A classification of nonlinear wave mode types in hyperbolic systems

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

$$\boxed{\frac{\partial U}{\partial t} + \mathsf{A}(U) \cdot \frac{\partial U}{\partial x} = 0}$$

hyperbolic $n \times n \Leftrightarrow \mathsf{A}(U) = \mathsf{R}(U) \cdot \Lambda(U) \cdot \mathsf{L}(U)$

n wave modes $\lambda_i(U)$ (n real characteristics)

$\Rightarrow$ there are different types of wave modes: in this lecture we classify them

(assume 'homogeneous medium', constant background)

Linear systems

- e.g., linearized MHD

- sine wave is always solution of linear system: $U(x,t) = \overline{U}\,\text{exp}(i(kx - \omega(k)t))$

  sine profile exactly preserved, arbitrary amplitude $\overline{U}$

  dispersion relation: $\omega(k)$

  phase velocity: $v_{phase} = \dfrac{\omega(k)}{k}$

  group velocity $v_g = \partial\omega(k)/\partial k$

- two types of waves:

- no dispersion: $v_g = v_{phase}$

  $\Rightarrow$ arbitrary profile exactly preserved, arbitrary amplitude

  example: MHD Alfvén wave

- dispersion: $v_g \neq v_{phase}$

  $\Rightarrow$ profile not exactly preserved

  example:

Nonlinear wave mode types

1. Linearly degenerate waves:

$$\boxed{\nabla\lambda_k(U)\cdot R_k(U)=0 \qquad \forall U} \qquad \left(\nabla\lambda_k(U)=\frac{\partial\lambda_k(U)}{\partial U}\right)$$

- arbitrary profile exactly preserved, arbitrary amplitude

- do not steepen nor rarefy (not compressive)

- example: entropy wave in Euler and MHD, Alfvén wave in MHD

2. Nonlinear waves: $\boxed{\nabla\lambda_k(U)\cdot R_k(U)\neq 0 \text{ for most } U}$

   - profiles not exactly preserved

   - do steepen or rarefy (compressive)

2A. Genuinely nonlinear waves: $\boxed{\nabla\lambda_k(U)\cdot R_k(U)\neq 0 \qquad \forall U}$

   - regular shocks and rarefactions

   - example: Euler sound waves

2B. Nonconvex nonlinear waves: $\boxed{\nabla\lambda_k(U)\cdot R_k(U)= 0 \text{ for some } U}$

   - compound shocks and rarefactions

   - example: MHD fast and slow waves

$\Rightarrow$ in this lecture we illustrate these three types in simple model systems

we use the concept of 'simple waves' (only kth mode is active in wave)

3.1 Simple waves

3.2 Linearly Degenerate waves

3.3 Genuinely Nonlinear waves

3.4 Nonconvex waves

3.1 Simple waves

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

Do solutions $U(x,t)$ and curves $x(t)$ exist such that $U(x,t)$ is constant on the curves $x(t)$ in a region of the (x,t) plane?

$$\boxed{\text{simple wave} \Leftrightarrow \frac{\mathrm{d}U(x(t),t)}{\mathrm{d}t} \equiv 0}$$

(U constant on curves $x(t)$)

$$\boxed{\text{simple wave} \Leftrightarrow \frac{\mathrm{d}U(x(t),t)}{\mathrm{d}t} \equiv 0}$$

(remark: scalar equation

$$\frac{\partial u}{\partial t} + f'(u)\,\frac{\partial u}{\partial x} = 0$$

characteristic curve $x(t)$: $\dfrac{\partial x(t)}{\partial t} = f'(u)$

$$\Rightarrow \frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} \equiv 0$$

every continuous flow is a simple wave

the curves $x(t)$ are the straight characteristics)

Simple waves in systems

$$\frac{\partial U}{\partial t} + F'(U) \cdot \frac{\partial U}{\partial x} = 0 \text{ and curve } x(t) : \frac{\partial x(t)}{\partial t} = \xi$$

simple wave: $$\frac{\mathrm{d}U(x(t),t)}{\mathrm{d}t} = \frac{\partial U(x(t),t)}{\partial t} + \frac{\partial U(x(t),t)}{\partial x}\frac{\partial x(t)}{\partial t} \equiv 0$$

$$\frac{\partial U(x(t),t)}{\partial t} + \frac{\partial U(x(t),t)}{\partial x}\xi = 0$$

$$F'(U) \cdot \frac{\partial U(x(t),t)}{\partial t} + F'(U) \cdot \frac{\partial U(x(t),t)}{\partial x}\xi = 0$$

$$F'(U) \cdot \frac{\partial U(x(t),t)}{\partial t} = \xi\,\frac{\partial U(x(t),t)}{\partial t}$$

$$F'(U) \cdot \frac{\partial U(x(t),t)}{\partial t} = \xi \, \frac{\partial U(x(t),t)}{\partial t}$$

$$\Rightarrow \xi = \lambda_k(u)$$

simple wave of the kth family

curve $x(t)$ is characteristic curve, and because U is constant on $x(t)$, the k–characteristic is a straight line

$$\Rightarrow \frac{\partial U(x(t),t)}{\partial t} = \alpha \, R_k$$

time derivative proportional to kth right eigenvector

$$\Rightarrow \frac{\partial U(x(t),t)}{\partial x} = -\frac{1}{\xi} \frac{\partial U(x(t),t)}{\partial t} = \beta \, R_k$$

space derivative proportional to kth right eigenvector

$\Rightarrow$ leads to equivalent definition of simple wave:

$\boxed{\text{simple wave of } k\text{th family} \Leftrightarrow U \text{ is constant on } k\text{–characteristic (straight line)}}$

$\Rightarrow$ only kth wave mode is 'active' in simple wave of kth family

k–Riemann Functions (k–RF)

for a given k, k–Riemann Functions are scalar functions

$$v_s^{(k)}(U) \quad s = 1 \ldots n-1$$

with property

$$\frac{\partial v_s^{(k)}(U)}{\partial U} \cdot R_k(U) = 0$$

$$(\nabla v_s^{(k)}(U) \cdot R_k(U) = 0)$$

theorem: there exist $n-1$ independent $v_s^{(k)}(U) \quad \forall k \quad (s = 1 \ldots n-1)$

the $\nabla v_s^{(k)}(U)$ span the orthogonal complement of R_k

k–Riemann Functions in k–simple wave

k–Riemann Functions:

$$\frac{\partial v_s^{(k)}(U)}{\partial U} \cdot R_k(U) \equiv 0$$

- k–simple wave:

$$\frac{\partial U}{\partial t} = \alpha\, R_k \quad \text{and} \quad \frac{\partial U}{\partial x} = \beta\, R_k$$

$$\Rightarrow \frac{\partial v_s^{(k)}(U)}{\partial t} = \frac{\partial v_s^{(k)(U)}}{\partial U} \cdot \frac{\partial U}{\partial t} = \alpha\, \frac{\partial v_s^{(k)(U)}}{\partial U} \cdot R_k = 0$$

$$\Rightarrow \frac{\partial v_s^{(k)}(U)}{\partial t} = \frac{\partial v_s^{(k)(U)}}{\partial U} \cdot \frac{\partial U}{\partial x} = \beta\, \frac{\partial v_s^{(k)(U)}}{\partial U} \cdot R_k = 0$$

$\Rightarrow$ in k–simple wave the $v_s^{(k)}(U)$ are constant in space and time ($s = 1 \ldots n-1$)

Characteristics and boundary conditions

scalar: $$\frac{\partial u}{\partial t} + f'(u)\,\frac{\partial u}{\partial x} = 0$$

in (x, t) plane: $\boxed{\nabla u \cdot (f', 1) = 0}$ with $\nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t})$

$(f', 1)$ gives slope of characteristic in (x, t) plane

$\Rightarrow$ conservation law gives information about how u changes in the direction of the characteristic

- if curve C is not characteristic: boundary conditions (BCs) specified on C allow to determine u in a neighborhood of C

  property: for continuous flow the local continuation is unique

- if curve C is a characteristic: BCs specified on C do NOT allow to determine u in a neighborhood of C!

  $\Rightarrow$ boundary conditions cannot be specified on characteristic

  (same reasoning for systems)

Property: every constant region is bordered by a characteristic in continuous flow

assume region in (x, t) plane with U constant in space and time, bordered by curve C

- if C is not characteristic $\Rightarrow$ the solution away from C can uniquely be continuated

  constant continuation is certainly possible $\Rightarrow$ the unique continuation is the constant solution!

