $C^2$-smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. 

The characterizations of variational sufficiency in (10)? For each $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ define the set of multipliers $\Lambda(x, v) := \{y \in \mathbb{R}^m \mid v = \nabla \varphi_0(x) + \nabla g(x)^* y, \ y \in \partial \psi(g(x))\}$. (11)
Here we consider the class of composite optimization problems given by:

$$\begin{align*}
\text{minimize } & \varphi(x) := \varphi_0(x) + \psi(g(x)) \quad \text{subject to } x \in \mathbb{R}^n, \\
\text{where } & \psi : \mathbb{R}^m \to \overline{\mathbb{R}} \text{ is an extended-real-valued l.s.c. function,} \\
& \varphi_0 : \mathbb{R}^n \to \mathbb{R} \text{ is a } C^2\text{-smooth function, and } g \text{ is a } C^2\text{-smooth mapping from } \mathbb{R}^n \to \mathbb{R}^m.
\end{align*}$$

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\text{minimize } \varphi(x) := \varphi_0(x) + \psi(g(x)) \quad \text{subject to } x \in \mathbb{R}^n, \quad (10)
\]

where \(\psi : \mathbb{R}^m \to \overline{\mathbb{R}}\) is an extended-real-valued l.s.c. function, \(\varphi_0 : \mathbb{R}^n \to \mathbb{R}\) is a \(C^2\)-smooth function, and \(g\) is a \(C^2\)-smooth mapping from \(\mathbb{R}^n\) to \(\mathbb{R}^m\).

The characterizations of variational sufficiency in (10)?

For each \((x, v) \in \mathbb{R}^n \times \mathbb{R}^n\) define the set of multipliers

\[
\Lambda(x, v) := \{ y \in \mathbb{R}^m \mid v = \nabla \varphi_0(x) + \nabla g(x)^* y, \ y \in \partial \psi(g(x)) \}. \quad (11)
\]
Theorem 6: Let \( \bar{x} \in \mathbb{R}^n \) be a stationary point of the composite optimization problem at which \( \text{rank} \ \nabla g(\bar{x}) = m \) and hence there exists a unique vector \( \bar{y} \in \mathbb{R}^m \) with

\[
\nabla \varphi_0(\bar{x}) + \nabla g(\bar{x})^* \bar{y} = 0 \quad \text{and} \quad \bar{y} \in \partial \psi(g(\bar{x})).
\]  

(12)

Suppose in addition that \( \psi \) is subdifferentially continuous at \( g(\bar{x}) \) for \( \bar{y} \). Then we have the following assertions:

(i) The variational sufficiency holds at \( \bar{x} \) if and only if \( \psi \) is prox-regular at \( g(\bar{x}) \) for \( \bar{y} \) and there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of 0 such that

\[
\langle \nabla^2 \varphi_0(x) w, w \rangle + \langle \nabla^2 \langle y, g \rangle(x) w, w \rangle + \langle u, \nabla g(x) w \rangle \geq 0
\]

(13)

for all \( x \in U, v \in V, y \in \Lambda(x, v), u \in \partial^2 \psi(g(x), y) (\nabla g(x) w) \) and \( w \in \mathbb{R}^n \), where \( \Lambda(x, v) \) is a singleton in this case.

(ii) The strong variational sufficiency holds at \( \bar{x} \) with modulus \( \sigma > 0 \) if and only if \( \psi \) is

prox-regular at \( g(\bar{x}) \) for \( \bar{y} \) and there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of 0 such that

\[
\langle \nabla^2 \varphi_0(x) w, w \rangle + \langle \nabla^2 \langle y, g \rangle(x) w, w \rangle + \langle u, \nabla g(x) w \rangle \geq \sigma \|w\|^2
\]

(14)

for all \( x \in U, v \in V, y \in \Lambda(x, v), u \in \partial^2 \psi(g(x), y) (\nabla g(x) w) \) and \( w \neq 0 \).

