University of Waterloo, Waterloo, ON

Faculty of Mathematics

Department of Applied Mathematics

AMATH 341 / CS 371

Introduction to Computational Mathematics

Midterm Test **October 28, 2004**
Instructor: Professor H. De Sterck **Time:** 2 hours
AIDS: AMATH 341 Course Notes, one book, calculator, four pages (sides) of notes

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[15] 1. (Errors in numerical computation.)
With $x = 101$ and $y = 100$, you are asked to calculate $z = x^2 - y^2$.

a) Calculate $z = x^2 - y^2$ in the floating point number system $F_1(b = 10, m = 5, e = 3)$ (assume chopping).

b) Calculate $z = x^2 - y^2$ in the floating point number system $F_2(b = 10, m = 3, e = 3)$ (assume chopping). Show your work and explain why the result is not the same as in a).

c) Devise a different, more stable method to calculate $z = x^2 - y^2$, and show that using this method you can obtain the exact result for z when calculating in floating point system $F_2(b = 10, m = 3, e = 3)$.

[20] 2. (Interpolation.)
The piecewise polynomial function $y(x)$ is defined as

$$y(x) = \begin{cases} ax^3 + bx^2 - 9/2x + c & \text{for} \quad x \in [-3, -1] \\ x^3 & \quad\quad\; x \in [-1, 0] \\ dx^3 + ex^2 & \quad\quad\; x \in [0, 1] \\ -x^3 + 6x^2 - 6x + f & \quad\quad\; x \in [1, 2] \end{cases}$$

Can the coefficients a, b, c, d, e and f be determined such that $y(x)$ is a natural cubic spline? If so, determine the coefficients a, b, c, d, e and f.

[25] 3. (Interpolation. Integration.)

a) Given f_0, f_1, f_1' in points $x_0 = 0$ and $x_1 = h$, determine the coefficients a, b and c of the interpolating polynomial $y(x) = a(x-h) + b(x-h)^2 + c$ that satisfies $y(x_0) = f_0, y(x_1) = f_1$ and $y'(x_1) = f_1'$.

b) Derive an integration rule for $I = \int_0^h f(x)dx$ by integrating the interpolating polynomial $y(x)$. Express your result as $\hat{I} = w_0 f_0 + w_1 f_1 + w_2 f_1'$ (determine w_0, w_1, w_2).

c) Using the integration rule $\hat{I}$, derive a composite integration rule $\hat{I}_{\text{composite}}$ for $I = \int_a^b f(x)dx$, using $n+1$ equidistant points $x_i = a + i(b-a)/n \quad (i = 0, \cdots, n)$, with $h = (b-a)/n$. Find the weights of $f_i (i = 0, \cdots, n)$, and $f_i' (i = 1, \cdots, n)$ in $\hat{I}_{\text{composite}}$.

[20] 4. (Differentiation.)

a) Given f_0 and f_1 in points x_0 and x_1, determine a and b in $\hat{f}_0' = af_0 + bf_1$ such that $\hat{f}_0'$ approximates f_0' as accurately as possible. Determine the leading order term of the truncation error.

b) Given f_0, f_1, and f_1' in point x_0 and x_1, determine a, b and c in $\hat{f}_0' = af_0 + bf_1 + cf_1'$ such that $\hat{f}_0'$ approximates f_0' as accurately as possible. Determine the leading order term of the truncation error.

[20] 5. (Integration.)
Given $\{(x_i, f_i = f(x_i))\}_{i=0}^n$ with $x_i \neq x_j$ when $i \neq j$, derive an integration rule $\hat{I} = \sum_{i=0}^{n} w_i f_i$ for approximating $I = \int_a^b f(x)dx$ with degree of precision n. (This means that the functions $f(x) = 1,\ f(x) = x,\ f(x) = x^2, \cdots, f(x) = x^n$ are integrated exactly). Show that the weights w_i can be found by solving a matrix equation $A \cdot w = r$, with w the vector containing the weights w_i. Find expressions for the matrix A and the right hand side r. Discuss existence and uniqueness of the solution w.