- if the solution is NOT constant away from C, then C has to be a characteristic

  (also, C will be a straight characteristic because U is constant on C)

Property: every constant region is bordered by a simple wave region

e.g., Lax ’57

border curve C is k–characteristic $\Rightarrow$ k–simple wave

j–characteristic $x_j(t)$: $\dfrac{\partial x_j(t)}{\partial t} = \lambda_j$

$$\mathsf{L}(U) \cdot \frac{\partial U}{\partial t} + \Lambda(U) \cdot \mathsf{L}(U) \cdot \frac{\partial U}{\partial x} = 0$$

$$L_j \cdot \left(\frac{\partial U}{\partial t} + \lambda_j \frac{\partial U}{\partial x}\right) = 0 \quad \text{or} \quad L_j \cdot \frac{dU}{dx_j} = 0$$

also $L_j \cdot R_k = 0 \quad (j \neq k)$

and thus $L_j = \sum_{s=1}^{n-1} b_{js} \nabla v_s^{(k)}$

because $\nabla v_s^{(k)}$ span orthogonal complement of R_k

$$\Rightarrow \sum_{s=1}^{n-1} b_{js}\, \nabla v_s^{(k)}\, \frac{dU}{dx_j} = 0$$

$$\Rightarrow \sum_{s=1}^{n-1} b_{js} \frac{dv_s^{(k)}}{dx_j} = 0 \quad \forall j \neq k \quad (n-1 \text{ system})$$

$$\Rightarrow \mathsf{B} \cdot \frac{\partial v_s^{(k)}}{\partial t} + \Lambda \cdot \mathsf{B} \cdot \frac{\partial v_s^{(k)}}{\partial x} = 0$$

$$\Rightarrow \frac{\partial v_s^{(k)}}{\partial t} + \mathsf{B}^{-1} \cdot \Lambda \cdot \mathsf{B} \cdot \frac{\partial v_s^{(k)}}{\partial x} = 0$$

linear hyperbolic system with $\lambda = \lambda_j \quad (j \neq k)$

border curve C is not a k–characteristic of this system!

we can uniquely continue the $v_s^{(k)}$ as a constant from the constant region

$v_s^{(k)}$ are constant on the other side of curve C

$\Rightarrow$ the constant region is bordered by a k–simple wave region!

this is stated as a general result in textbooks on hyperbolic systems

BUT: is this always true?

NO! what if curve C is a DOUBLE characteristic

$$x_C(t) : \frac{\partial x_C(t)}{\partial t} = \lambda_k = \lambda_{k+1}$$

$\Rightarrow$ proof is not valid

- in Euler: strictly hyperbolic: $v + c \neq v - c$!

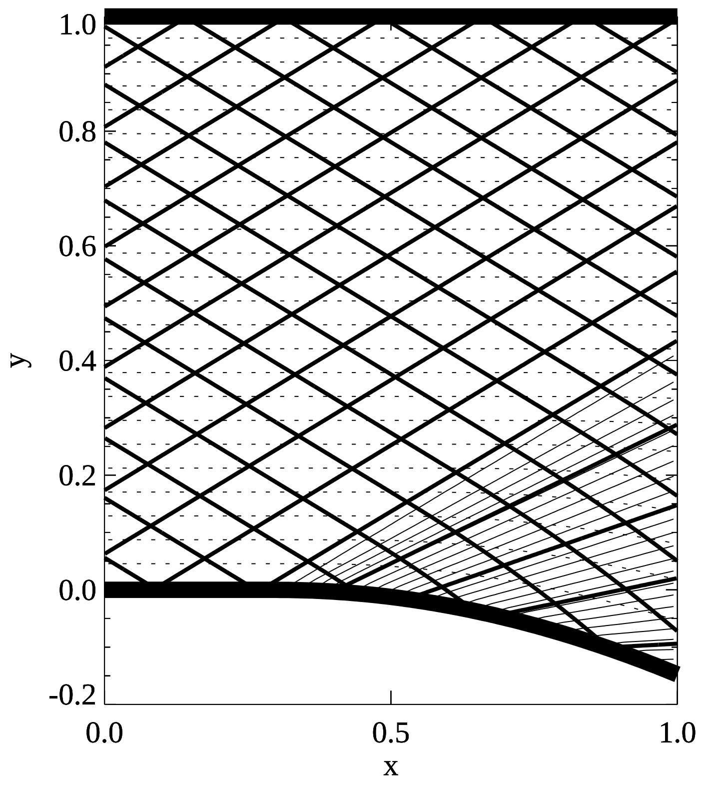
  this case can never happen; property is generally valid

- in MHD: non-strictly hyperbolic: $v + c_s = v + c_f$ is possible

  theorem not generally valid, but remains special case

  example in 2D simulation results of MHD bow shock

  (stationary characteristics $\sim$ space-time characteristics)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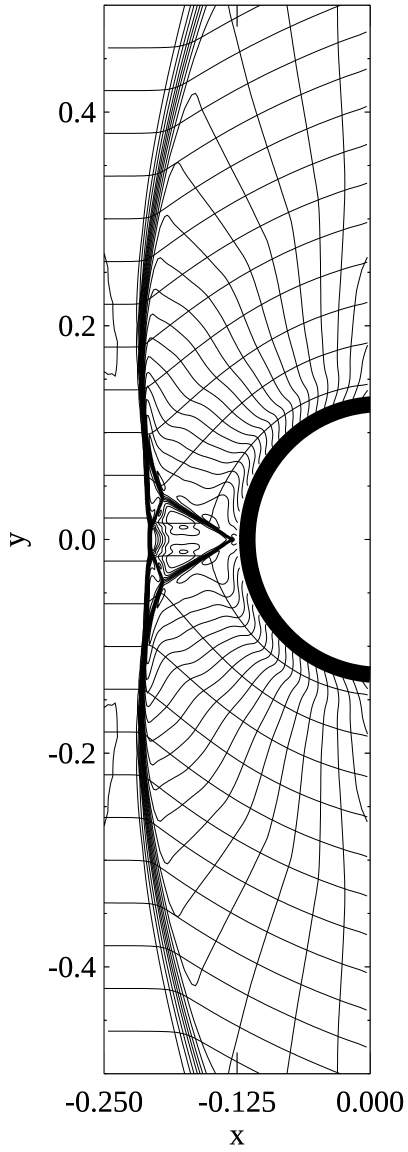
0.6
0.4
0.2
0.0
-0.2
y
0.0
0.5
1.0
x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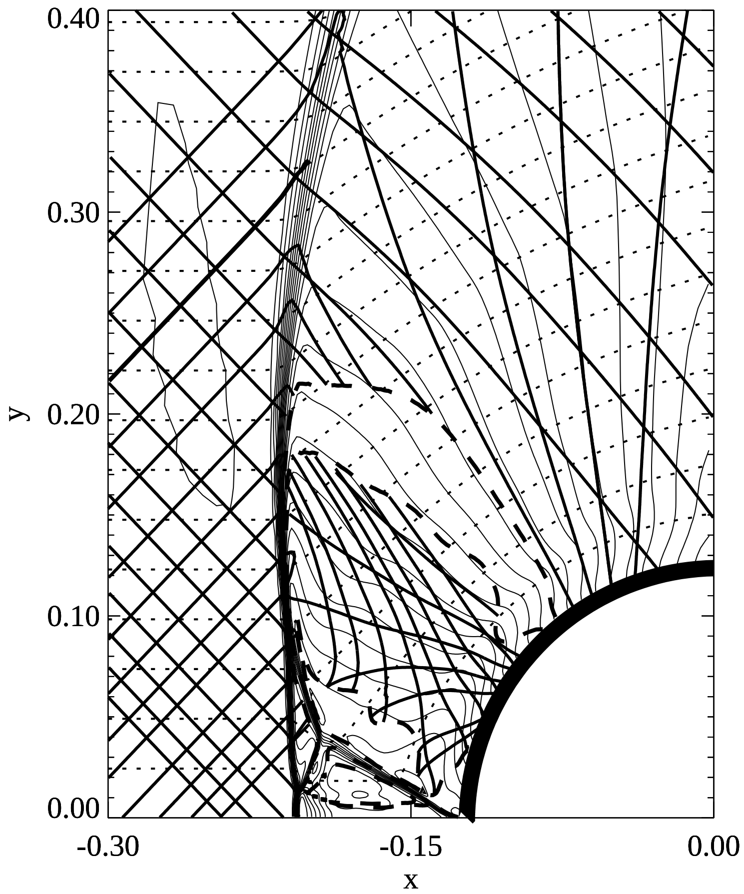
0.2
y 0.0
-0.2
-0.4
-0.250
-0.125
0.000
x