Furthermore, the strong variational sufficiency in (ii) with some modulus \( \sigma > 0 \) is equivalent to the prox-regularity of \( \psi \) at \( g(\bar{x}) \) for \( \bar{y} \) together with the fulfillment of the point-based condition

\[
\langle \nabla^2 \varphi_0(\bar{x}) w, w \rangle + \langle \nabla^2 \langle \bar{y}, g \rangle(\bar{x}) w, w \rangle + \langle u, \nabla g(\bar{x}) w \rangle > 0
\]

(15)

whenever \( u \in \partial^2 \psi(g(\bar{x}), \bar{y}) (\nabla g(\bar{x}) w) \) and \( w \neq 0 \).
Theorem 6: Let $\bar{x} \in \mathbb{R}^n$ be a stationary point of the composite optimization problem at which $\text{rank} \nabla g(\bar{x}) = m$ and hence there exists a unique vector $\bar{y} \in \mathbb{R}^m$ with

$$\nabla \varphi_0(\bar{x}) + \nabla g(\bar{x})^* \bar{y} = 0 \quad \text{and} \quad \bar{y} \in \partial \psi(g(\bar{x})).$$

(12)

Suppose in addition that $\psi$ is subdifferentially continuous at $g(\bar{x})$ for $\bar{y}$. Then we have the following assertions:

(i) The variational sufficiency holds at $\bar{x}$ if and only if $\psi$ is prox-regular at $g(\bar{x})$ for $\bar{y}$ and there exist neighborhoods $U$ of $\bar{x}$ and $V$ of 0 such that

$$\langle \nabla^2 \varphi_0(x)w, w \rangle + \langle \nabla^2 \langle y, g \rangle(x)w, w \rangle + \langle u, \nabla g(x)w \rangle \geq 0$$

(13)

for all $x \in U$, $v \in V$, $y \in \Lambda(x, v)$, $u \in \partial^2 \psi(g(x), y)(\nabla g(x)w)$, $w \in \mathbb{R}^n$, where $\Lambda(x, v)$ is a singleton in this case.

Furthermore, the strong variational sufficiency in (ii) with some modulus $\sigma > 0$ is equivalent to the prox-regularity of $\psi$ at $g(\bar{x})$ for $\bar{y}$ together with the fulfillment of the point-based condition

$$\langle \nabla^2 \varphi_0(\bar{x})w, w \rangle + \langle \nabla^2 \langle \bar{y}, g \rangle(\bar{x})w, w \rangle + \langle u, \nabla g(\bar{x})w \rangle > 0$$

(15)

whenever $u \in \partial^2 \psi(g(\bar{x}), \bar{y})(\nabla g(\bar{x})w)$ and $w \neq 0$. 

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Theorem 6: Let $\bar{x} \in \mathbb{R}^n$ be a stationary point of the composite optimization problem at which \( \text{rank} \ \nabla g(\bar{x}) = m \) and hence there exists a unique vector $\bar{y} \in \mathbb{R}^m$ with

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(i) The variational sufficiency holds at $\bar{x}$ if and only if $\psi$ is prox-regular at $g(\bar{x})$ for $\bar{y}$ and there exist neighborhoods $U$ of $\bar{x}$ and $V$ of 0 such that

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(13)

for all $x \in U$, $v \in V$, $y \in \Lambda(x, v)$, $u \in \partial^2 \psi(g(x), y)(\nabla g(x)w)$, $w \in \mathbb{R}^n$, where $\Lambda(x, v)$ is a singleton in this case.

(ii) The strong variational sufficiency holds at $\bar{x}$ with modulus $\sigma > 0$ if and only if $\psi$ is prox-regular at $g(\bar{x})$ for $\bar{y}$ and there exist neighborhoods $U$ of $\bar{x}$ and $V$ of 0 such that

$$\langle \nabla^2 \varphi_0(x)w, w \rangle + \langle \nabla^2 \langle y, g \rangle(x)w, w \rangle + \langle u, \nabla g(x)w \rangle \geq \sigma \|w\|^2$$

(14)

for all $x \in U$, $v \in V$, $y \in \Lambda(x, v)$, $u \in \partial^2 \psi(g(x), y)(\nabla g(x)w)$, $w \in \mathbb{R}^n$, where $\Lambda(x, v)$ is a singleton in this case.
**Theorem 6:** Let $\bar{x} \in \mathbb{R}^n$ be a stationary point of the composite optimization problem at which $\text{rank} \, \nabla g(\bar{x}) = m$ and hence there exists a unique vector $\bar{y} \in \mathbb{R}^m$ with

$$\nabla \varphi_0(\bar{x}) + \nabla g(\bar{x})^* \bar{y} = 0 \quad \text{and} \quad \bar{y} \in \partial \psi(g(\bar{x})). \quad (12)$$

