CO350 Linear Programming
Chapter 6: The Simplex Method

8th June 2005

Minimization Problem (§6.5)

We can solve minimization problems by transforming it into a maximization problem.

Another way is to change the selection rule for entering variable.

Since we want to minimize z, we would now choose a reduced cost $\bar{c}_k$ that is negative, so that increasing the nonbasic variable x_k decreases the objective value.

We now stop with an optimal solution only when $\bar{c}_j \geq 0$ for all $j \in N$.

Final remark:

We can always transform the LP into a maximization problem, and use the simplex method as normal.

Choice Rules (§6.6)

In the simplex method, we need to make two choices at each step: entering and leaving variables.

When choosing entering variable, there may be more than one reduced cost $\bar{c}_j > 0$.

When choosing leaving variable, there may be more than one ratio $\bar{b}_i/\bar{a}_{ik}$ that matches the minimum ratio.

We may pick entering and leaving variable arbitrarily in the event of multiple choices.

On the other hand, we may devise choice rules for choosing entering and leaving variable.

We leave the discussion of the choice rules for leaving variable to a later chapter.

Rules for entering variables

We discuss four commonly used choice rules for entering variables.

We shall use the following tableau (T) for illustration.

$$(T) \qquad \begin{array}{rcrcrcrcrcl} z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 2 \\ & & x_1 & - & x_2 & & & + & x_4 & = & 3 \end{array}$$

Largest coefficient rule.

- This rule was first suggested by George Dantzig (the inventor of the simplex method).
- As a result, it is also known as Dantzig's rule.
- The rule states
  "Pick the nonbasic variable with the largest reduced cost. Break tie arbitrarily".
- E.g., in (T), since $\bar{c}_2 = 3 > 2 = \bar{c}_1$, we choose x_2 to enter according to this rule.
- This rule is well-motivated and widely used since it picks a variable that gives the largest increase in objective value per unit of increase in the variable.

Rules for entering variables

We discuss four commonly used choice rules for entering variables.

We shall use the following tableau (T) for illustration.

$$(T) \quad \begin{array}{rrrrrrrrrrr} z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 2 \\ & & x_1 & - & x_2 & & & + & x_4 & = & 3 \end{array}$$

Smallest subscript rule.

- This rule assumes that the variable are in pre-arranged (before starting the simplex method) in a certain order.
- When the variables are labelled $x_1, x_2, \ldots, x_n$, we order them in ascending order of their subscript (hence the name for the rule).
- The rule states
  "Pick the nonbasic variable with the least subscript among those with positive reduced cost".
- E.g., in (T), both x_1 and x_2 have positive reduced costs. Since x_1 has a smaller subscript than x_2, we choose x_1 to enter according to this rule.
- This rule is not so well-motivated as Dantzig's rule, but it is unambiguous — there is no need to break tie.

Rules for entering variables

We discuss four commonly used choice rules for entering variables.

We shall use the following tableau (T) for illustration.

$$(T) \quad \begin{array}{rcrcrcrcrcl} z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\ & & x_1 & + & 2x_2 & + & x_3 & & & = & 2 \\ & & x_1 & - & x_2 & & & + & x_4 & = & 3 \end{array}$$

Largest improvement rule.

- This rule has the same motivation as Dantzig's rule — it aims for large increase in objective value.
- We need to compute the amount of possible increase t_j for each potential entering variable x_j.
- The rule states
  "Pick the nonbasic variable with the largest $\bar{c}_j t_j$ among those with positive reduced cost".
- E.g., in (T), both x_1 and x_2 have positive reduced costs.
  For x_1 we can increase it to $t_1 = \min\{2/1, 3/1\} = 2$.
  For x_2 we can increase it to $t_2 = \min\{2/2, -\} = 1$.
  Since $\bar{c}_1 t_1 = 2 \times 2 > 3 \times 1 = \bar{c}_2 t_2$, we choose x_1 to enter according to this rule.
- This rule picks the variable that results in the largest overall increase in objective value.

Rules for entering variables

We discuss four commonly used choice rules for entering variables.

We shall use the following tableau (T) for illustration.

$$
(T) \quad \begin{array}{rcrcrcrcrcl}
z & - & 2x_1 & - & 3x_2 & & & & & = & 0 \\
 & & x_1 & + & 2x_2 & + & x_3 & & & = & 2 \\
 & & x_1 & - & x_2 & & & + & x_4 & = & 3
\end{array}
$$

Steepest edge rule.

- This rule also aim for large increase in objective value; however, it does so geometrically.
- We need to compute, for each potential entering variable, the increase in objective value per unit distance moved.
- For each $\bar{c}_k > 0$, increasing x_k from 0 to t changes each basic variable x_i from x_i^* to $x_i^* - \bar{a}_{ik}t$ and leaves the other nonbasic variables unchanged.