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0.20
0.10
0.00
y
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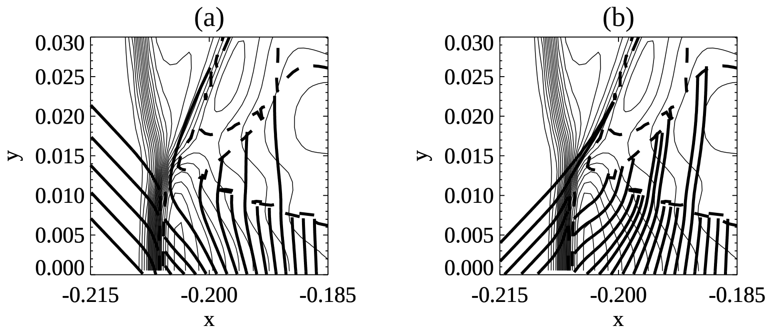
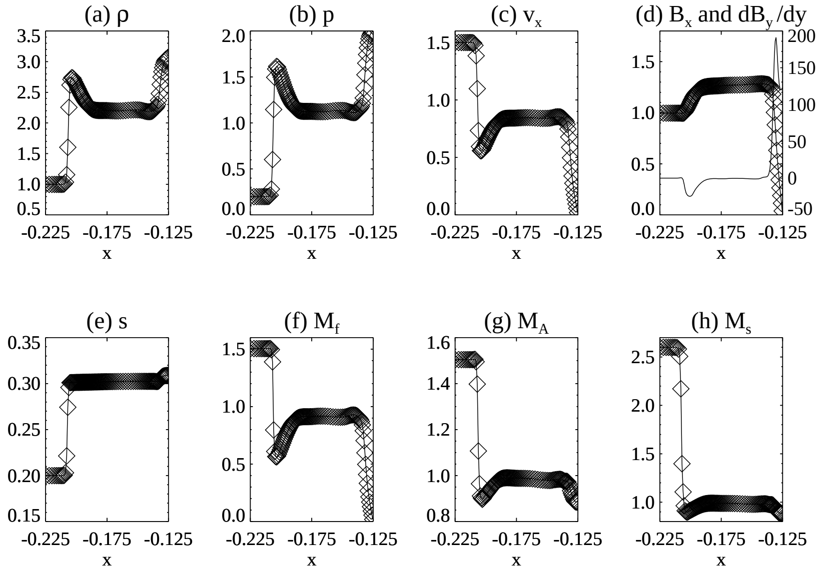
-0.15
0.00
x


(a)
(b)
0.030
0.025
0.020
0.015
0.010
0.005
0.000
y
-0.215
-0.200
-0.185
x


(a) ρ
(b) p
(c) v_x
(d) B_x and dB_y/dy
(e) s
(f) M_f
(g) M_A
(h) M_s
-0.225
-0.175
-0.125
x

3.2 Linearly degenerate waves

$$\boxed{\nabla \lambda_k(U) \cdot R_k(U) = 0 \qquad \forall U}$$

Simple waves

- k–simple wave:

$$\frac{\partial U}{\partial t} = \alpha\, R_k \quad \text{and} \quad \frac{\partial U}{\partial x} = \beta\, R_k$$

$$\Rightarrow \frac{\partial \lambda_k}{\partial t} = \nabla \lambda_k(U) \cdot \frac{\partial U}{\partial t} = \alpha\, \nabla \lambda_k(U) \cdot R_k = 0$$

$$\Rightarrow \frac{\partial \lambda_k}{\partial x} = \nabla \lambda_k(U) \cdot \frac{\partial U}{\partial x} = \beta\, \nabla \lambda_k(U) \cdot R_k = 0$$

$\Rightarrow$ in k–simple waves λ_k is constant in space and time

all k–characteristics are straight (simple wave) and parallel (λ_k constant) simple waves are not distorted, arbitrary profile exactly preserved

no steepening nor rarefaction

Linearly degenerate wave modes behave as dispersionless linear waves, but in nonlinear system!

Example: entropy wave in Euler equations

- $$\frac{\partial V}{\partial t} + \mathsf{A}(V)_V \cdot \frac{\partial V}{\partial x} = 0$$

  with $V = \begin{bmatrix} \rho \\ v_x \\ v_y \\ p \end{bmatrix}$ and $\mathsf{A}_V = \begin{bmatrix} v_x & \rho & 0 & 0 \\ 0 & v_x & 0 & 1/\rho \\ 0 & 0 & v_x & 0 \\ 0 & c^2\rho & 0 & v_x \end{bmatrix}$

  $c = \sqrt{\frac{\gamma p}{\rho}}$

  $\Rightarrow$

  $\lambda_1 = v_x$: entropy wave

  $\lambda_2 = v_x$: shear wave

  $\lambda_3 = v_x + c$: sound wave, right traveling

  $\lambda_4 = v_x - c$: sound wave, left traveling

$$\Rightarrow \mathsf{R} = \left[\begin{array}{c|c|c|c} 1 & 0 & \rho & \rho \\ 0 & 0 & c & -c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho\, c^2 & \rho\, c^2 \end{array}\right]$$

$$\nabla\lambda_1 \cdot R_1 = \left[\begin{array}{c|c|c|c} 0 & 1 & 0 & 0 \end{array}\right] \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \equiv 0$$

$\Rightarrow$ entropy wave is a linearly degenerate wave (density profile is advected with v_x)

also in MHD

other example: Alfvén wave in MHD

Discontinuities

any linearly degenerate simple wave profile is advected with constant λ_k

$\Rightarrow$ also discontinuous profile!

should then satisfy Rankine-Hugoniot relation $s(U_r - U_l) = F(U_r) - F(U_l)$

advection of profile:

$U(x,t) = U^*(y(x,t))$ with $y = x - \lambda_k\, t$

$$\frac{\partial U^*}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F(U^*)}{\partial y}\frac{\partial y}{\partial x} = 0$$

$$\frac{\partial U^*}{\partial y}(-\lambda_k) + \frac{\partial F(U^*)}{\partial y} = 0$$

$$\boxed{-\lambda_k\, U^* + F(U^*) = F_c}$$

take limit of discontinuous profile with states U_l, U_r:

$\Rightarrow -\lambda_k\, U_r + F(U_r) = -\lambda_k\, U_l + F(U_l)$

RH relation $s(U_r - U_l) = F(U_r) - F(U_l)$ is satisfied with $s = \lambda_k$

phase plane: jump has arbitrary amplitude ($\sim$ linear system)

$\lambda_k = \lambda_{kl} = \lambda_{kr}$: contact discontinuity

3.3 Genuinely nonlinear waves

$$\boxed{\nabla\lambda_k(U)\cdot R_k(U)\neq 0 \qquad \forall U}$$

The isothermal Euler equations

$$\boxed{\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=0}$$

- Euler (dissipationless hydrodynamics or gasdynamics, compressible)

$$U=\begin{bmatrix}\rho\\ \rho v_x\\ \rho\, e\end{bmatrix} \qquad F(U)=\begin{bmatrix}\rho v_x\\ \rho v_x^2+p\\ (\rho\, e+p)v_x\end{bmatrix}$$

with $\rho\, e=\dfrac{p}{\gamma-1}+\dfrac{1}{2}\rho v^2$

- isothermal: $p = \rho\, R\, T \;\Rightarrow \dfrac{p}{\rho} = R\, T =$ constant

  energy equation drops out

  $\Rightarrow c = \sqrt{\dfrac{\gamma p}{\rho}} =$ constant, and define $m = \rho\, v_x$

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \\ m \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} m \\ \\ \dfrac{m^2}{\rho} + \dfrac{c^2\rho}{\gamma} \end{bmatrix} = 0$$

$\Rightarrow$ quasi-linear form A $= \begin{bmatrix} 0 & 1 \\ \\ \dfrac{-m^2}{\rho^2} + \dfrac{c^2}{\gamma} & \dfrac{2m}{\rho} \end{bmatrix}$

$\Rightarrow \lambda_{1,2} = v_x \pm \dfrac{c}{\sqrt{\gamma}} \;\Rightarrow \gamma = 1$ for isothermal flow (hyperbolic)

$$\Rightarrow \mathsf{R} = \left[\begin{array}{c|c} 1 & 1 \\ \frac{m}{\rho} + c & \frac{m}{\rho} - c \end{array}\right]$$

$$\Rightarrow \mathsf{L} = \left[\begin{array}{cc} -\frac{m}{2\rho c} + \frac{1}{2} & \frac{1}{2c} \\ \hline \frac{m}{2\rho c} + \frac{1}{2} & -\frac{1}{2c} \end{array}\right]$$

- genuinely nonlinear: $\nabla \lambda_1(U) \cdot R_1(U) \neq 0 \qquad \forall U$

$$\nabla \lambda_1 \cdot R_1 = \left[-\frac{m}{\rho^2} \middle| \frac{1}{\rho} \right] \cdot \left[\begin{array}{c} 1 \\ \frac{m}{\rho} + c \end{array}\right] = -\frac{m}{\rho^2} + \frac{m}{\rho^2} + \frac{c}{\rho} \neq 0$$

same for λ_2

Riemann Invariants

$$\frac{\mathsf{L}(U)\cdot\partial U}{\partial t}+\Lambda(U)\cdot\frac{\mathsf{L}(U)\cdot\partial U}{\partial x}=0$$

- $x_i(t):\dfrac{\partial x_i(t)}{\partial t}=\lambda_i(U)$: ith characteristic

$\Rightarrow \dfrac{\mathrm{d}w_i(x_i(t),t)}{\mathrm{d}t}=0$: w_i Riemann Invariant (RI) on characteristic x_i

- $i=1$: find $w_1(U)$ such that

$$L_1(U)\cdot\partial U=(-\frac{m}{2\rho c}+\frac{1}{2})\partial\rho+\frac{1}{2\,c}\partial m$$

$$=\alpha(U)\,\partial w_1(U)=\alpha(U)\,(\frac{\partial w_1}{\partial\rho}\,\partial\rho+\frac{\partial w_1}{\partial m}\,\partial m)$$

try $\alpha=\rho$

$$\Rightarrow w_1 = \frac{m}{2\rho c} + \frac{1}{2}\ln\rho$$

is a Riemann Invariant on $x_1(t)$ with $\dfrac{\partial x_1(t)}{\partial t} = \lambda_1 = v_x + c$

$$\Rightarrow w_2 = -\frac{m}{2\rho c} + \frac{1}{2}\ln\rho$$

is a Riemann Invariant on $x_2(t)$ with $\dfrac{\partial x_2(t)}{\partial t} = \lambda_2 = v_x - c$

