Basic properties of nonlinear magnetohydrodynamic flows: theory and simulations

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Contents

Introduction – motivation

Conservative form ideal MHD equations

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\vec{v} \\ \frac{\rho v^2}{2} + \rho e + \frac{B^2}{2} \\ \vec{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\vec{v} \\ \rho\vec{v}\vec{v} + \left(p + \frac{B^2}{2}\right)\vec{I} - \vec{B}\vec{B} \\ \left(\frac{\rho v^2}{2} + \rho e + p\right)\vec{v} - (\vec{v}\times\vec{B})\times\vec{B} \\ \vec{v}\vec{B} - \vec{B}\vec{v} \end{bmatrix} = 0$$

- MHD describes macroscopic behavior of many plasmas in space, laboratory, ...
- plasma $=$ ionized gas $\Rightarrow$ electromagnetic effects

theory

- eight complex equations: very ‘old’
- MHD physics $\neq$ derive new equations

  $=$ find and understand solutions ($\sim$ general relativity)
- mathematical nature (complex!):

  - conservation law

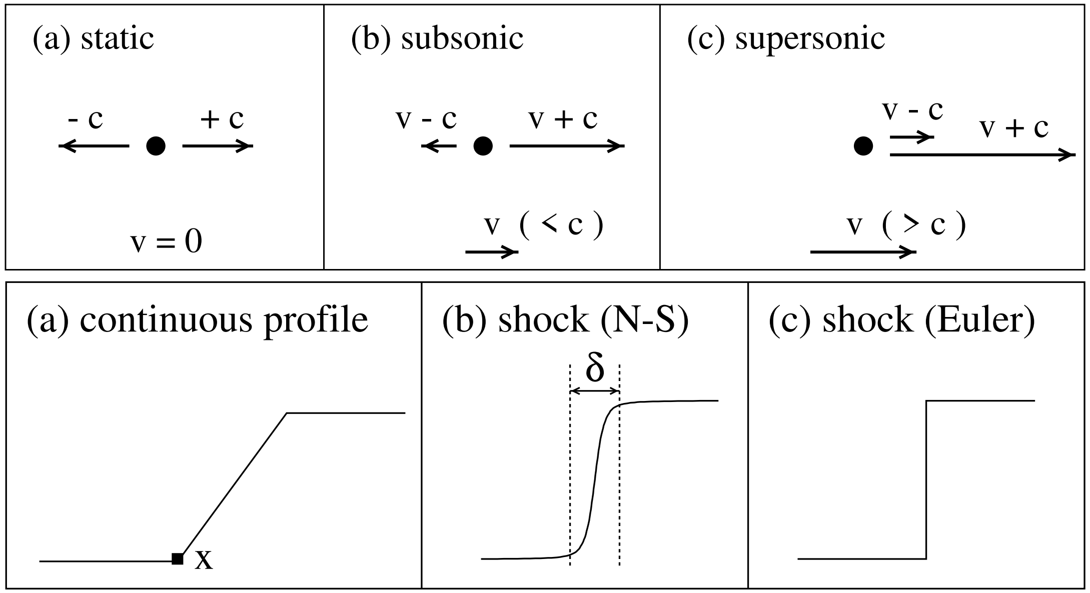
  - hyperbolic $\Rightarrow$ waves

  three anisotropic waves (gasdynamics: one isotropic wave, sound wave)

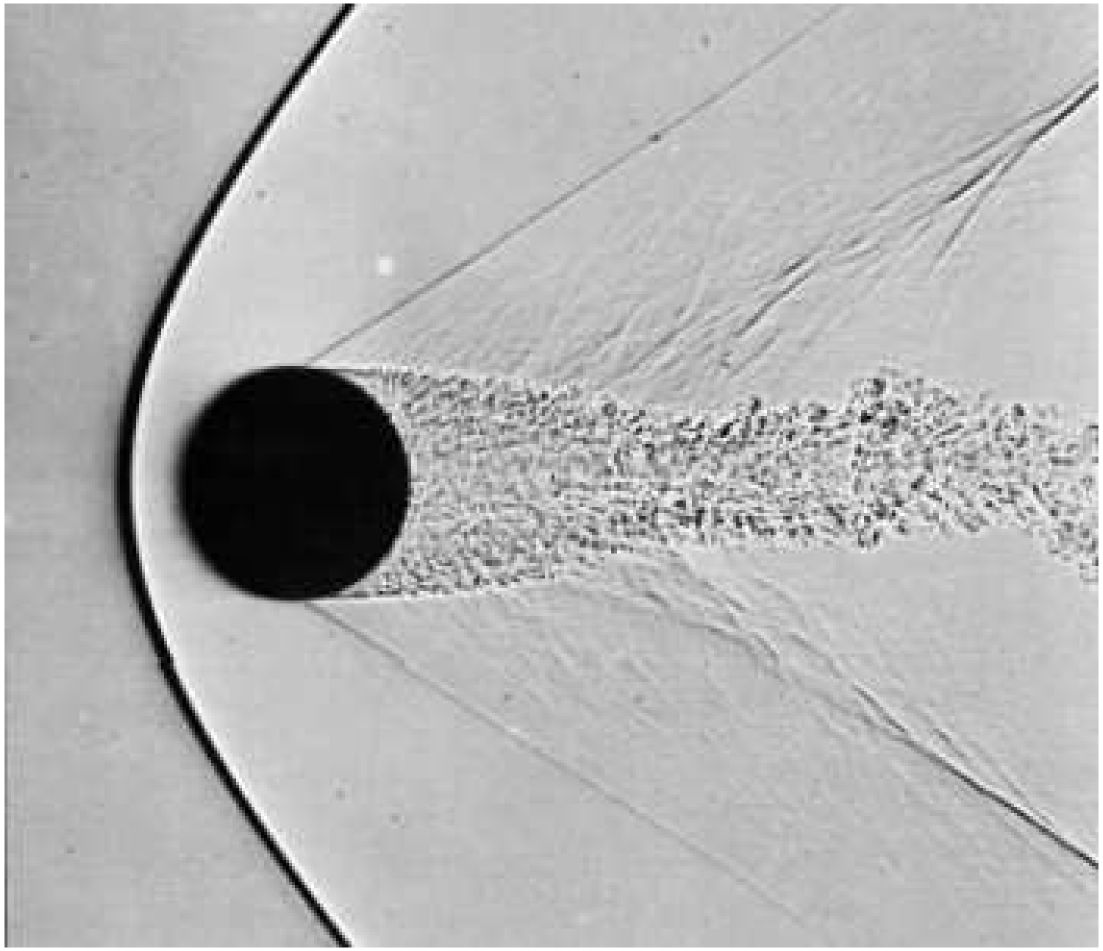
  - nonlinear $\Rightarrow$ waves can steepen into shocks (discontinuities)
- learn from simpler systems! (nonlinear hyperbolic conservation laws)