Suppose in addition that $\psi$ is subdifferentially continuous at $g(\bar{x})$ for $\bar{y}$. Then we have the following assertions:

(i) The variational sufficiency holds at $\bar{x}$ if and only if $\psi$ is prox-regular at $g(\bar{x})$ for $\bar{y}$ and there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $0$ such that

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$$\langle \nabla^2 \varphi_0(x)w, w \rangle + \langle \nabla^2 \langle y, g \rangle(x)w, w \rangle + \langle u, \nabla g(x)w \rangle \geq \sigma \|w\|^2 \quad (14)$$

for all $x \in U$, $v \in V$, $y \in \Lambda(x,v)$, $u \in \partial^2 \psi(g(x), y)(\nabla g(x)w)$, $w \in \mathbb{R}^n$, where $\Lambda(x, v)$ is a singleton in this case.

Furthermore, the strong variational sufficiency in (ii) with some modulus $\sigma > 0$ is equivalent to the prox-regularity of $\psi$ at $g(\bar{x})$ for $\bar{y}$ together with the fulfillment of the pointbased condition

$$\langle \nabla^2 \varphi_0(\bar{x})w, w \rangle + \langle \nabla^2 \langle \bar{y}, g \rangle(\bar{x})w, w \rangle + \langle u, \nabla g(\bar{x})w \rangle > 0 \quad (15)$$

whenever $u \in \partial^2 \psi(g(\bar{x}), \bar{y})(\nabla g(\bar{x})w)$ and $w \neq 0$. 
An l.s.c. function $\theta : \mathbb{R}^n \rightarrow \mathbb{R}$ is strongly amenable at $\bar{x}$ if there exists neighborhood $U$ of $\bar{x}$ on which $\theta$ can be represented in the composition form $\theta = \psi \circ g$ with a $C^2$-smooth mapping $g : U \rightarrow \mathbb{R}^m$ and a proper l.s.c. convex function $\psi : \mathbb{R}^m \rightarrow \mathbb{R}$ such that the following first-order qualification condition holds:

$$\partial^\infty \psi(\bar{z}) \cap \ker \nabla g(\bar{x})^* = \{0\} \text{ with } \bar{z} := g(\bar{x})$$ (16)

The second-order qualification condition (SOQC) for problem (10) at $\bar{x}$, which is formulated as follows:

$$\partial^2 \psi(\bar{z}, \bar{y})(0) \cap \ker \nabla g(\bar{x})^* = \{0\} \text{ with } \bar{y} \in \partial \psi(\bar{z}) \text{ and } \bar{z} := g(\bar{x}).$$ (17)
**Theorem 7:** Let \( \bar{x} \in \mathbb{R}^n \) be a **stationary point** of the composite optimization problem. Suppose in addition that \( \psi \) and \( g \) be mappings from the composite representation of a strongly amenable function at \( \bar{x} \) and that the **second-order qualification condition** is satisfied at \( \bar{x} \). Then we have the following assertions:

(i) The variational sufficiency holds at \( \bar{x} \) if there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of 0 such that (13) is satisfied for all \( x \in U, \ v \in V, \ y \in \Lambda(x, v), \ u \in \partial^2 \psi(g(x), y)(\nabla g(x)w), \) and \( w \in \mathbb{R}^n. \)

(ii) The strong variational sufficiency holds at \( \bar{x} \) with modulus \( \sigma > 0 \) if there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of 0 such that the neighborhood condition (14) is satisfied for all \( x \in U, \ v \in V, \ y \in \Lambda(x, v), \ u \in \partial^2 \psi(g(x), y)(\nabla g(x)w), \) and \( w \in \mathbb{R}^n. \)