- also: λ_1 and λ_2 as a function of w_1 and w_2

$$\lambda_1 = c\,(w_1 - w_2 + 1)$$

$$\lambda_2 = c\,(w_1 - w_2 - 1)$$

Flow regimes

- uniform flow:

  m and ρ constant $\Leftrightarrow$ w_1 and w_2 constant $\Leftrightarrow$ λ_1 and λ_2 constant $\Leftrightarrow$ all characteristics are straight lines

- k simple wave: $\dfrac{\mathrm{d}U(x_k(t),t)}{\mathrm{d}t} \equiv 0$, or U is constant on $x_k(t)$, and the k–characteristic is a straight line

- take $w_1(x,0) =$ constant ($\Rightarrow w_1(x,t) =$ constant), choose free profile for $w_2(x,0)$

  $\Rightarrow \lambda_2 = c(w_1 - w_2 - 1)$ is constant on $x_2(t)$, because w_2 invariant on characteristic and w_1 constant everywhere

  $\Rightarrow \lambda_1 = c(w_1 - w_2 + 1)$ is NOT constant on $x_1(t)$, because w_2 is not constant

$\Rightarrow$ x_2 characteristics are straight, x_1 characteristics are not: simple wave

$\Rightarrow$ only 2nd wave mode is ‘active’ in simple wave of 2nd family ($w_1 =$ constant!)

- non-simple wave:

- choose free profiles for $w_1(x,0)$ and $w_2(x,0)$

$\Rightarrow \lambda_1$ and λ_2 are not constant, characteristics not straight

Domain of dependence – Domain of influence

- nonlinear: domain of dependence – domain of influence

- linear: rays of dependence – rays of influence

- hyperbolic region – elliptic region

3.4 Nonconvex waves

$\nabla\lambda_k(U)\cdot R_k(U)=0$ for some U

scalar ($k=1$): $\nabla\lambda(U)=\dfrac{\partial\lambda}{\partial u}=\dfrac{\partial^2 f(u)}{\partial u}=f''(u)$, $R_k=1$

nonconvex $\Leftrightarrow$ $f''(u)=0$ for some u

The Riemann problem

- Riemann problem $=$ how does initial discontinuity between two constant states u_l and u_r evolve in time?

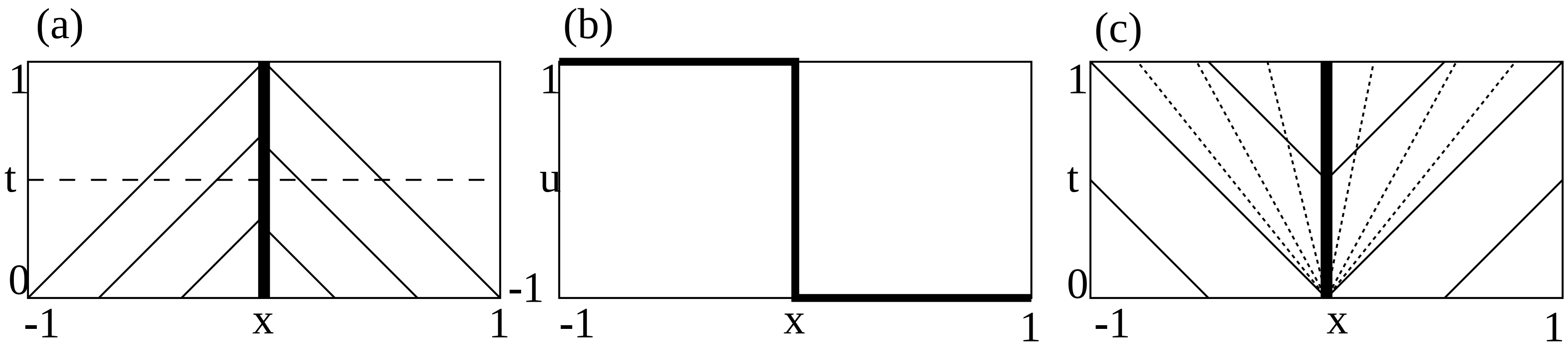
$\Rightarrow$ Burgers ($f(u) = u^2/2$):

- $f'(u_l) > f'(u_r)$, characteristics converge

$\Rightarrow$ shock with shock speed s from RH relation

- $f'(u_l) < f'(u_r)$, characteristics diverge

$\Rightarrow$ (continuous) rarefaction wave



Convexity of flux functions

- for Burgers: every Riemann problem results in either a shock or a rarefaction

  - this is true for all convex flux functions $f(u)$

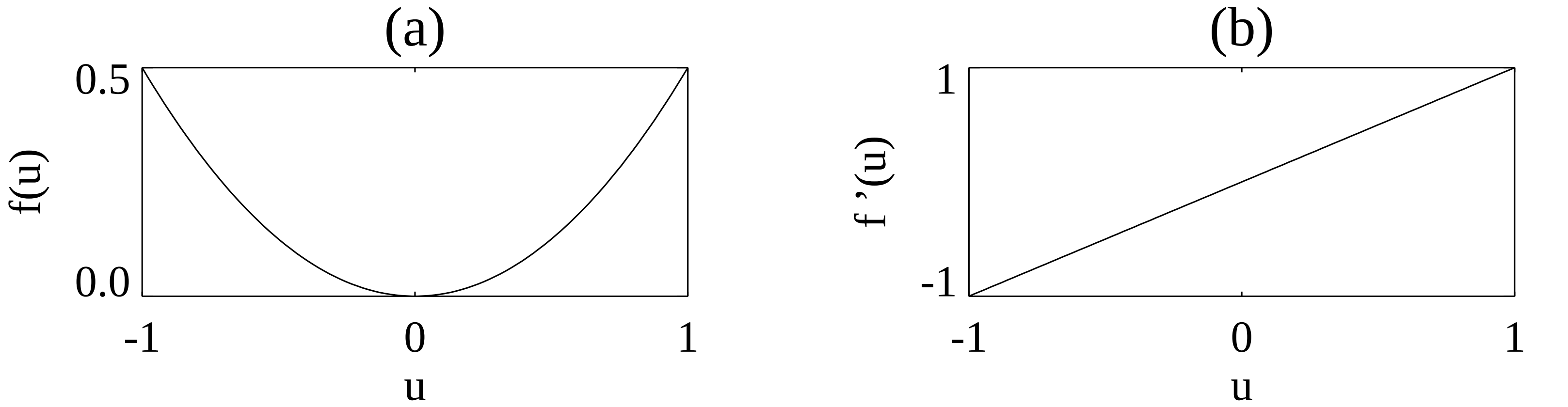
  - $f(u)$ is convex $\Leftrightarrow$ $f'(u)$ is monotone $\Leftrightarrow$ $f''(u)$ does not change sign

  - $f(u)$ is convex $\Rightarrow f'(u_l) > s > f'(u_r)$ when characteristics converge

  reason: $s = \dfrac{f(u_r) - f(u_l)}{u_r - u_l}$ and middle value theorem

  - Burgers is convex ($f(u) = u^2/2$)

- if $f(u)$ is non-convex: Riemann problems may have more complicated solutions than just a shock or a rarefaction


(a)
0.5
f(u)
0.0
-1
0
1
u
(b)
1
f '(u)
-1
-1
0
1
u

Nonconvex nonlinear scalar conservation laws

The Riemann problem

- example: $f(u) = \dfrac{u^3}{3}$, $f'(u) = 3\,u^2$

$\Rightarrow$ non-convex

Riemann problem: $u_l = 1$ and $u_r = -0.75$

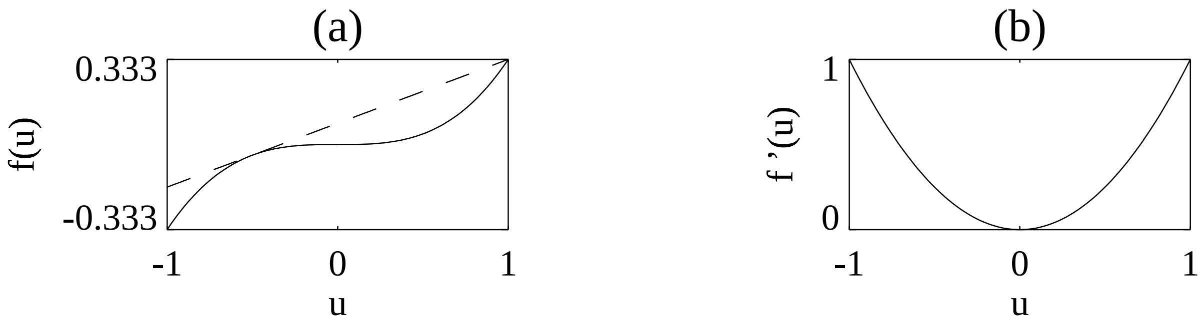
RH: $f'(u_l) > f'(u_r) > s = 37/168$

right characteristics diverge from shock : unstable

solution: rarefaction starting from right state until right characteristic becomes parallel to shock (with speed $\neq s$!)