Shock phenomena

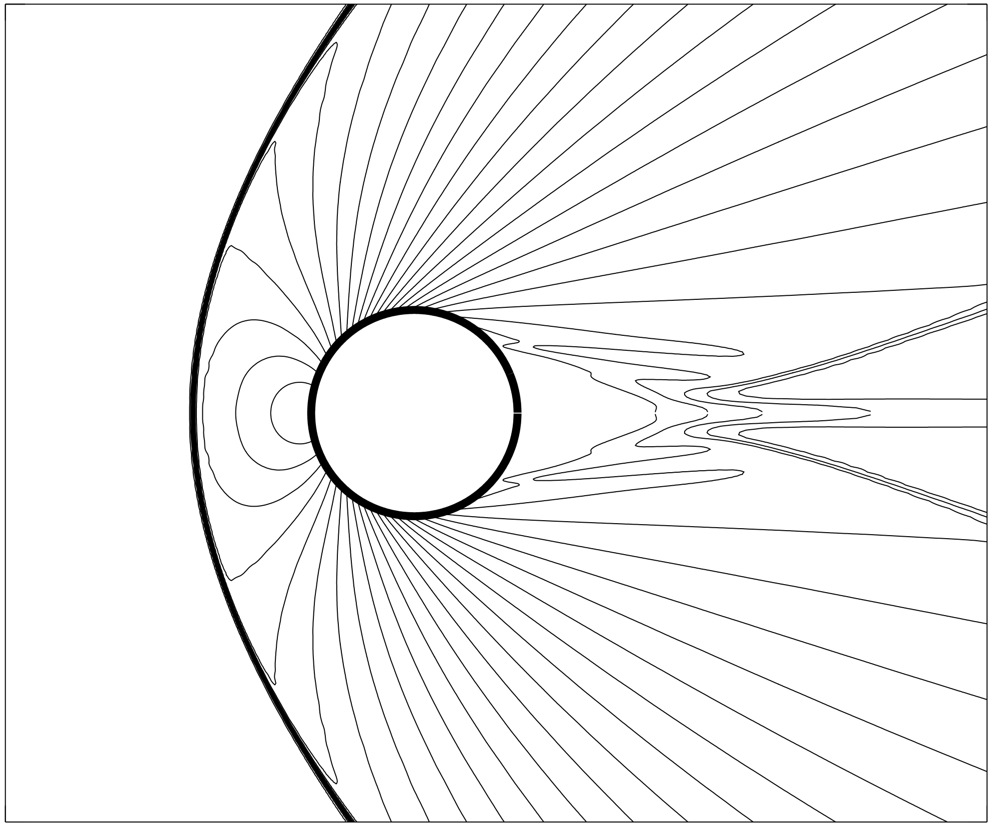
Sound waves and shocks in 1D



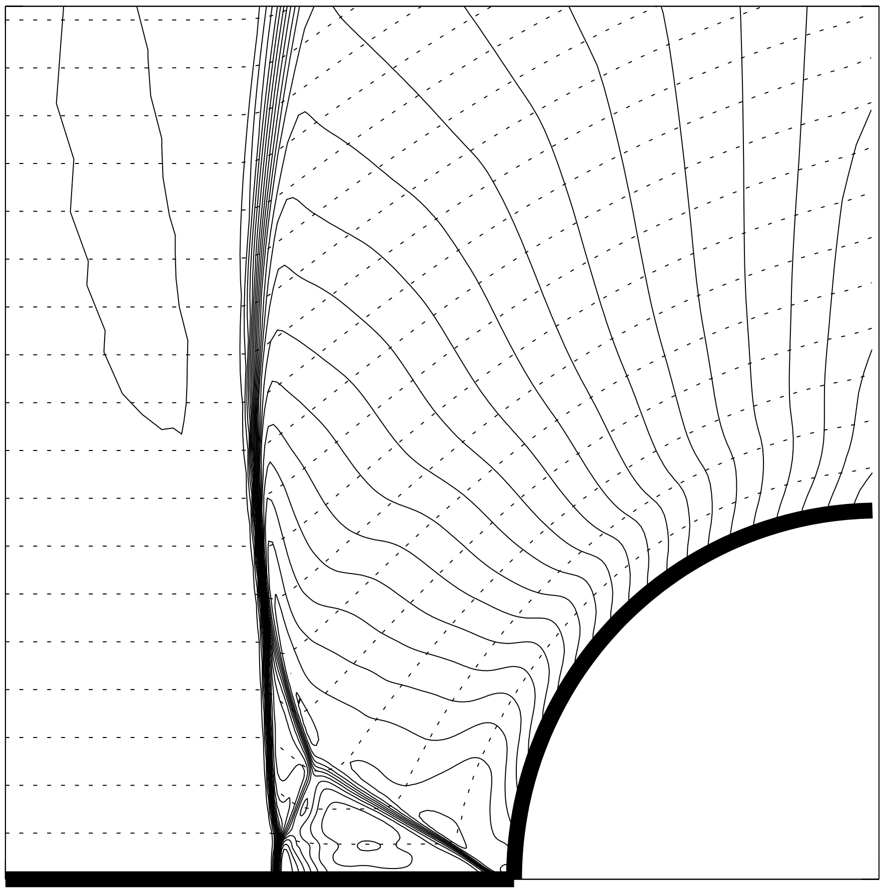
Supersonic airflow over sphere



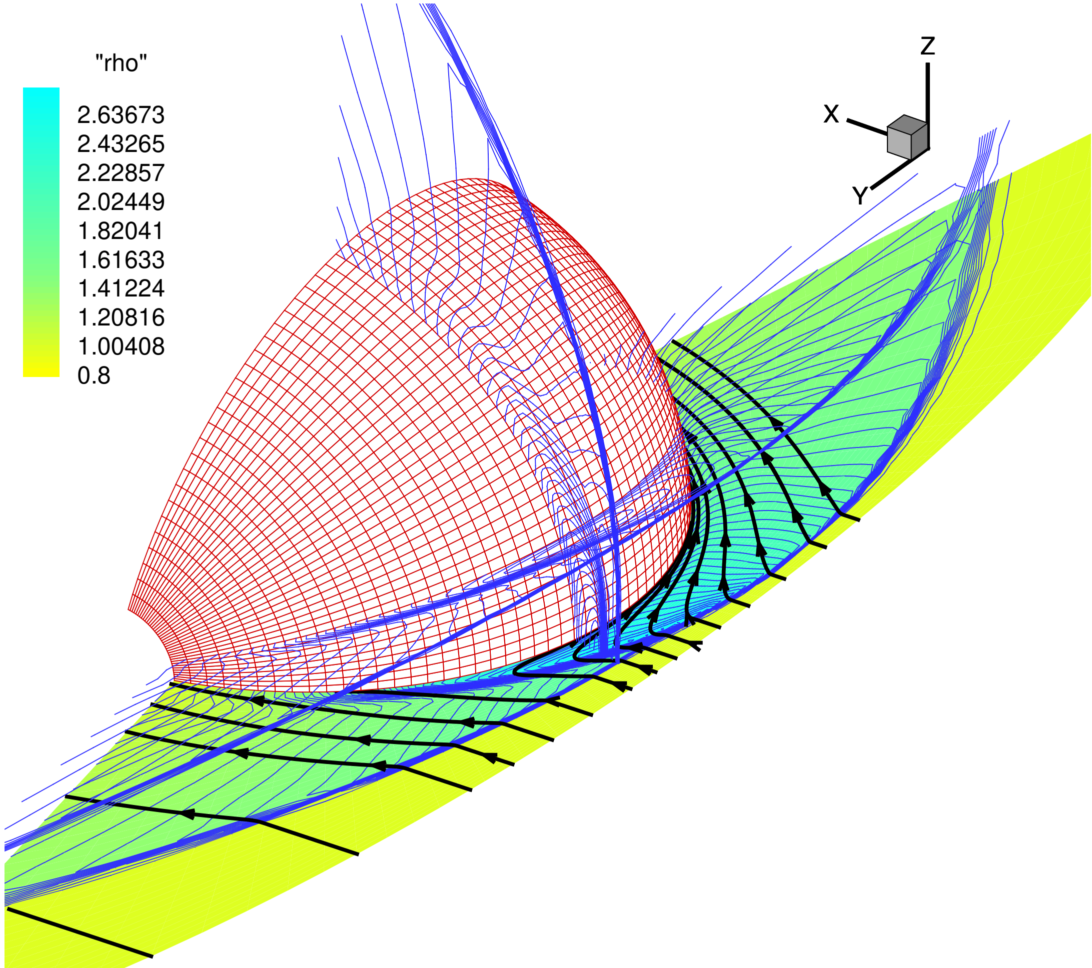
Numerical simulation of bow shock (gasdynamic)



MHD simulation of bow shock (2D)



MHD simulation of bow shock (3D)