(iii) The strong variational sufficiency holds at \( \bar{x} \) if the pointbased condition (15) is satisfied for any \( \bar{y} \in \Lambda(\bar{x}, 0). \)
Theorem 7: Let \( \bar{x} \in \mathbb{R}^n \) be a stationary point of the composite optimization problem. Suppose in addition that \( \psi \) and \( g \) be mappings from the composite representation of a strongly amenable function at \( \bar{x} \) and that the second-order qualification condition is satisfied at \( \bar{x} \). Then we have the following assertions:

(i) The variational sufficiency holds at \( \bar{x} \) if there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of 0 such that (13) is satisfied for all \( x \in U, \ v \in V, \ y \in \Lambda(x, v), \ u \in \partial^2 \psi(g(x), y)(\nabla g(x)w), \) and \( w \in \mathbb{R}^n \).
**Theorem 7:** Let $\bar{x} \in \mathbb{R}^n$ be a stationary point of the composite optimization problem. Suppose in addition that $\psi$ and $g$ be mappings from the composite representation of a strongly amenable function at $\bar{x}$ and that the second-order qualification condition is satisfied at $\bar{x}$. Then we have the following assertions:

(i) The variational sufficiency holds at $\bar{x}$ if there exist neighborhoods $U$ of $\bar{x}$ and $V$ of 0 such that (13) is satisfied for all $x \in U$, $v \in V$, $y \in \Lambda(x, v)$, $u \in \partial^2 \psi(g(x), y)(\nabla g(x)w)$, and $w \in \mathbb{R}^n$.

(ii) The strong variational sufficiency holds at $\bar{x}$ with modulus $\sigma > 0$ if there exist neighborhoods $U$ of $\bar{x}$ and $V$ of 0 such that the neighborhood condition (14) is satisfied for all $x \in U$, $v \in V$, $y \in \Lambda(x, v)$, $u \in \partial^2 \psi(g(x), y)(\nabla g(x)w)$, and $w \in \mathbb{R}^n$. 

(iii) The strong variational sufficiency holds at $\bar{x}$ if the point-based condition (15) is satisfied for any $\bar{y} \in \Lambda(\bar{x}, 0)$. 


**Theorem 7:** Let \( \bar{x} \in \mathbb{R}^n \) be a stationary point of the composite optimization problem. Suppose in addition that \( \psi \) and \( g \) be mappings from the composite representation of a strongly amenable function at \( \bar{x} \) and that the second-order qualification condition is satisfied at \( \bar{x} \). Then we have the following assertions:

(i) The variational sufficiency holds at \( \bar{x} \) if there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of 0 such that (13) is satisfied for all \( x \in U, \ v \in V, \ y \in \Lambda(x, v), \ u \in \partial^2 \psi(g(x), y)(\nabla g(x)w), \) and \( w \in \mathbb{R}^n \).

(ii) The strong variational sufficiency holds at \( \bar{x} \) with modulus \( \sigma > 0 \) if there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of 0 such that the neighborhood condition (14) is satisfied for all \( x \in U, \ v \in V, \ y \in \Lambda(x, v), \ u \in \partial^2 \psi(g(x), y)(\nabla g(x)w), \) and \( w \in \mathbb{R}^n \).

(iii) The strong variational sufficiency holds at \( \bar{x} \) if the pointbased condition (15) is satisfied for any \( \bar{y} \in \Lambda(\bar{x}, 0) \).
**Theorem 8:** In addition to the assumptions of Theorem 7, suppose that  
(a) either \( \psi \) is piecewise linear,  
(b) or \( \psi \) is of class

\[
\psi(z) := \sup_{v \in P} \left\{ \langle v, z \rangle - \frac{1}{2} \langle Qv, v \rangle \right\},
\]

where \( P \subset \mathbb{R}^m \) is a nonempty polyhedral set, \( Q \) is positive-definite, and the inner mapping \( g \) is open around \( \bar{x} \).