condition: $f'(u^*) = s^* = \dfrac{f(u^*) - f(u_l)}{u^* - u_l}$

tangent hull construction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f(u)
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0
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u
(b)
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0
f '(u)
-1
0
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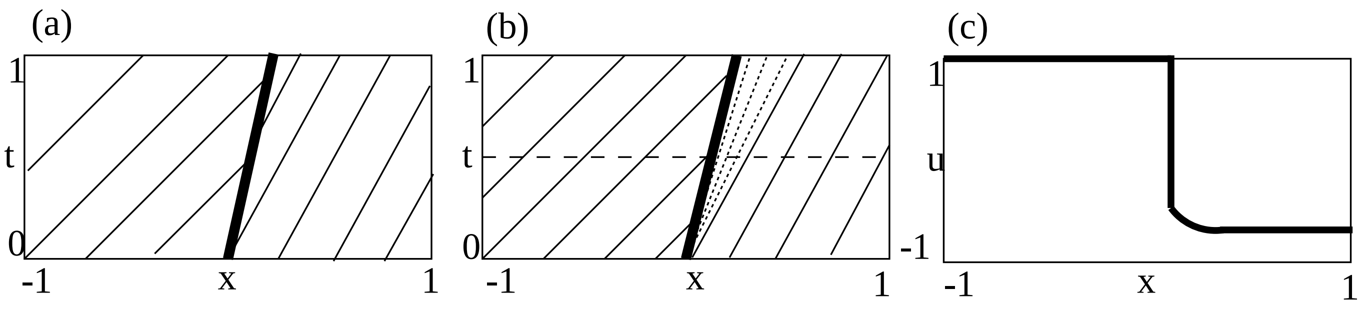
u


(a)
1
t
0
-1
x
1
(b)
1
t
0
-1
x
1
(c)
1
u
-1
-1
x
1

$\Rightarrow$ result: compound shock

- shock with attached rarefaction

- characteristic is parallel to shock where rarefaction is attached

- rarefaction is sonic where it is attached: shock speed = wave speed (characteristic speed)

$\Rightarrow$ Riemann problem (if one inflection point):

- $f'(u_l) > f'(u_r)$, characteristics converge

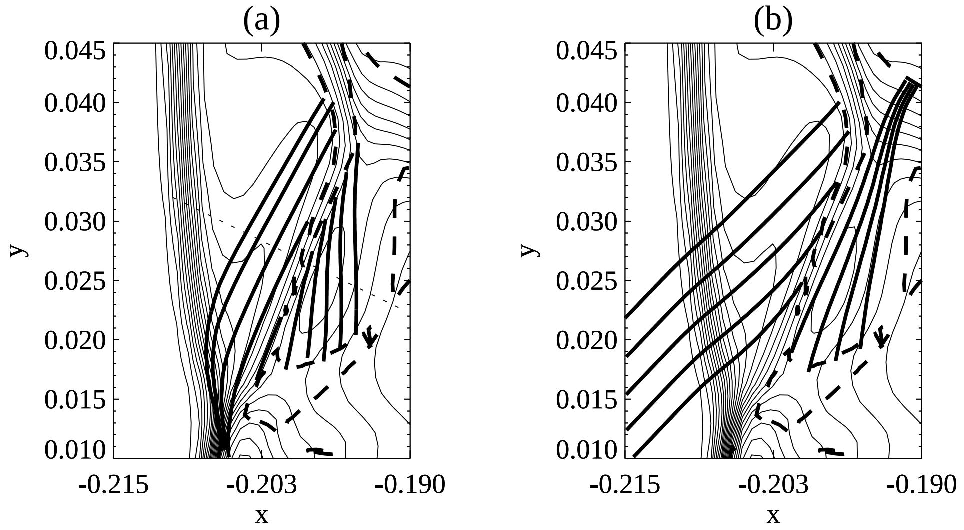
  - if $f'(u_l) > s > f'(u_r)$: shock

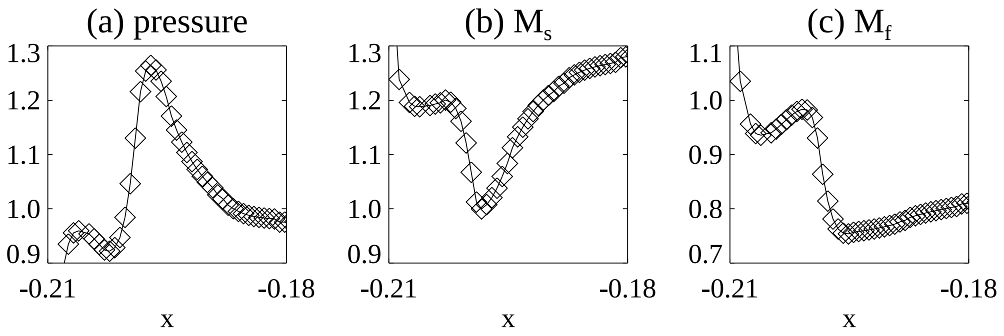
  - if $f'(u_l) > f'(u_r) > s$ or $s > f'(u_l) > f'(u_r)$: compound shock

- $f'(u_l) < f'(u_r)$, characteristics diverge

  - (continuous) rarefaction wave

for systems: analogous, rarefaction is k–simple wave example: MHD: fast and slow modes are nonconvex, can have compound shocks





MHD discontinuities

4.1 The MHD equations

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0}$$

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ \rho\, e \end{bmatrix} \qquad F(U) = \begin{bmatrix} \rho\, v_x \\ \rho\, v_x^2 + p + B^2/2 - B_x^2 \\ \rho\, v_x v_y - B_x B_y \\ \rho\, v_x v_z - B_x B_z \\ 0 \\ B_y v_x - B_x v_y \\ B_z v_x - B_x v_z \\ (\rho\, e + p + B^2/2) v_x - B_x\, (\vec{v} \cdot \vec{B}) \end{bmatrix}$$

4.2 Rankine-Hugoniot relations

$$\boxed{s(U_r - U_l) = F(U_r) - F(U_l)}$$

- given U_l, find curves $U(\xi)$ in phase space of points that can be connected to U_l by a k-discontinuity with speed $s(\xi)$

$$s(\xi)\,(U(\xi) - U_l) = F(U(\xi)) - F(U_l)$$

- Riemann problem: given U_1, U_2, find intermediate states U^{*i} such that U_1 and U_2 can be connected by shocks and rarefactions
- given U_l, s: find U_r, and how many U_r exist for a given s and U_l?

4.2.1 MHD Rankine-Hugoniot relations

go through succession of simplifications, using properties

- Galilean Invariance: transform to the shock frame

$$x \to x - s\,t$$

$$v \to v - s$$

$$(s \to s - s = 0)$$

$$B \to B$$

$$\rho \to \rho$$

$$p \to p$$

$$\Rightarrow F(U_r) - F(U_l) = 0$$

$$[\rho\, v_x] = 0$$

$$[\rho\, v_x^2 + p + B^2/2 - B_x^2] = 0$$

$$[\rho\, v_x v_y - B_x B_y] = 0$$

$$[\rho\, v_x v_z - B_x B_z] = 0$$

$$[B_y v_x - B_x v_y] = 0$$

$$[B_z v_x - B_x v_z] = 0$$

$$[(\rho\, e + p + B^2/2) v_x - B_x\, (\vec{v} \cdot \vec{B})] = 0$$

also: $[B_x] = 0$ (follows from $\nabla \cdot \vec{B} = 0$)

given U_l, 8 equations to obtain 8 U_r state variables

but: nonlinear: 1, 2, ... , many solutions!