  So the solution has moved a distance of

$$\sqrt{\sum_{i \in B} \left((x_i^* - \bar{a}_{ik}t) - x_i^*\right)^2 + (t-0)^2} = \sqrt{\sum_{i \in B} \bar{a}_{ik}^2 t^2 + t^2}$$

  Also, the objective value changes from $\bar{v}$ to $\bar{v} + \bar{c}_k t$.

- So the increase in objective value per unit distance is

$$\frac{(\bar{v} + \bar{c}_k t) - \bar{v}}{\sqrt{\sum_{i \in B} \bar{a}_{ik}^2 t^2 + t^2}} = \frac{\bar{c}_k t}{t\sqrt{\sum_{i \in B} \bar{a}_{ik}^2 + 1}} = \frac{\bar{c}_k}{\sqrt{\sum_{i \in B} \bar{a}_{ik}^2 + 1}}$$

Rules for entering variables

We discuss four commonly used choice rules for entering variables.

We shall use the following tableau (T) for illustration.

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Steepest edge rule.

- So the increase in objective value per unit distance is

$$\frac{(\bar{v}+\bar{c}_k t)-\bar{v}}{\sqrt{\sum_{i\in B}\bar{a}_{ik}^2 t^2 + t^2}} = \frac{\bar{c}_k t}{t\sqrt{\sum_{i\in B}\bar{a}_{ik}^2+1}} = \frac{\bar{c}_k}{\sqrt{\sum_{i\in B}\bar{a}_{ik}^2+1}}$$

- The rule states

  "Pick the nonbasic variable x_k with the largest value of $\dfrac{\bar{c}_k}{\sqrt{\sum_{i\in B}\bar{a}_{ik}^2+1}}$ among those with positive reduced cost".

- E.g., in (T), both x_1 and x_2 have positive reduced costs.

  For x_1, we have $\dfrac{\bar{c}_k}{\sqrt{\sum_{i\in B}\bar{a}_{ik}^2+1}} = \dfrac{2}{\sqrt{1^2+1^2+1}} = \dfrac{2}{\sqrt{3}}$.

  For x_2, we have $\dfrac{\bar{c}_k}{\sqrt{\sum_{i\in B}\bar{a}_{ik}^2+1}} = \dfrac{3}{\sqrt{2^2+(-1)^2+1}} = \dfrac{3}{\sqrt{6}}$.

  Since $\dfrac{3}{\sqrt{6}} > \dfrac{2}{\sqrt{3}}$, we choose x_2 to enter.

The Simplex Method and Duality (§6.7)

Suppose at the end of the simplex method, we have an optimal solution x^* determined by a basis B and the corresponding tableau

$$(T) \qquad \begin{array}{rcrcl} z & - & \sum_{j\in N} \bar{c}_j x_j & = & \bar{v} \\ x_i & + & \sum_{j\in N} \bar{a}_{ij} x_j & = & \bar{b}_i \quad (i \in B) \end{array}$$

From (T), we can give a proof of optimality of x^*.

Recall that we may also prove the optimality of x^* using the C.S. Theorem.

We only need to demonstrate a y^* that is feasible for the dual LP, and satisfy the C.S. condition with x^*.

It turns out that the simplex method is implicitly computing this y^* at the same time!

The simplex method uses elementary row operations to move from the initial tableau to the final optimal tableau

So the z-row in the final tableau must be obtained by taking a linear combination of the equations $Ax = b$ and add it to the equation $z - c^T x = 0$.

Suppose that this linear combination is

$$\hat{y}_1 \times (\text{eqn. } 1) + \hat{y}_2 \times (\text{eqn. } 2) + \cdots + \hat{y}_m \times (\text{eqn. } m)$$

If we let $\hat{y} = [\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_m]^T$, then we can write the linear combination as $$\hat{y}^T A x = \hat{y}^T b$$

In another words, the z-row $[z - \sum_{j \in N} \bar{c}_j x_j = \bar{v}]$ is actually

$$z - c^T x + \hat{y}^T A x = \hat{y}^T b$$

$$\text{i.e. } z - (c - A^T \hat{y})^T x = b^T \hat{y}$$

Comparing the above expression with the z-row, we conclude that

$$c_i - A_i^T \hat{y} = 0 \quad (i \in B)$$

$$c_j - A_j^T \hat{y} = \bar{c}_j \quad (j \in N)$$

Since $$x_i^* > 0 \implies i \in B \implies A_i^T \hat{y} = c_i,$$
$\hat{y}$ satisfies the C.S. condition with x^*.

Since $\bar{c}_j \leq 0$ for all $j \in N$ in the optimal tableau (T), $\hat{y}$ satisfies the dual constraints

$$A_i^T \hat{y} \geq c_i \ (i = 1, 2, \ldots, m)$$

By the C.S. Theorem, x^* is optimal for the primal and $\hat{y}$ is optimal for the dual.