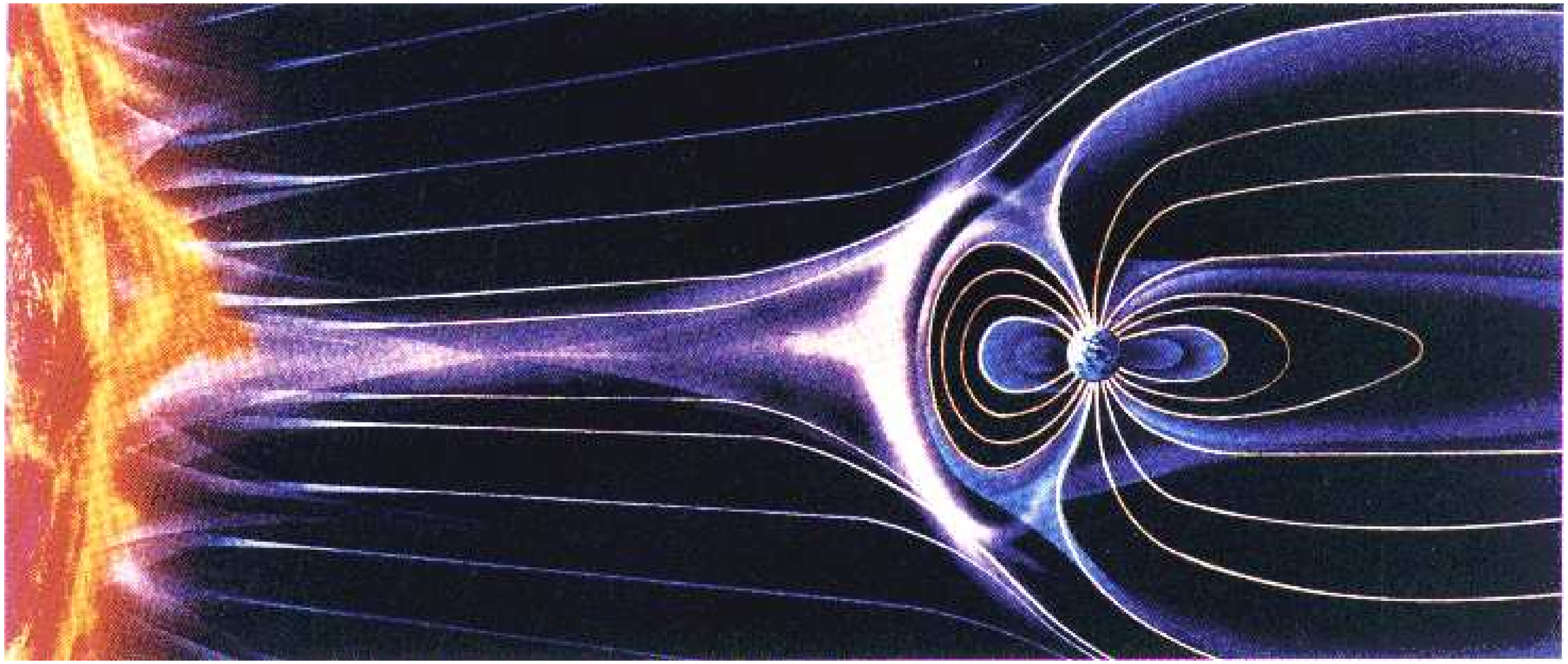


Numerical simulation techniques

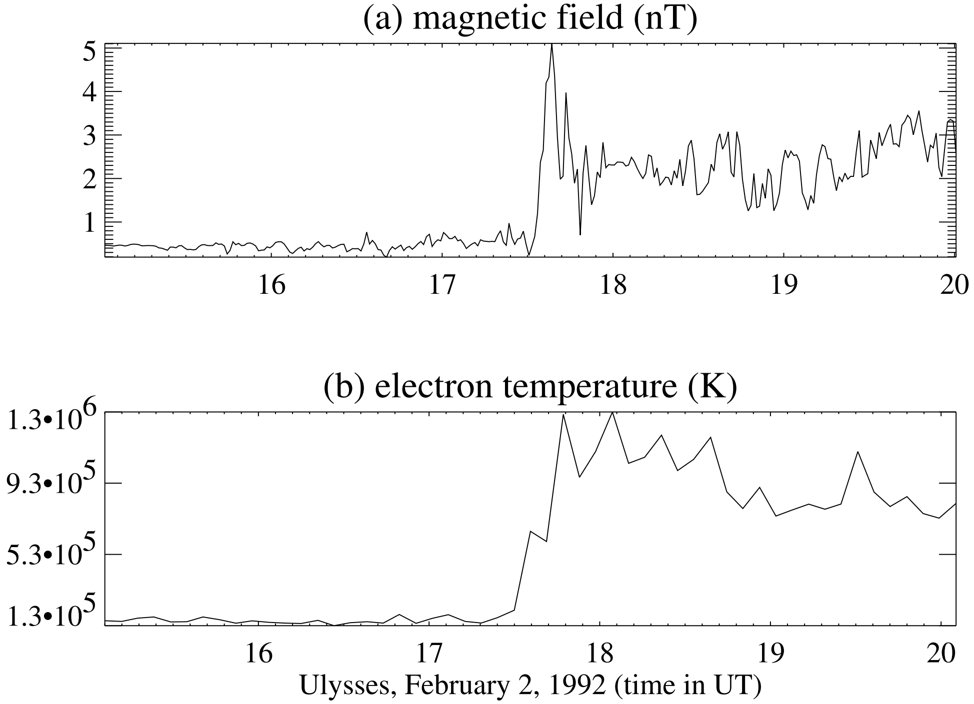
- nonlinear hyperbolic conservation law
- methods borrowed from Computational Fluid Dynamics (airplanes, ...)
- introduction to shock-capturing methods for MHD
- parallel computing using MPI