- B_x constant, $m = \rho\, v_x$ constant, $\tau = 1/\rho$ new variables

  $\Rightarrow$ six remaining variables: $\tau, \vec{v}_t = (v_y, v_z), \vec{B}_t = (B_y, B_z), p$

$$m^2[\tau] + [p] + [\frac{\vec{B}_t^2}{2}] = 0$$

$$m[\vec{v}_t] - B_x[\vec{B}_t] = 0$$

$$m[\vec{B}_t\tau] - B_x[\vec{v}_t] = 0$$

$$m[\epsilon(p,\rho) + \frac{1}{2}m^2\tau^2 + \frac{1}{2}\vec{v}_t^2 + p\tau + \vec{B}_t^2\tau] - B_x[\vec{v}_t \cdot \vec{B}_t] = 0$$

4.2.2 Classification of discontinuities

A: $v_x = 0$: no mass flow

A1: $v_x = 0$ and $B_x \neq 0$: contact discontinuity

$\Rightarrow$ corresponds to LD entropy wave

$\Rightarrow$ $[\rho]$ arbitrary, all other variables constant

A2: $v_x = 0$ and $B_x = 0$: tangential discontinuity

$\Rightarrow$ $[\rho]$, $[\vec{v}_t]$, $[\vec{B}_t]$ arbitrary, $[p_{tot}]$ constant ($p_{tot} = p + B^2/2$)

B: $v_x \neq 0$: mass flow

$$m[\vec{v}_t] - B_x[\vec{B}_t] = 0$$

$$m[\vec{B}_t\tau] - B_x[\vec{v}_t] = 0$$

$$\Rightarrow [\vec{B}_t] \times [\vec{B}_t\tau] = 0$$

$$(\tau_r - \tau_l)(\vec{B}_{tr} \times \vec{B}_{tl}) = 0$$

B1: $v_x \neq 0$ and $\tau_r = \tau_l$: rotational discontinuity

$\Rightarrow$ corresponds to LD Alfven wave

$[\rho] = 0$, $[v_x] = 0$, $[p] = 0$, $[B^2] = 0$, $[v^2] = 0$

$$v_x = \pm \frac{B_x}{\sqrt{\rho}}$$

B2: $v_x \neq 0$ and $\tau_r \neq \tau_l$: SHOCKS

$\Rightarrow (\vec{B}_{tr} \times \vec{B}_{tl}) = 0$: coplanarity theorem

choose coordinate system $B_{zl} = B_{zr} \equiv 0$

one more Galilean transformation along the shock front makes $v_{zl} = 0$, and then also $v_{zr} = 0$

$\Rightarrow$ four remaining equations:

$$m^2[\tau] + [p] + [\frac{B_y^2}{2}] = 0$$

$$m[v_y] - B_x[B_y] = 0$$

$$m[B_y\tau] - B_x[v_y] = 0$$

$$m[\epsilon(p,\rho) + \frac{1}{2}m^2\tau^2 + \frac{1}{2}v_y^2 + p\tau + B_y^2\tau] - B_x[v_yB_y] = 0$$

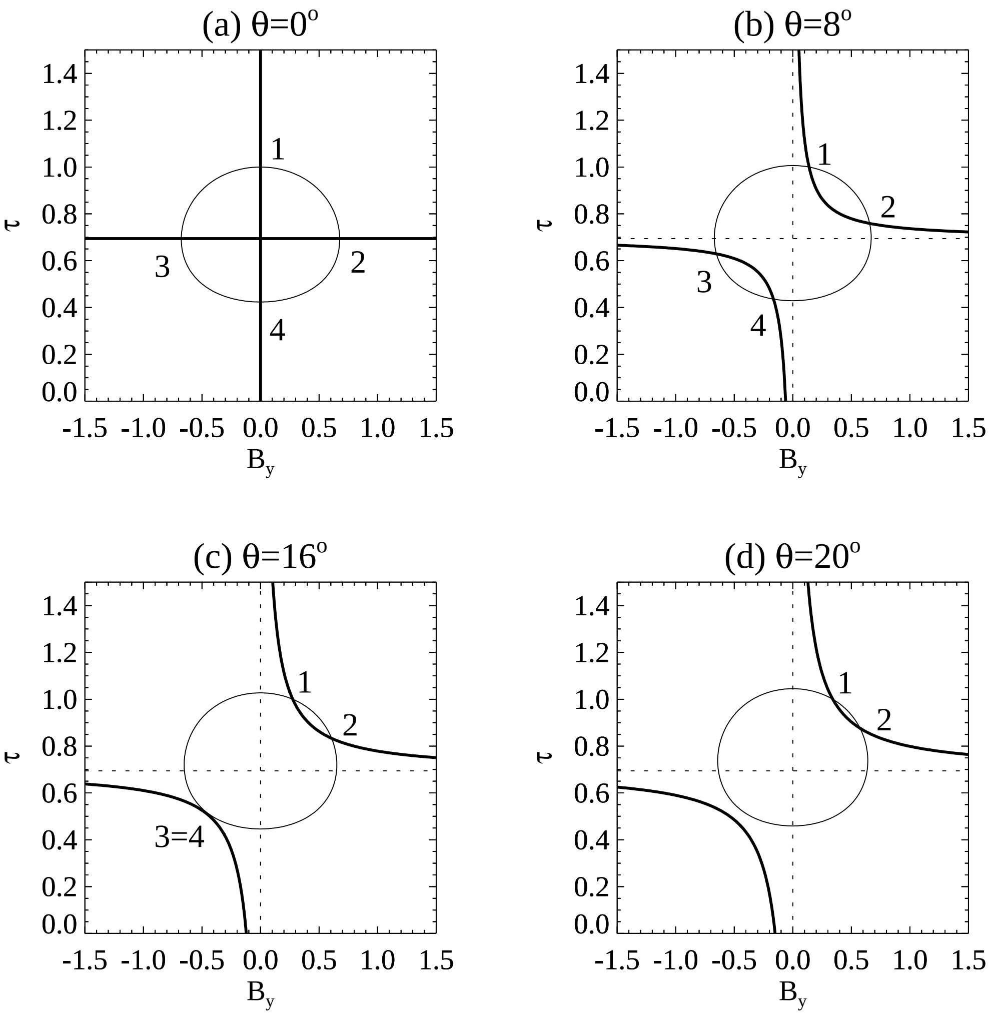
one more Galilean transformation along the shock front makes $E_{zl} = 0$, and then also $E_{zr} = 0$ ($E_z = (\vec{v} \times \vec{B})_z = v_yB_x - v_xB_y$)

use this to eliminate v_y, three remaining equations

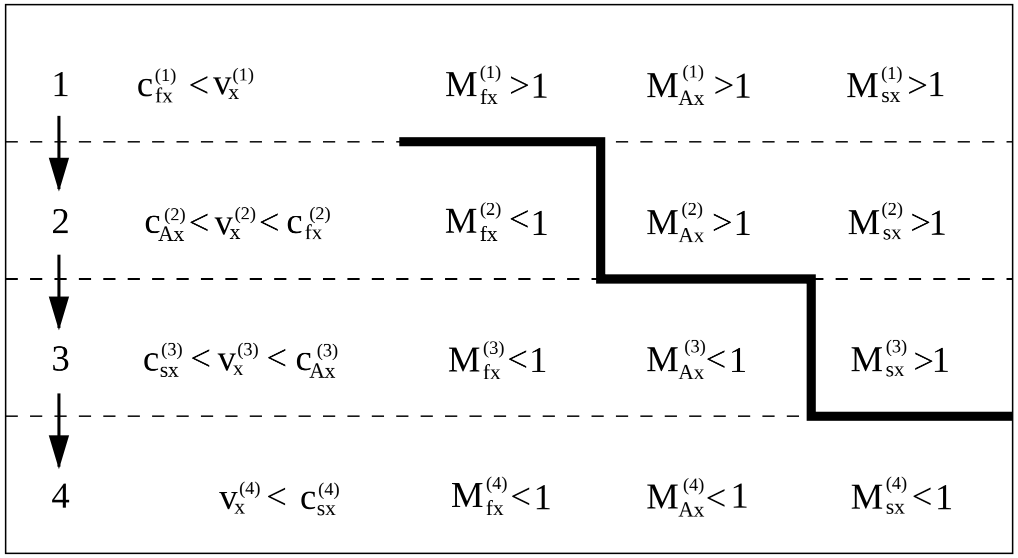
assume ideal gas, eliminate p

$\Rightarrow$ two remaining equations in τ and B_y

solve this graphically


(a) θ=0°
(b) θ=8°
(c) θ=16°
(d) θ=20°
τ
B_y
1
2
3
4
3=4
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-1.5 -1.0 -0.5 0.0 0.5 1.0 1.5

Solutions of the RH relations for parameter values $M_A = 1.2$ and $\beta = 0.4$ for state 1 with the field aligned to the flow. These are magnetically dominated parameters for which switch-on shocks occur. θ is the angle between the fields and the shock normal. (a) For $\theta = 0^\circ$, switch-on shocks, switch-off shocks and hydrodynamic shocks arise. (b–d) For $\theta < 16^\circ$, intermediate shocks occur, but they cease to exist for larger θ.



Possible states that can be connected through a MHD shock. They are ordered by increasing entropy, with the lowest entropy for state 1. The normal fast, Alfvén, and slow wave speeds are c_{fx}, c_{Ax}, and c_{sx}, respectively. The normal plasma speed is v_x. The fast, Alfvén, and slow normal Mach numbers are M_{fx}, M_{Ax}, and M_{sx}, respectively. State 1 is superfast, because its normal velocity in the shock frame is larger than the normal fast MHD wave speed. Therefore the fast, Alfvénic and slow Mach numbers are all greater than one. State 2 is subfast but super-Alfvénic. State 3 is sub-Alfvénic but superslow. State 4 is subslow. Possible shock transitions are 1–2 (fast), 3–4 (slow), and 1–3, 1–4, 2–3, 2–4 (intermediate).

