HYPERBOLIC SYSTEMS: assignment 1

1 Burgers equation

(a) take $u_l = 1$, $u_r = 2$ for the left and the right state in a Riemann problem

-calculate the shock speed

-draw an xt diagram with characteristics

-draw the solution in an xu diagram for $t = 1$ and $t = 2$

(b) switch the left and the right state, and do the same as above

2 The Euler equations

For each of the following wave modes, determine if they are linear or nonlinear

-entropy wave

-shear wave

-sound waves

3 Nonlinear 2×2 systems

(a) the shallow water equations

$$\frac{\partial}{\partial t}\left[\begin{array}{c} v \\ \phi \end{array}\right] + \frac{\partial}{\partial x}\left[\begin{array}{c} v^2/2 + \phi \\ v\,\phi \end{array}\right] = 0$$

(b) the isentropic Euler equations

$$\frac{\partial}{\partial t}\left[\begin{array}{c} \rho \\ \rho\, v \end{array}\right] + \frac{\partial}{\partial x}\left[\begin{array}{c} \rho\, v \\ \rho\, v^2 + s\,\rho^\gamma \end{array}\right] = 0$$

(s and γ are constants)

For (a) or (b):

-is the system hyperbolic? give eigenvalues, left and right eigenvectors

-are the wave modes linear or nonlinear?

-determine the Riemann Invariants

-write down the Rankine-Hugoniot relations; consider state $U_1 = (1, 2)$; which state U_2 can be connected with U_1 through a shock with speed $s = 0$? (take all constants in the equations $= 1$)

-draw and xt diagram with two families of characteristics with states U_1 and U_2 either left/right or right/left; you get a shock when one family of characteristics converges, a rarefaction when one family diverges; determine when you get a shock and when a rarefaction

-express the eigenvalues as a function of the Riemann Invariants; will characteristics be straight lines in general?; will they be straight in simple waves?

Now choose either 4 or 5

4 The shallow water MHD equations

The 'Shallow water MHD equations' are given by

$$
\begin{aligned}
\frac{\partial h}{\partial t} + \nabla \cdot (h\,\vec{v}) &= 0, \\
\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} - (\vec{B}\cdot\nabla)\vec{B} + g\,\nabla h &= 0, \\
\frac{\partial \vec{B}}{\partial t} + (\vec{v}\cdot\nabla)\vec{B} - (\vec{B}\cdot\nabla)\vec{v} &= 0, \\
\nabla \cdot (h\,\vec{B}) &= 0.
\end{aligned}
$$

Here $V = (h, v_x, v_y, B_x, B_y)^T$ are the 'primitive variables', g is a constant. If $\nabla \cdot (h\,\vec{B}) = 0$ is true at time $t = 0$, then it remains true automatically for all later times (it is called the divergence 'constraint').

Questions:

-write the equations in 1D quasi-linear form; is the system hyperbolic?; derive the eigenvalues and left and right eigenvectors (I suggest you use Mathematica or so); what does the divergence constraint become in 1D?

-write the 1D equations in conservation law form (hint: the conserved variables are $U = (h, m_x, m_y, \phi_x, \phi_y)^T = (h, h\,v_x, h\,v_y, h\,B_x, h\,B_y)^T$, and use the divergence constraint); show that the divergence constraint is indeed conserved in time

-find Riemann Invariants (for two of the waves it will be hard to find them in closed form, although they exist; find the closed form for the other three)

-which of the wave modes are linear, which are nonlinear?

(This equation set may be used later on for computational work in 1D (we can add a source term to describe rotation) or in 2D (we can calculate bow shock flows).)

5 Non-convex scalar equation (towards the sedimentation problem)

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

(a) Consider $f(u) = u(u-1)/2$, $u \in [0,1]$

-sketch/plot $f(u)$, $f'(u)$, $f''(u)$

-is $f(u)$ strictly convex or concave? (or, is $f'(u)$ monotone?)

-consider a Riemann problem with $u_l = 1$ and $u_r = 0.95$; what does the solution of the Riemann problem look like?; now consider Riemann problems by keeping u_l fixed, but changing u_r gradually from 0.95 to 0; what do the solutions of those Riemann problems look like? (draw characteristics in xt diagrams)

-consider a Riemann problem with $u_r = 0$ and $u_l = 0.05$; what does the solution of the Riemann problem look like?; now consider Riemann problems by keeping u_r fixed, but changing u_l gradually from 0.05 to 1; what do the solutions of those Riemann problems look like? (draw characteristics in xt diagrams)

(b) Consider $f(u) = u^2(u-1)/2$, $u \in [0,1]$

-sketch/plot $f(u)$, $f'(u)$, $f''(u)$

-is $f(u)$ strictly convex or concave? (or, is $f'(u)$ monotone?)

-consider a Riemann problem with $u_l = 1$ and $u_r = 0.95$; what does the solution of the Riemann problem look like?; now consider Riemann problems by keeping u_l fixed, but changing u_r gradually from 0.95 to 0; what do the solutions of those Riemann problems look like? (draw characteristics in xt diagrams)

-consider a Riemann problem with $u_l = 0$ and $u_r = 0.05$; what does the solution of the Riemann problem look like?; now consider Riemann problems by keeping u_r fixed, but changing u_r gradually from 0.05 to 0.5; what do the solutions of those Riemann problems look like?; what happens when $u_r = 0.5$? (draw characteristics in xt diagrams); what happens when $u_r > 0.5$?

You need compound shocks. Read pp. 48–50 from Leveque's book (attached).

-sketch the solution of the Riemann problem $u_l = 1$, $u_r = 0.7$; (xu diagrams for t=1, 2; draw characteristics in xt diagram)

-sketch the solution of the Riemann problem $u_l = 0.7$, $u_r = 0$; (xu diagrams for t=1, 2; draw characteristics in xt diagram)

-consider the double Riemann problem with two discontinuities: at $x = 0$, $u_l = 1$ and $u_r = 0.7$; at $x = 1$, $u_l = 0.7$ and $u_r = 0$; what happens when waves originating from the left and right discontinuity merge?; is there a final, steady state solution?; can you easily find out where the final discontinuity is located?

(This problem may lead to 1D computational work to try and reproduce lab eperiments of waves and shocks in a sedimentation process (so-called settling column tests by Gert Bartholomeeusen at Oxford (gert.bartholomeeusen@eng.ox.ac.uk), see http://www-civils.eng.ox.ac.uk/ gb/FrameSet_settcol.htm). For these simulations we will try to use an experimentally fitted flux curve $f(u)$.)