University of Waterloo, Waterloo, ON

Faculty of Mathematics

Department of Applied Mathematics

AMATH 341 / CS 371

Introduction to Computational Mathematics

Final Examination **Friday, December 10, 2004**

Instructor: Professor H. De Sterck **Time:** 3 hours

AIDS: AMATH 341 Course Notes, one book, calculator, four pages (sides) of notes

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[15] 1. (Errors in numerical computation.)

(a) Given $p_0 = 1/3$ and $p_1 = 1/5$, a recurrence relation for calculating p_n is given by

$$p_n = \frac{5}{6}p_{n-1} - \frac{1}{6}p_{n-2} \quad (n \geq 2).$$

Assume that rounding errors occur in assigning the floating point values $\hat{p}_0$ and $\hat{p}_1$, but that no further errors occur in the subsequent calculations. Analyse the propagation of the absolute error $\Delta p_n = p_n - \hat{p}_n$. Is the recurrence relation stable with respect to the absolute error?

(b) On a base $-$ 2 machine, the distance between 7 and the next larger floating point number is 2^{-12}. What is the (exact) distance between 71 and the next largest floating point number?

[15] 2. (Interpolation.)

(a) Let $f(x) = x^6 - 3x^5 + 7x^4 - 3x^3 + 2x^2 + x - 1$.
Let $p_5(x)$ be the polynomial of degree 5 that interpolates $f(x)$ in the points $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$ and $x_5 = 5$. What is the leading coefficient of $p_5(x)$?
(Hint: Consider the error formula $E_5(x) = f(x) - p_5(x)$. You should be able to compute this precisely without constructing $p_5(x)$.)

(b) With $l_i(x)$, $i = 0, \cdots, n$ being the Lagrange polynomials, for which values of m is it true that $\sum_{k=0}^{n} x_k^m l_k(x) = x^m$. Explain why.

[15] 3. (Differentiation.)

(a) It is known that $\hat{y}_0'(h) = \frac{y_h - y_0}{h}$ is an approximation for y_0' with truncation error $T(h) = ch + O(h^2)$. Find a formula that approximates the truncation error $T(h)$, given two approximations $\hat{y}_0'(h)$ and $\hat{y}_0'(2h)$ of y_0'. (Hint: assume that $T(h) = ch$.)

(Integration.)

(b) Let $f(x) = \frac{1}{x}$. What is the fewest number of intervals and function evaluations necessary to approximate $I = \int_1^2 f(x)dx$ to within an error that is guaranteed to be less than $\frac{24}{2880} \cdot 10^{-4}$, using the Composite Simpson rule?
(Hint: recall that the global truncation error for the Composite Simpson rule is bounded as follows: $|T_{\text{global}}| \leq \frac{(b-a)}{2880} h^4 \max_{\zeta \in [a,b]} |f^{(4)}(\zeta)|, \quad \text{with } h = \frac{b-a}{n}$).

[15] 4. (Fourier Methods.)

(a) Calculate the DFT of $(f[n]) = (3, 2, 1)$.

(b) The convolution of two time domain functions $h(t)$ and $u(t)$ is given by $y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)d\tau$. Show that the Fourier Transforms $Y(q), H(q)$ and $U(q)$ satisfy the relation $Y(q) = H(q) \cdot U(q)$.

[15] 5. (Fourier Methods.)
Let the DFT of $f[n]$ be given by $F[k]$.
Find the DFT $G[k]$ of time series $g[n] = f[n] \cdot (-1)^n$, in terms of $F[k]$.
(Hint: $G[k]$ is related to $F[k]$ by a shift in the frequency domain.) (Assume that the length N of time series $f[n]$ is even.)

[10] 6. (Linear Algebra.)
Find a permutation matrix P, and unit - lower and upper triangular matrices L and U, such that $P \cdot A = L \cdot U$, for

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

[15] 7. (Linear Algebra.)

(a) Show that if A is orthonormal, then the two-norm condition number $\kappa_2(A) = 1$.

(b) Assume that you are given the decomposition $A = L \cdot U$. Describe how you can use the decomposition to solve the system $A^2\vec{x} = \vec{b}$ in an efficient way. What is the compuational complexity in terms of floating point operations? (Specify the coefficient of the highest-order term precisely.)