Applications

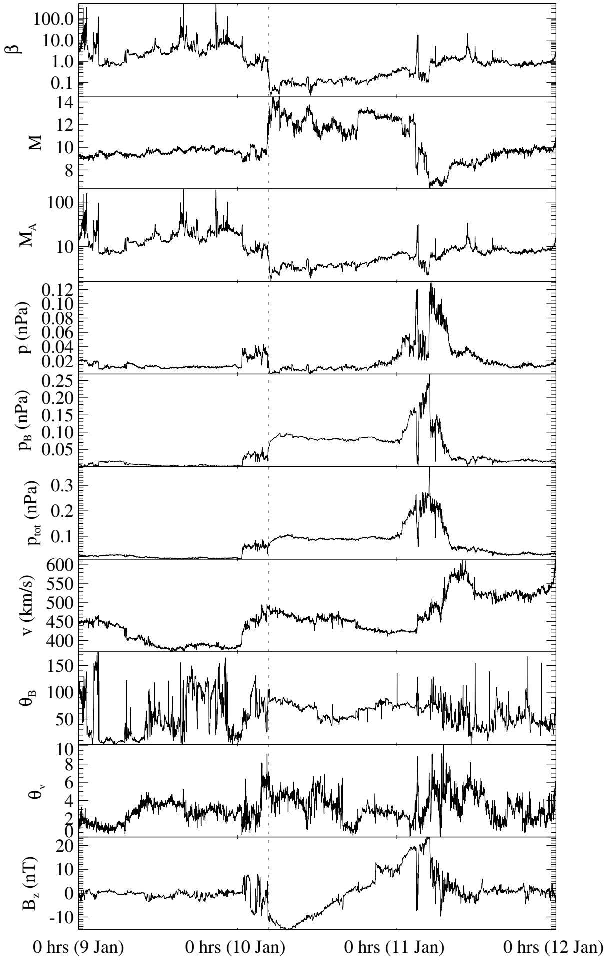
Earth's bow shock



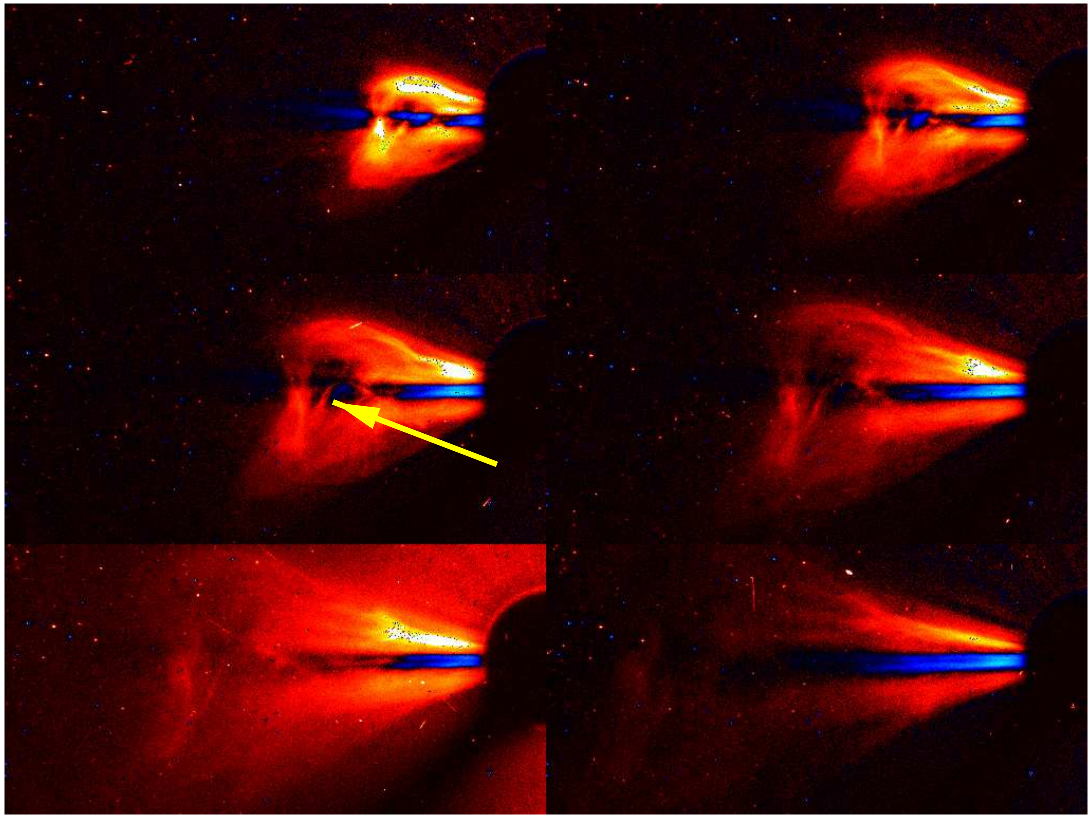
Jupiter's bow shock crossing



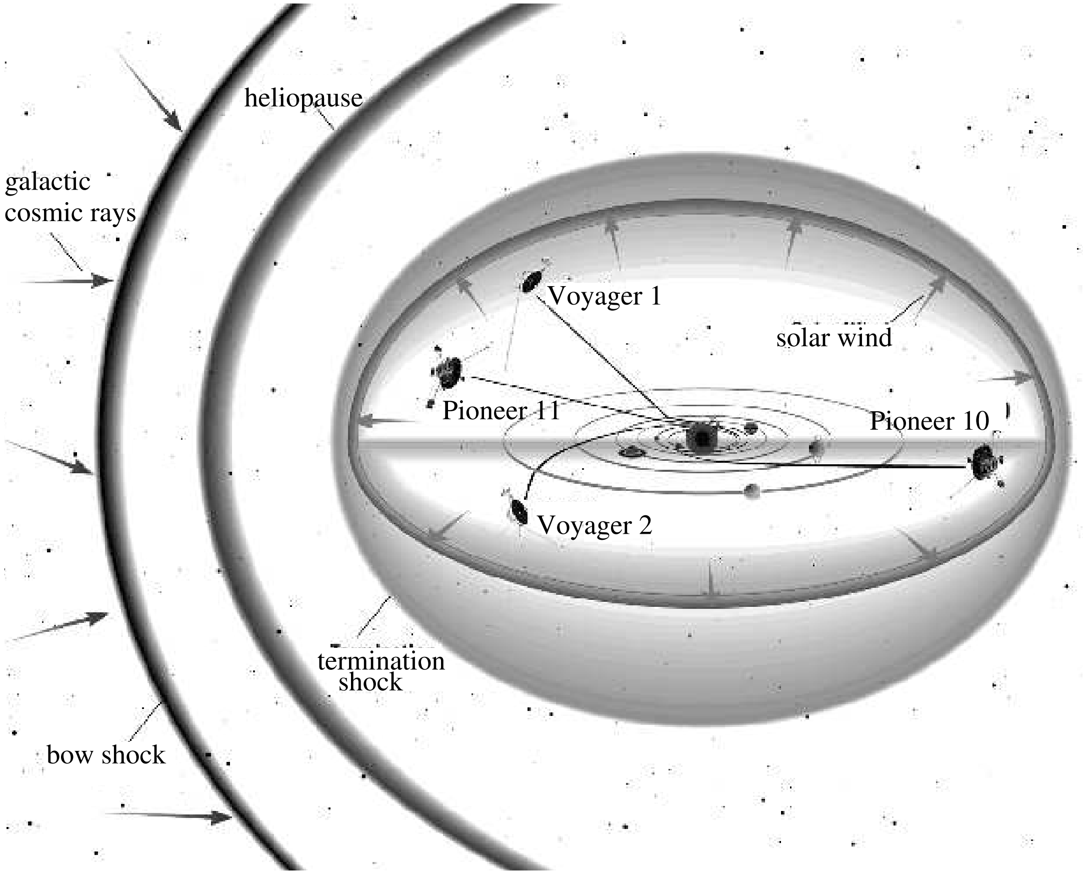
Magnetic cloud at Earth



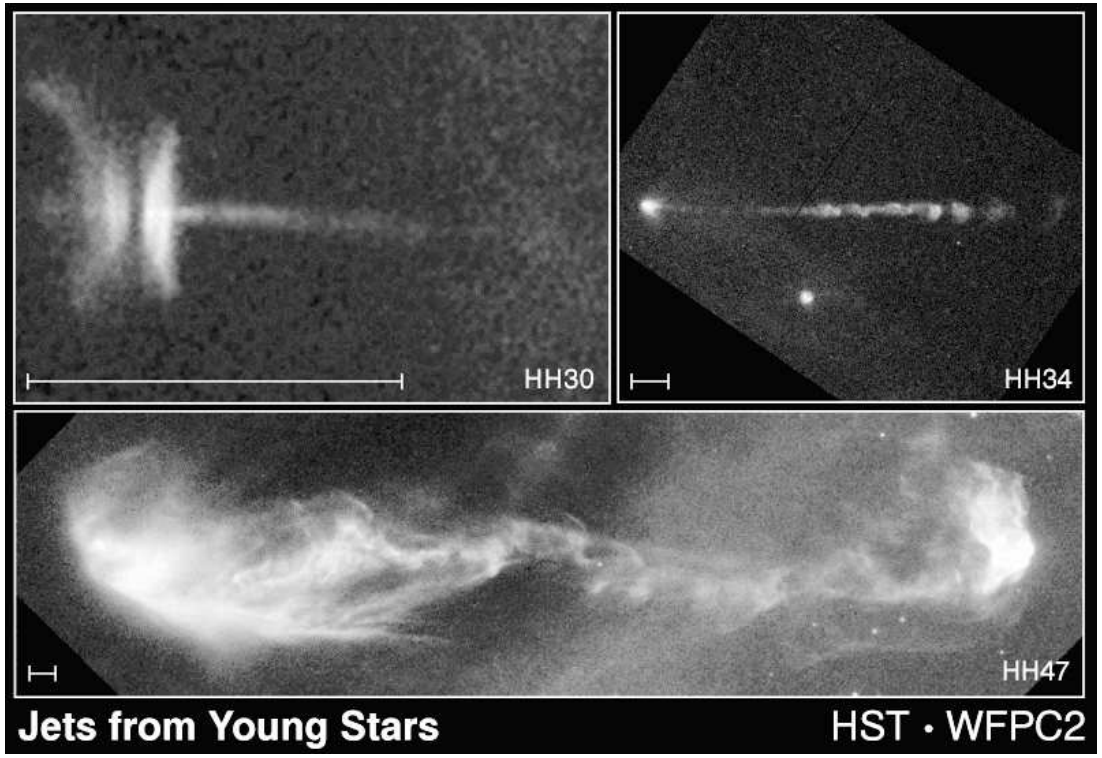
Solar Coronal Mass Ejections



Heliospheric bow shock



Astrophysical jets



This course

- basic properties of basic MHD flows

  - basic properties: ‘classical’ properties of full nonlinear equations, as far as they can be analyzed (we will not solve linearized equations)