Then all the three characterizations (i)–(iii) of Theorem 7 hold.
Applications to Nonlinear Programming
The conventional model of nonlinear programming (NLP) is formulated as follows:

$$\text{minimize } \varphi_0(x) \text{ subject to } \begin{cases} 
\varphi_i(x) \leq 0 & \text{for } i = 1, \ldots, s, \\
\varphi_i(x) = 0 & \text{for } i = s + 1, \ldots, m,
\end{cases}$$

(19)

where $\varphi_i$, $i = 0, \ldots, m$, are $C^2$-smooth functions around the references points. Problem (19) can be obviously written in the form of composite optimization (10) with $\psi = \delta_\Omega$, where $\Omega$ is given by

$$\Omega := \{ u \in \mathbb{R}^m \mid u_i \leq 0 \text{ for } i = 1, \ldots, s \text{ and } u_i = 0 \text{ for } i = s + 1, \ldots, m \},$$

(20)

and where $g(x) := (\varphi_1(x), \ldots, \varphi_m(x))$. 
Lagrangian functions: $L(x, y) = \varphi_0(x) + y_1\varphi_1(x) + \ldots + y_m\varphi_m(x)$.

For each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ consider the subspace

$$S(x, y) := \{ w \in \mathbb{R}^n \mid \langle \nabla \varphi_i(x), w \rangle = 0 \text{ for } i \in I^+(x, y) \cup \{ s + 1, \ldots, m \} \}$$

(21)

together with the index collections

$$I^+(x, y) := \{ i \in I(x) \mid y_i > 0 \} \text{ and } I(x) := \{ i \in \{ 1, \ldots, s \} \mid \varphi_i(x) = 0 \}.$$  

(22)
**Corollary 9:** Let $\bar{x}$ be a feasible solution to the NLP problem satisfying the first-order optimality condition under the fulfillment of LICQ at $\bar{x}$. Then we have the following assertions:

(i) The variational sufficiency holds at $\bar{x}$ if and only if there exist neighborhoods $U$ of $\bar{x}$, $V$ of 0 such that

$$\langle \nabla^2_{xx} L(x, y) w, w \rangle \geq 0 \quad \text{whenever} \quad x \in U, \ v \in V, \ \text{and} \ w \in S(x, y),$$

where $y \in \mathbb{R}^s_+ \times \mathbb{R}^{m-s}$ is a unique solution to the system

$$\nabla_x L(x, y) = v, \ y_1 \varphi_1(x) + \ldots + y_m \varphi_m(x) = 0.$$ (23)

(ii) The strong variational sufficiency holds at $\bar{x}$ with modulus $\sigma > 0$ if and only if there exist neighborhoods $U$ of $\bar{x}$, $V$ of 0 such that

$$\langle \nabla^2_{xx} L(x, y) w, w \rangle \geq \sigma \|w\|^2 \quad \text{whenever} \quad x \in U, \ v \in V, \ \text{and} \ w \in S(x, y),$$

where $y \in \mathbb{R}^s_+ \times \mathbb{R}^{m-s}$ is a unique solution to the system

$$\nabla_x L(x, y) = v, \ y_1 \varphi_1(x) + \ldots + y_m \varphi_m(x) = 0.$$ (24)
(iii) The strong variational sufficiency holds at $\bar{x}$ if and only if

$$\langle \nabla^2_{xx} L(\bar{x}, \bar{y}) w, w \rangle > 0 \quad \text{whenever} \quad \bar{y} \in \Lambda(\bar{x}, 0) \quad \text{and} \quad w \in S(\bar{x}, \bar{y}) \setminus \{0\},$$

where $\bar{y} \in \mathbb{R}_+^s \times \mathbb{R}^{m-s}$ is a unique solution to the system

$$\nabla_x L(\bar{x}, y) = v, \quad y_1 \varphi_1(\bar{x}) + \ldots + y_m \varphi_m(\bar{x}) = 0.$$
Future Investigations

- **Numerical methods** that benefit from the local convexity/local strong convexity of Moreau envelopes of variationally convex/variationally strongly convex functions.
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- Explicit characterizations of **variational sufficiency vs. strong variational sufficiency** of polyhedral and nonpolyhedral optimization problems.
Future Investigations

- **Numerical methods** that benefit from the local convexity/local strong convexity of Moreau envelopes of variationally convex/variationally strongly convex functions.
- Explicit characterizations of *variational sufficiency* vs. *strong variational sufficiency* of polyhedral and nonpolyhedral optimization problems.
- **Graphical derivative** characterizations of variational convexity and strong variational convexity for extended-real-valued functions with applications in NLP.
THANK YOU FOR YOUR ATTENTION