perfect wine glass

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1 Introduction

The following is a description of the *perfect wine glass* variational problem. The objective is to find the surface which can enclose a maximal volume. What makes a possible solution different from the sphere is that we allow it to be *open* like a glass. I think one can restrict to surfaces of revolution, so the simplified version becomes:

Problem 1.1 *Let $y : [0, \infty[\to \Re$ be a continuous function with weak first derivative and compact support. The enclosed volume of the surface of revolution is (up to a constant)*

$$V(y) = \int_0^\infty y(x)^2 dx$$

and the surface area is

$$A(y) = \int_0^\infty \sqrt{y(x)^2(1 + y'(x)^2)} dx$$

Now to get rid of the units take

$$v(y) = V(y)^{1/3}, a(y) = A(y)^{1/2}$$

and minimize the ratio

$$r(y) = a(y)/v(y). \tag{1}$$

Note that the functional r is invariant under scaling

$$y(x) \to c * y(x/c) \tag{2}$$

for positive real c and also invariant under $y \to -y$.

So one might want to consider the problem in a kind of projective space. I do not know whether minimizing r is equivalent to the problem

$$\min\{a(y) : v(y) = 1\} \tag{3}$$

but one could scale a minimizer for (1) as in (2) to obtain a function satifying the constraint in (3).

To get rid of the condition of compact support we could alter the problem in the following way:

Consider pairs (y, x_0) where y is a real-valued weakly differentiable function on the positive real axis and x_0 is some value > 0.

Then alter the functional A to

$$A(y, x_0) = \int_0^{x_0} \sqrt{y(x)^2(1 + y'(x)^2)} dx + 1/2 * y(x_0)^2,$$

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which is thought of a surface of revolution with a flat bottom at $x = x_0$. Of course in the case $y(x_0) = 0$ the new A is the same as the old one.

What might be a problem is that r is not defined for the zero function, and by the invariance (2) one can see that r attains any possible value in any neighborhood of zero, since a for a function y of bounded support

$$\lim_{c \to 0} c * y(x/c) = 0$$

for all $x \geq 0$, which means that we find a path where r is constant that links y and the zero function (at least in the topology of pointwise convergence).

I already did a simple discretization using B-splines and a gradient method, and my guess is that the half-sphere

$$y(x) = \sqrt{1 - x}, x_0 = 1$$

is the global optimum.