  - basic flows: ’abstract’ setting, most simple flows

- double focus

  - continuous flow

  - discontinuities in flow

- approach:

  - theoretical properties of equations and solutions

  - numerical techniques

  - simulation of basic flows (+ link to observations)

Overview

1) Basic concepts of Hyperbolic Conservation Laws

2) Numerical simulation of flows with shocks

3) Stability of shock waves

4) Analysis and simulation of continuous MHD flow

5) Analysis and simulation of MHD flow with discontinuities

References

- introductory:

- Leveque, Numerical methods for conservation laws, Birkhauser, 1992. (no MHD)

- (also De Sterck, PhD thesis, 1999).

- general hyperbolic systems, advanced:

- Courant and Hilbert, Methods of mathematical physics, vol. 2, Interscience, 1962.

- Courant and Friedrichs, Supersonic flow and shock waves, Interscience, 1948.

- Whitham, Linear and nonlinear waves, Wiley-Interscience, 1974.

- MHD:

- Landau and Lifshitz, Electrodynamics of Continuous Media, Pergamon, 1984.

- Jeffrey and Taniuti, Nonlinear wave propagation, Academic Press, 1964.

- Anderson, Magnetohydrodynamic shock waves, MIT Press, 1963.

course dates

- Wednesday, March 29, 14.00 – 17.00, room B01.07.
- Wednesday, April 5, 14.00 – 17.00, room B01.07.
- Monday, May 15, 14.00 – 17.00, room B01.07.
- Wednesday, May 17, 14.00 – 17.00, room B01.05.
- Wednesday, May 31, 14.00 – 17.00, room B01.05.

homework/exam

- computational, starting from provided programs

  - MAPLE derivations

  - parallel simulations of simple problems (F90+MPI)

- analytical

  - try some of the (semi-)open problems encountered

Basic concepts of Hyperbolic Conservation Laws

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

- general flow properties of $u(x,t)$ (nonlinear!)

  - continuous flow

  - flow with discontinuities

- hyperbolic $\Rightarrow$ waves

1.1 Conservation Laws: Introduction

1.2 Scalar Conservation Laws

1.3 Systems of Conservation Laws

1.1 Conservation Laws: Introduction

1.1.1 Conservation laws

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \qquad u(x,t)}$$

- u is conserved variable (or rather, $\int u(x,t)dx$)
- $f(u)$ is the flux of u

Why is this equation called a ‘conservation law’?

define $\overline{u}(t) = \displaystyle\int_{x_0}^{x_1} u(x,t)dx$ and $\overline{\overline{f}}(x) = \displaystyle\int_{t_0}^{t_1} f(u(x,t))dt$, then

$$\int\int \left(\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x}\right) dx\ dt = 0$$

$$\Downarrow$$

$$\int_{t_0}^{t_1} \frac{\partial \overline{u}(t)}{\partial t} dt + \int_{x_0}^{x_1} \frac{\partial \overline{\overline{f}}(x)}{\partial x} dx = 0$$

$$\Downarrow$$

$$\boxed{\overline{u}(t_1) - \overline{u}(t_0) + \overline{\overline{f}}(x_1) - \overline{\overline{f}}(x_0) = 0}$$

(fig)

$\Rightarrow \overline{u}(t) = \displaystyle\int u(x,t)dx$ conserved in time (if no flux through boundaries)

$\Rightarrow$ integral form of conservation law, also for discontinuous solutions: more general

1.1.2 A scalar example: the linear advection equation

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

$$f(u) = a\,u \quad \Rightarrow \quad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

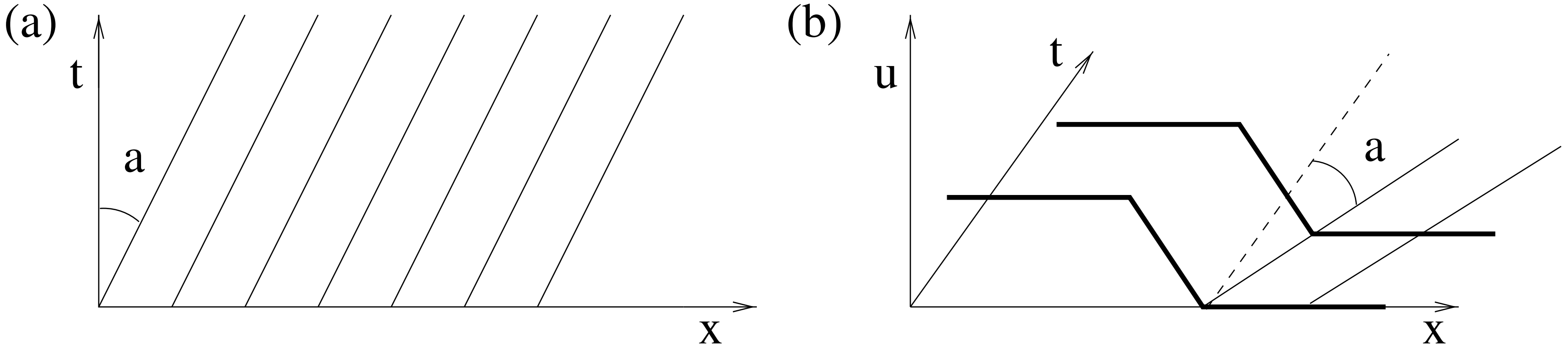
- general solution:

$$u(x,t) = u^*(y(x,t)) \text{ with } y = x - at$$

$$\frac{\partial u^*}{\partial y}\,\frac{\partial y}{\partial t} + a\,\frac{\partial u^*}{\partial y}\,\frac{\partial y}{\partial x} = 0$$

$$\frac{\partial u^*}{\partial y}\,(-a) + a\,\frac{\partial u^*}{\partial y}\,1 \equiv 0\,!$$

- linear advection of arbitrary profile $u^*(y)$: traveling wave


(a)
t
a
x
(b)
u
t
a
x

1.1.3 A system example: the Euler equations

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0} \quad \text{with } U(x,t) \in R^n$$

- Euler (dissipationless hydrodynamics or gasdynamics, compressible)

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ e \end{bmatrix} \qquad F(U) = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p \\ \rho v_x v_y \\ (e+p) v_x \end{bmatrix}$$

with $e = \dfrac{p}{\gamma - 1} + \dfrac{1}{2}\rho v^2$

- also MHD, shallow water, relativistic hydrodynamics, general relativity, ...