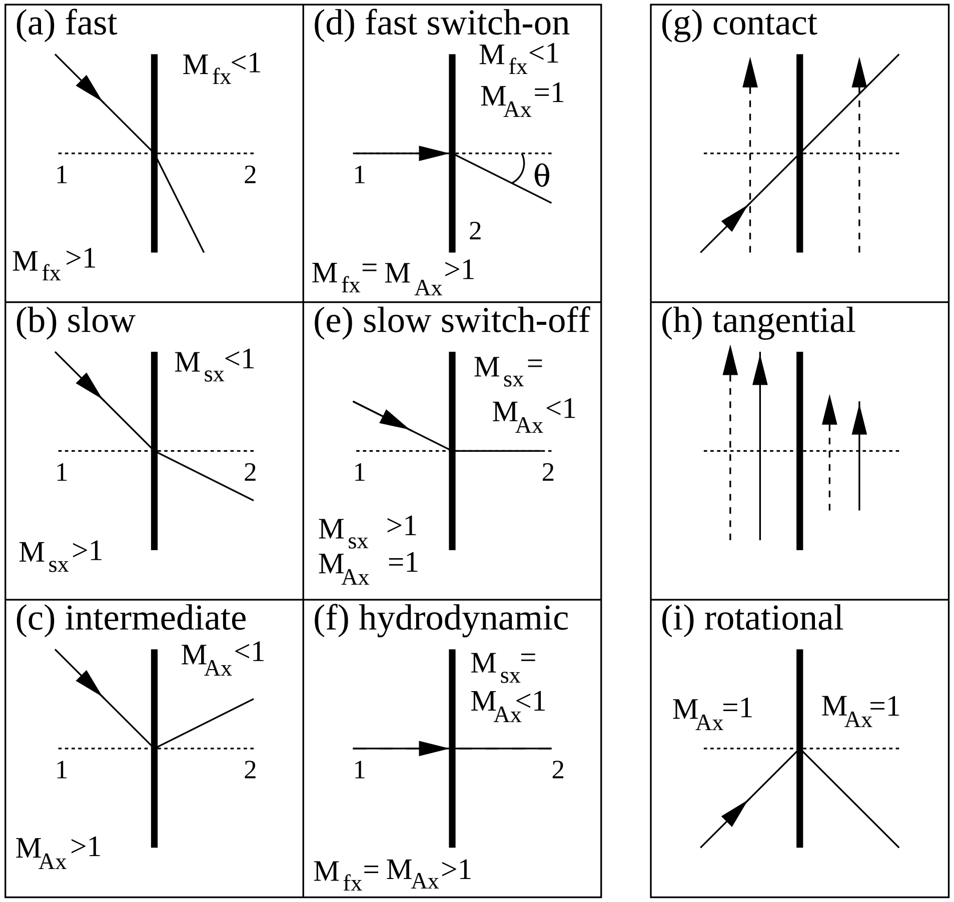
- switch-on shocks exist when

$$B_1^2 > \gamma\, p_1$$

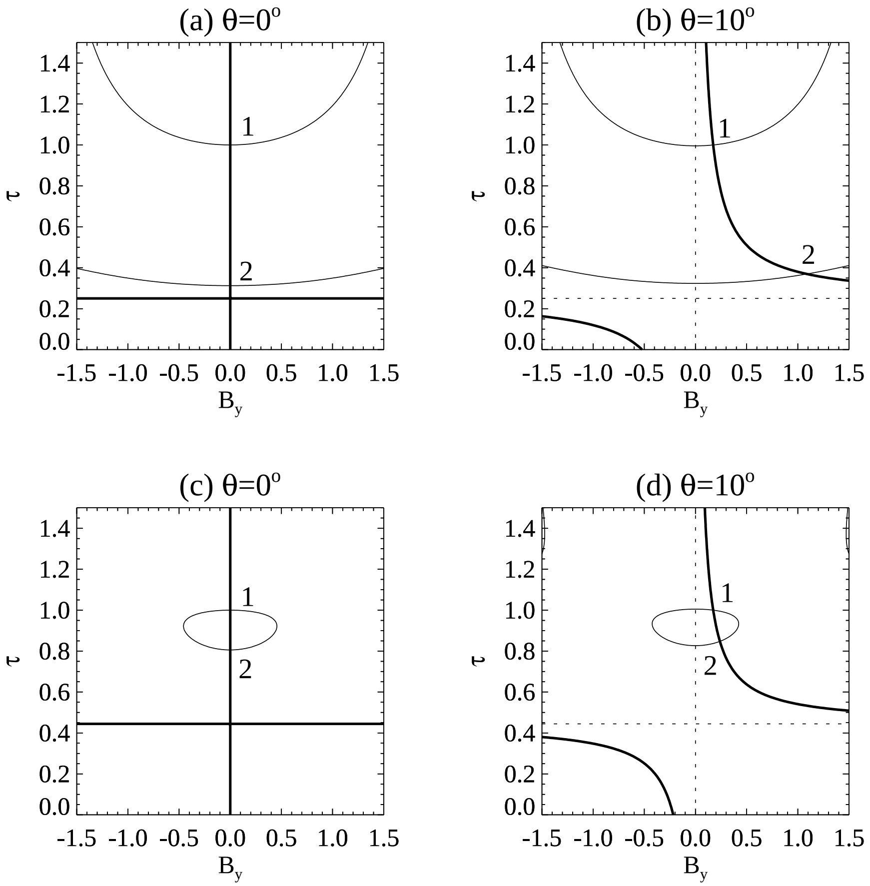
and

$$B_1^2 > \rho_1\, v_{x,1}^2\, \frac{\gamma-1}{\gamma(1-\beta_1)+1}$$

(with B along magnetic field line, and v_x along shock normal)

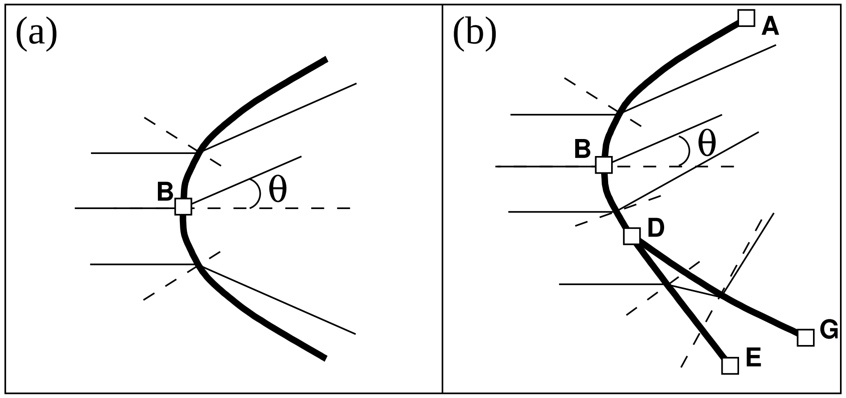


Some properties of MHD shocks and discontinuities. The thick vertical line is the shock surface. The shock normal is dotted. The full arrowed lines are magnetic field lines that are refracted through the shock surface. The dashed arrowed lines are velocity vectors. Region 1 is upstream, 2 is downstream.