1.1.4 Generalization to 2D

$$\frac{\partial u}{\partial t} + \frac{\partial f_x(u)}{\partial x} + \frac{\partial f_y(u)}{\partial y} = 0 \qquad \text{with } u(x,y,t)$$

or

$$\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) = 0} \qquad \text{with } \vec{f}(u) = (f_x(u), f_y(u))$$

define $\overline{u}(t) = \int\int u(x,y,t)dx\,dy$ and $\overline{\overline{\vec{f}}}(x,y) = \int_{t_0}^{t_1} \vec{f}(u(x,y,t))dt$, then

$$\overline{u}(t_1) - \overline{u}(t_0) + \int\int \nabla \cdot \overline{\overline{\vec{f}}}(x,y)dx\,dy = 0$$

$$\Downarrow$$

$$\boxed{\overline{u}(t_1) - \overline{u}(t_0) + \oint \overline{\overline{\vec{f}}}(x,y) \cdot \vec{n}dl = 0}$$

(fig)

1.1.5 The rest of this lecture

properties of (hyperbolic) conservation laws:

- continuous flow (characteristics, invariants, linear waves)
- flow with discontinuities (shocks, jump relations, Riemann problem)

1.2 Scalar Conservation Laws

- linear
- nonlinear
- nonconvex nonlinear

1.3 Systems of Conservation Laws

- linear system
- wave equation
- nonlinear 2×2 system
- Euler
- MHD

1.2 Scalar Conservation Laws

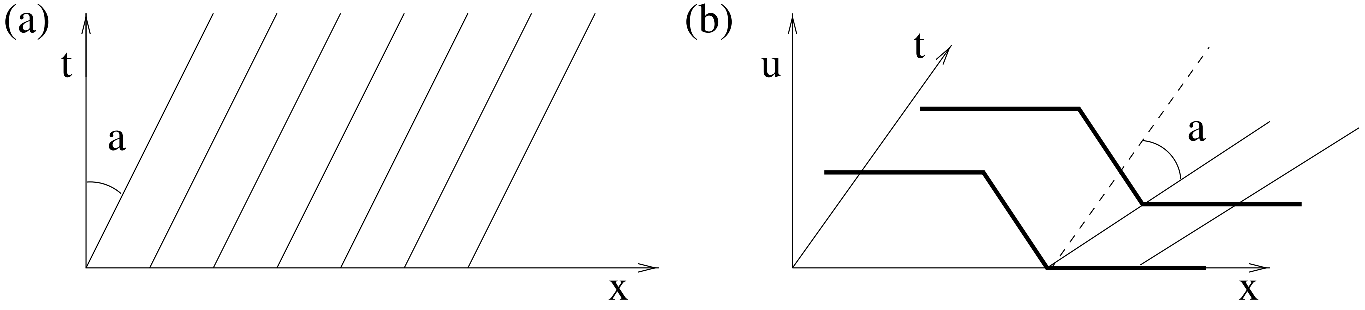
1.2.1 Linear advection equation

$$f(u) = a\,u \quad \Rightarrow \quad \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

- general solution:

$$u(x,t) = u^*(y(x,t)) \text{ with } y = x - at$$

- linear advection of arbitrary profile $u^*(y)$, also discontinuous profile (integral form of conservation law) : traveling wave



Characteristics and Riemann Invariants

$$\frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x} = 0$$

- consider curve $x(t)$ in xt-plane, how does u vary along curve?

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = \frac{\partial u(x(t),t)}{\partial t} + \frac{\partial u(x(t),t)}{\partial x}\,\frac{\partial x(t)}{\partial t}$$

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} \equiv 0 \qquad \text{if} \qquad \frac{\partial x(t)}{\partial t} = a$$

- $\boxed{x(t) : \dfrac{\partial x(t)}{\partial t} = a}$ is called a **characteristic curve** (straight line!)

  slope of characteristic $\equiv$ characteristic speed = (advection) wave speed

- $\boxed{\dfrac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} \equiv 0}$ along the characteristic $\Rightarrow$ u is a **Riemann Invariant** (RI)

1.2.2 Nonlinear scalar conservation laws

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

$$\frac{\partial u}{\partial t} + f'(u)\,\frac{\partial u}{\partial x} = 0 \qquad \text{with} \qquad f'(u) = \frac{\partial f(u)}{\partial u}$$

- example: (inviscid) Burgers equation

$$f(u) = \frac{u^2}{2}$$

$$f'(u) = u$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0$$

Characteristics and Riemann Invariants (RIs)

$$\frac{\partial u}{\partial t} + f'(u)\,\frac{\partial u}{\partial x} = 0$$

- consider curve $x(t)$ in xt-plane, how does u vary along curve?

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = \frac{\partial u(x(t),t)}{\partial t} + \frac{\partial u(x(t),t)}{\partial x}\,\frac{\partial x(t)}{\partial t}$$

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} \equiv 0 \qquad \text{if} \qquad \frac{\partial x(t)}{\partial t} = f'(u)$$

- $\boxed{x(t) : \dfrac{\partial x(t)}{\partial t} = f'(u)}$ is called a **characteristic curve**

- $\boxed{\dfrac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} \equiv 0}$ along the characteristic $\Rightarrow$ u is a **Riemann Invariant** (RI)

- $\boxed{x(t) : \dfrac{\partial x(t)}{\partial t} = f'(u)}$ $\boxed{\dfrac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} \equiv 0}$

$\Rightarrow$ u is advected along characteristic with slope $f'(u)$ $\Rightarrow$ $f'(u)$ is wave speed

$\Rightarrow$ slope of characteristic $\equiv$ characteristic speed = wave speed

$\Rightarrow$ characteristic is straight line !

- example:

  Burgers, $f(u) = u^2/2$, $f'(u) = u$

  slope between $u_l = 0$ and $u_r = 1$

  $f'(u_l) = 0$ and $f'(u_r) = 1$

$\Rightarrow$ expansion or rarefaction wave

- weak discontinuity $\equiv$ discontinuity in slope

  travels with characteristic speed

(fig)

Linear waves

$$\frac{\partial u}{\partial t} + f'(u)\,\frac{\partial u}{\partial x} = 0$$

- assume $u(x,t) = u_0 + u_1(x,t)$

  with u_0 constant background,

  $u_1(x,t)$ small perturbation

  $$f'(u) = f'(u_0) + f''(u_0)\,u_1 + O(u_1^2)$$

$$\Rightarrow \quad \frac{\partial u_1}{\partial t} + f'(u_0)\,\frac{\partial u_1}{\partial x} \approx 0$$

(fig)

linear advection equation for $u_1(x,t)$ with constant wave speed $f'(u_0)$!

profile u_1 is advected, wave speed (phase speed) is $f'(u_0)$

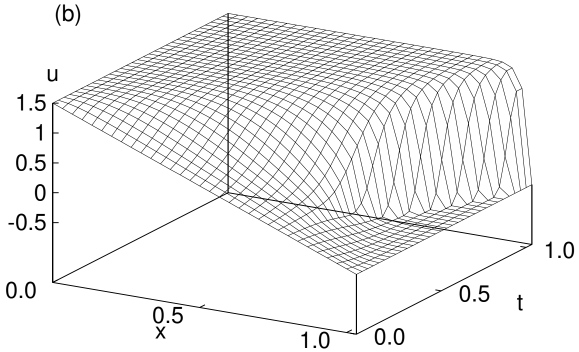
Steepening into shocks

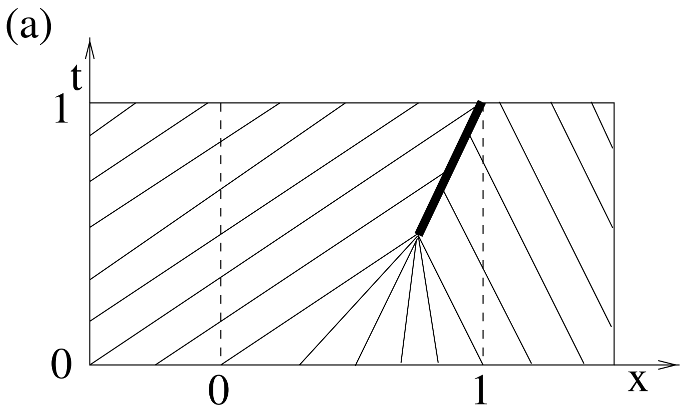
- Burgers, $f(u) = u^2/2$, $f'(u) = u$

  switch u_l and u_r, such that characteristics converge

  slope between $u_l = 1.5$ and $u_r = -0.5$

  $f'(u_l) = 1.5$ and $f'(u_r) = -0.5$





$\Rightarrow$ a shock wave is formed

shock wave $=$ discontinuity in $u(x)$

characteristics enter into shock

shock propagates with constant speed, but how fast?

remark: for the (linear) advection equation the characteristics do not converge, so shocks cannot be formed through steepening

nonlinearity is necessary for shock formation!