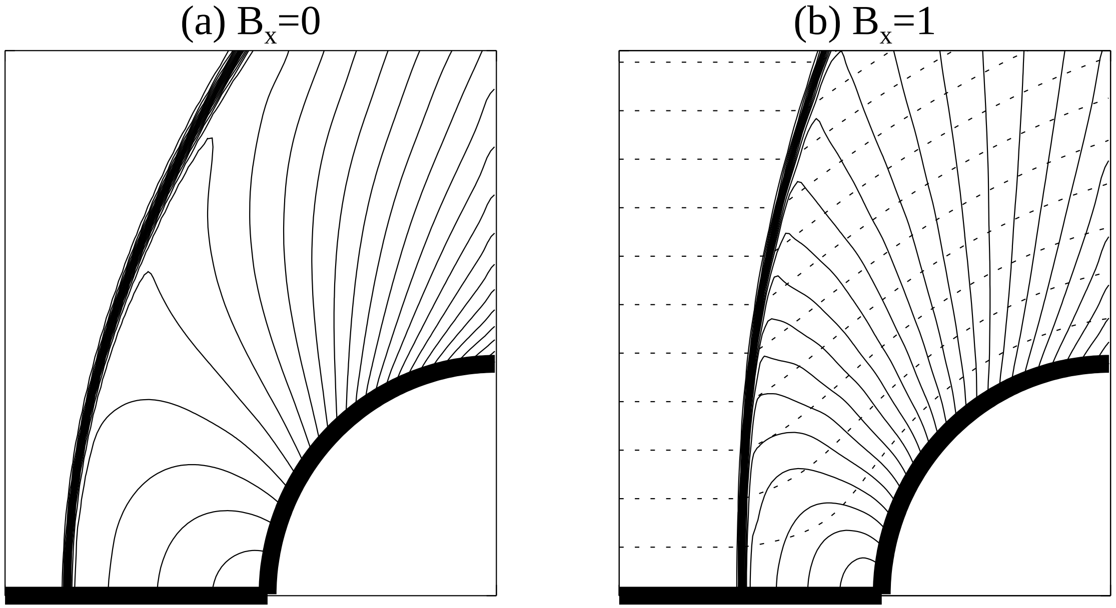


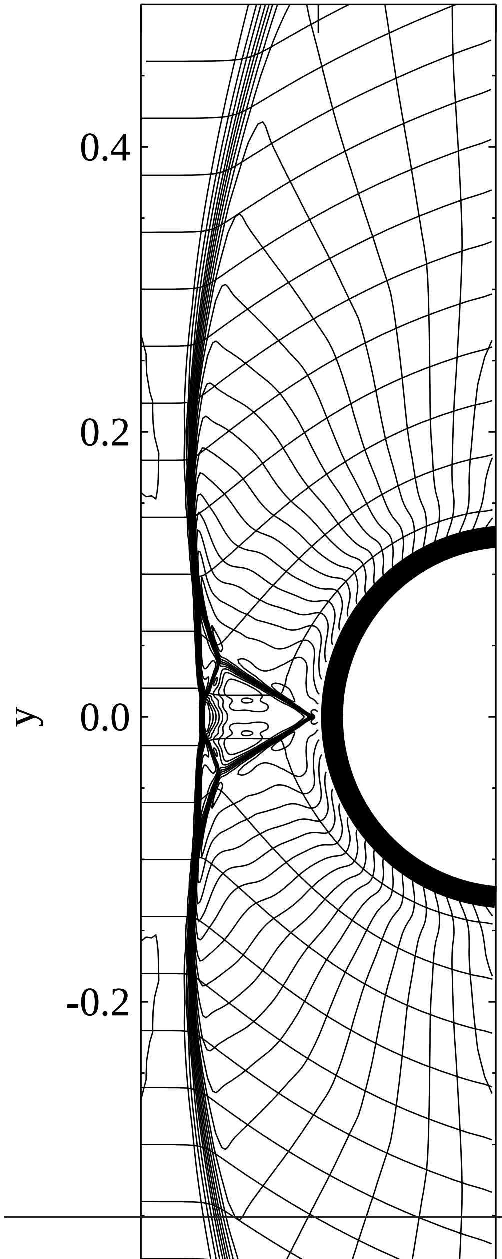
Solutions of the RH relations for pressure-dominated parameter values for state 1. Switch-on shocks do not occur. Only 1–2 fast shocks exist. (a–b) $M_A = 2$ and $\beta = 0.4$. (c–d) $M_A = 1.5$ and $\beta = 2$.

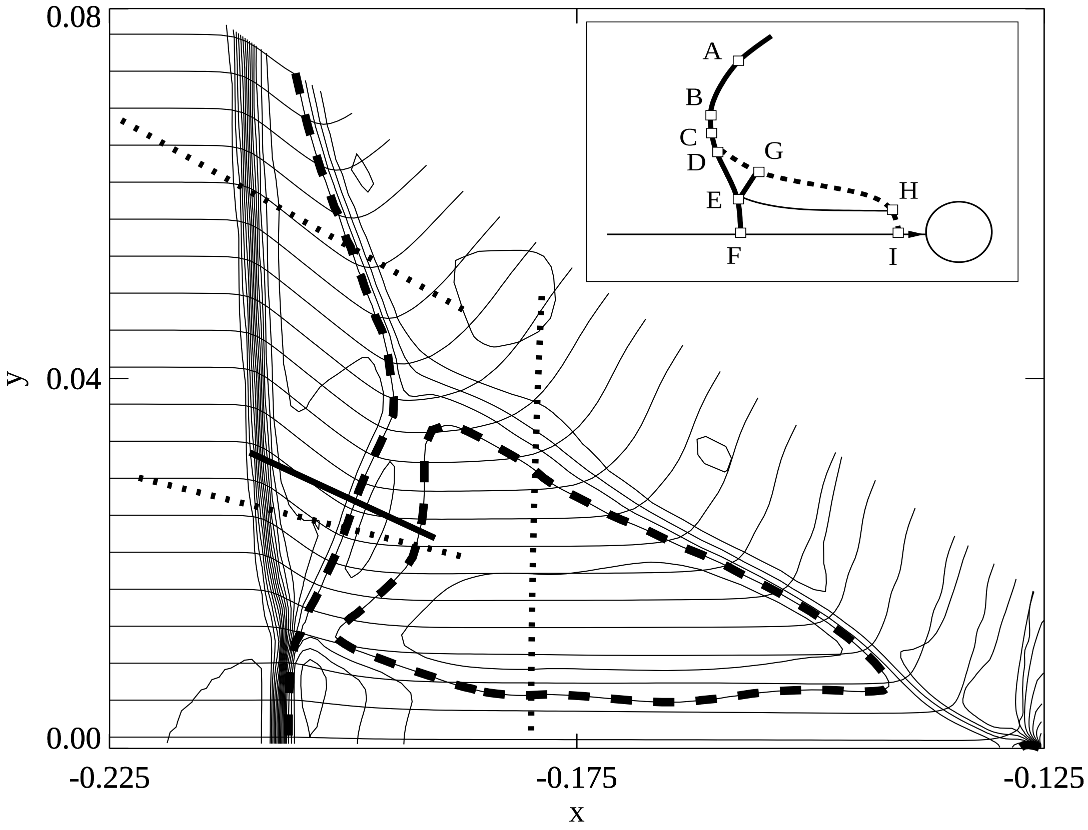
4.2.3 MHD bow shock flow topology

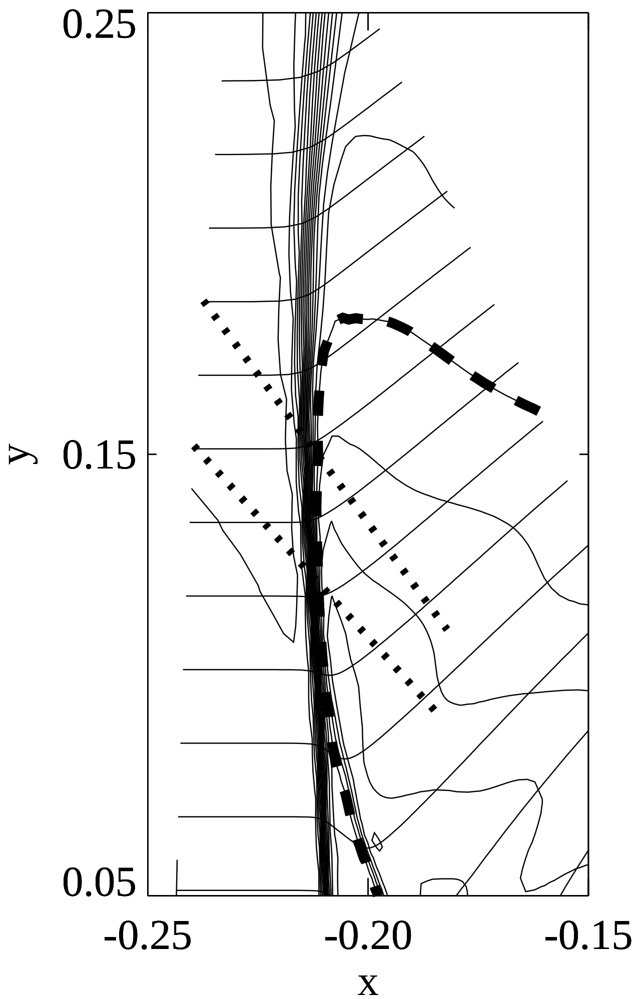


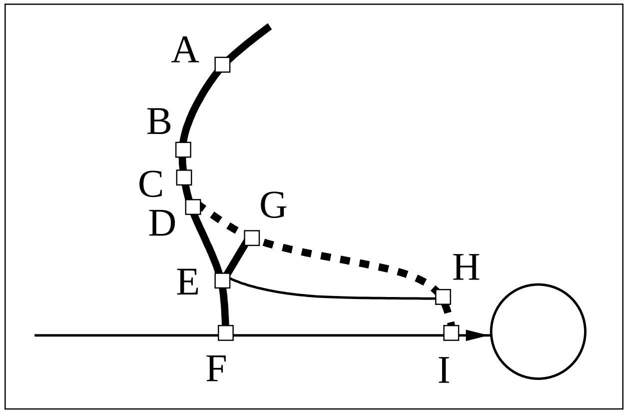
Two proposed topologies for a shock front with a switch-on shock at perpendicular point B for the case of a magnetically dominated upstream flow. Thick lines are shock fronts, thin lines are magnetic field lines, and shock normals are dashed. Point B is a perpendicular point, where the magnetic field is normal to the shock front. The shock at point B is a switch-on shock, which deflects the magnetic field with downstream angle θ. (a) The shock front cannot entirely be of the 1–2 fast type. (b) A complex shock topology is necessary to channel the flow. Shock segment AB is of 1–2 fast type, BD is 1–3 intermediate, DE is 1–2 fast, and DG is 2–4 intermediate, evolving into 2=3–4 switch-off and 3–4 slow along the front.


(a) B_x=0
(b) B_x=1


0.4
0.2
y
0.0
-0.2


0.08
0.04
0.00
y
-0.225
-0.175
-0.125
x
A
B
C
D
G
E
H
F
I


0.25
0.15
0.05
y
-0.25
-0.20
-0.15
x


A
B
C
D
G
E
H
F
I