Rankine-Hugoniot (RH) jump relations

- what is the shock speed s? $\Rightarrow$ use the integral form of the conservation law

$$\overline{u}(t_1) - \overline{u}(t_0) + \overline{\overline{f}}(x_1) - \overline{\overline{f}}(x_0) = 0$$

$$\Downarrow$$

$$-s\,\Delta t\,(u_r - u_l) + \Delta t\,(f(u_r) - f(u_l)) = 0$$

$$-s\,\Delta u + \Delta f = 0$$

$$\boxed{s = \frac{\Delta f}{\Delta u} = \frac{f(u_r) - f(u_l)}{u_r - u_l}}$$

(fig)

- linear advection: $s = a$
- Burgers: $s = \dfrac{u_r^2 - u_l^2}{2\,(u_r - u_l)} = \dfrac{u_r + u_l}{2}$

The Riemann problem

- Riemann problem $=$ how does initial discontinuity between two constant states u_l and u_r evolve in time?

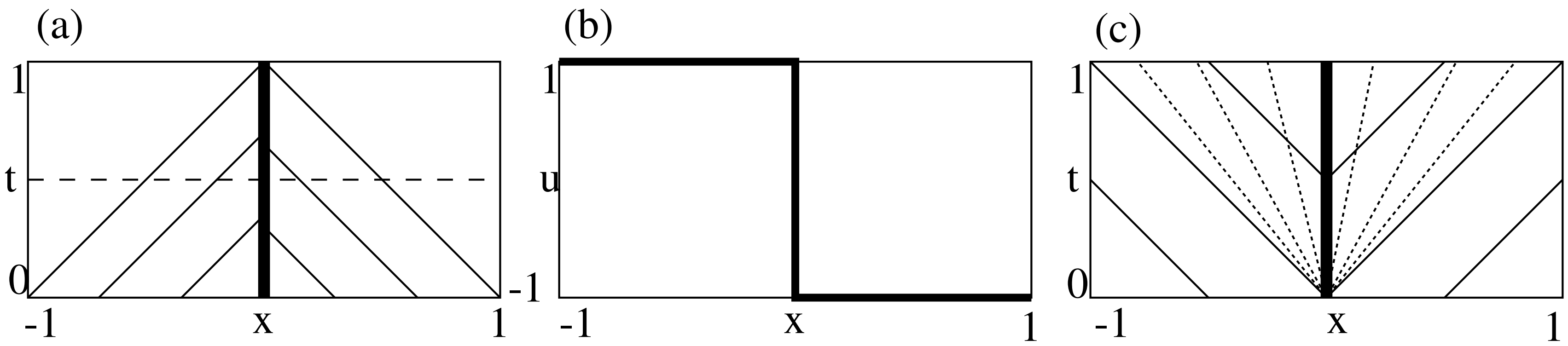
$\Rightarrow$ Burgers:

- $f'(u_l) > f'(u_r)$, characteristics converge

$\Rightarrow$ shock with shock speed s from RH relation

- $f'(u_l) < f'(u_r)$, characteristics diverge

$\Rightarrow$ (continuous) rarefaction wave



Instability of expansion shocks

- remark: the rarefaction case initially also satisfies the RH relation with shock speed s
- the characteristics would diverge from this shock : expansion shock
- BUT: this shock solution is intrinsically unstable

  - formation argument: cannot be formed from wave steepening

  - perturbation argument: ‘infinitesimally’ small or ‘generic’ perturbation makes the shock into a slope, which expands like the rarefaction wave

  (for a compressional wave, the slope resulting from perturbation would steepen again!)
- characteristics have to converge in a shock
- instability of expansion shocks is related to entropy (see Lecture 3)

Convexity of flux functions

- for Burgers: every Riemann problem results in either a shock or a rarefaction

  - this is true for all convex flux functions $f(u)$

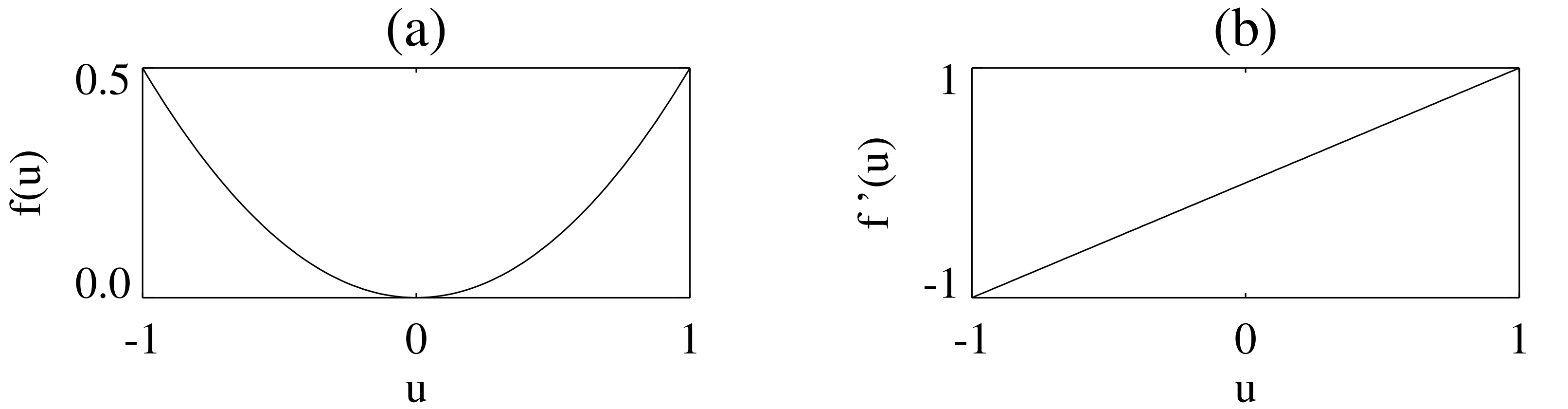
  - $f(u)$ is convex $\Leftrightarrow$ $f'(u)$ is monotone $\Leftrightarrow$ $f''(u)$ does not change sign

  - $f(u)$ is convex $\Rightarrow f'(u_l) > s > f'(u_r)$ when characteristics converge

  reason: $s = \dfrac{f(u_r) - f(u_l)}{u_r - u_l}$ and middle value theorem

  - Burgers is convex ($f(u) = u^2/2$)

- if $f(u)$ is non-convex: Riemann problems may have more complicated solutions than just a shock or a rarefaction


(a)
0.5
0.0
f(u)
-1
0
1
u
(b)
1
-1
f '(u)
-1
0
1
u

1.2.3 Nonconvex nonlinear scalar conservation laws

The Riemann problem

- example: $f(u) = \dfrac{u^3}{3}$, $f'(u) = 3\,u^2$

$\Rightarrow$ non-convex

Riemann problem: $u_l = 1$ and $u_r = -0.75$

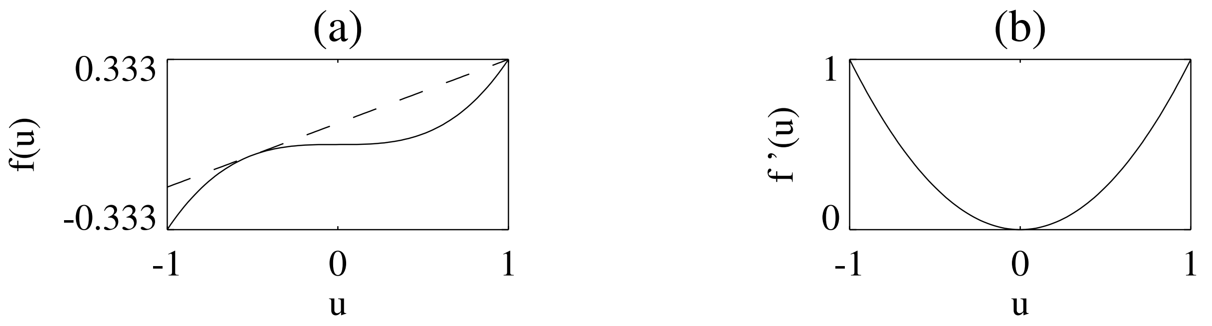
RH: $f'(u_l) > f'(u_r) > s = 37/168$

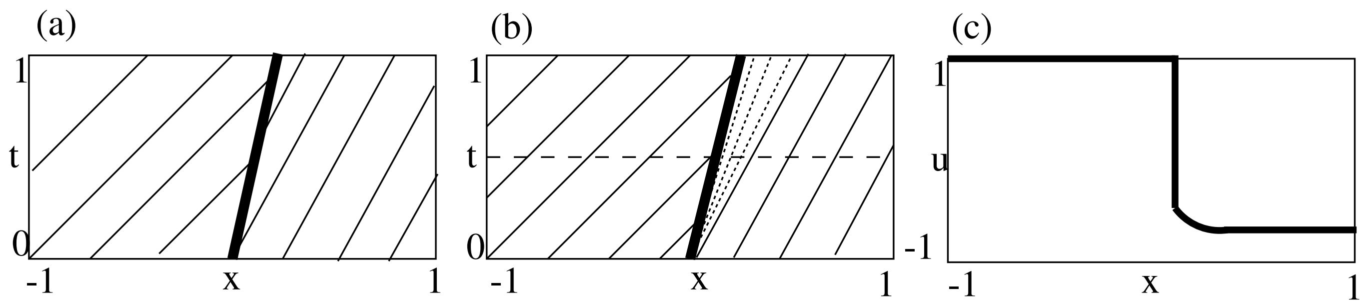
right characteristics diverge from shock : unstable

solution: rarefaction starting from right state until right characteristic becomes parallel to shock (with speed $\neq s$!)

condition: $f'(u^*) = s^* = \dfrac{f(u^*) - f(u_l)}{u^* - u_l}$

tangent hull construction


(a)
0.333
f(u)
-0.333
-1
0
1
u
(b)
1
f '(u)
0
-1
0
1
u


(a)
1
t
0
-1
x
1
(b)
1
t
0
-1
x
1
(c)
1
u
-1
-1
x
1

$\Rightarrow$ result: compound shock

- shock with attached rarefaction

- characteristic is parallel to shock where rarefaction is attached

- rarefaction is sonic where it is attached: shock speed = wave speed (characteristic speed)

$\Rightarrow$ Riemann problem (if one inflection point):

- $f'(u_l) > f'(u_r)$, characteristics converge

- if $f'(u_l) > s > f'(u_r)$: shock

- if $f'(u_l) > f'(u_r) > s$ or $s > f'(u_l) > f'(u_r)$: compound shock

- $f'(u_l) < f'(u_r)$, characteristics diverge

- (continuous) rarefaction wave

recapitulation: scalar conservation laws

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0}$$

- continous flow

$$\Rightarrow \boxed{x(t) : \frac{\partial x(t)}{\partial t} = f'(u)} \quad : \quad \text{characteristic}$$

$$\Rightarrow \boxed{\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} \equiv 0} \quad : \quad \text{Riemann Invariant (RI)}$$

- flow with discontinuities:

$$\Rightarrow \boxed{s = \frac{\Delta f}{\Delta u} = \frac{f(u_r) - f(u_l)}{u_r - u_l}} \quad : \quad \text{Rankine Hugoniot (RH) relation}$$

$\Rightarrow$ Riemann problem: shock or rarefaction (or compound shock)

1.3 Systems of Conservation Laws

1.4 Systems of Conservation Laws

recapitulation: scalar conservation laws

$$\boxed{\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0} \qquad \boxed{\overline{u}(t_1) - \overline{u}(t_0) + \overline{\overline{f}}(x_1) - \overline{\overline{f}}(x_0) = 0}$$

- continous flow

$$\Rightarrow \boxed{x(t) : \frac{\partial x(t)}{\partial t} = f'(u)} \quad : \quad \text{characteristic}$$

$$\Rightarrow \boxed{\frac{\mathrm{d}u(x(t), t)}{\mathrm{d}t} \equiv 0} \quad : \quad \text{Riemann Invariant (RI)}$$

- flow with discontinuities:

$$\Rightarrow \boxed{s = \frac{\Delta f}{\Delta u} = \frac{f(u_r) - f(u_l)}{u_r - u_l}} \quad : \quad \text{Rankine Hugoniot (RH) relation}$$

$\Rightarrow$ Riemann problem: shock or rarefaction

1.4.1 Linear systems

Linear hyperbolic systems

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

$$\Downarrow F(U) = \mathsf{A} \cdot U$$

$$\boxed{\frac{\partial U}{\partial t} + \mathsf{A} \cdot \frac{\partial U}{\partial x} = 0 \quad (1)}$$

with $U(x,t) = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in R^n$

and $F(U) = \begin{bmatrix} f_1(U) \\ \vdots \\ f_n(U) \end{bmatrix}$

(1) is a hyperbolic system of equations

$\Updownarrow$

A has n real eigenvalues and a complete set of eigenvectors

$\Updownarrow$

the system has n real characteristic curves

$\Rightarrow$ $\mathsf{A} = \mathsf{R} \cdot \Lambda \cdot \mathsf{L}$ with

- $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$

  diagonal eigenvalue (e-val) matrix

- $\mathsf{R} = \left[\begin{array}{c|c|c} R_1 & \cdots & R_n \end{array} \right]$: right eigenvectors (e-vect), $\mathsf{A} \cdot R_i = \lambda_i \, R_i$

- $\mathsf{L} = \left[\begin{array}{c} L_1 \\ \hline \vdots \\ \hline L_n \end{array} \right]$: left eigenvectors (e-vect), $L_i \cdot \mathsf{A} = \lambda_i \, L_i$

  normalization: $\mathsf{R} = \mathsf{L}^{-1}$

  (if, additionally, $\mathsf{R} \cdot \mathsf{R}^T = \mathsf{I} \Rightarrow \mathsf{R}^T = \mathsf{L}$)

Characteristic variables

$$\frac{\partial U}{\partial t} + \mathsf{A} \cdot \frac{\partial U}{\partial x} = 0 \quad \text{with } \mathsf{A} = \mathsf{R} \cdot \Lambda \cdot \mathsf{L}$$

$$\frac{\partial U}{\partial t} + \mathsf{R} \cdot \Lambda \cdot \mathsf{L} \cdot \frac{\partial U}{\partial x} = 0$$

$$\frac{\mathsf{L} \cdot \partial U}{\partial t} + \Lambda \cdot \frac{\mathsf{L} \cdot \partial U}{\partial x} = 0 \quad \text{with } \mathsf{L} \cdot \partial U = \partial \mathsf{L} \cdot U \text{ (linear, } \mathsf{L} \text{ constant!)}$$

define $\boxed{W = \mathsf{L} \cdot U : \quad \text{characteristic variables } W}$ $\Rightarrow$ $\boxed{\frac{\partial W}{\partial t} + \Lambda \cdot \frac{\partial W}{\partial x} = 0}$

$$\forall i : \boxed{\frac{\partial w_i}{\partial t} + \lambda_i \cdot \frac{\partial w_i}{\partial x} = 0} \qquad (i = 1..n)$$

$\Rightarrow$ n scalar linear advection equations for $w_i(y_i = x - \lambda_i\, t)$ with wave speed λ_i

$\Rightarrow$ the equations fully decouple

Characteristics and Riemann Invariants

$$\forall i : \frac{\partial w_i}{\partial t} + \lambda_i \cdot \frac{\partial w_i}{\partial x} = 0 \qquad (i = 1..n)$$

$\forall i$: (n characteristic fields)

- $\boxed{x_i(t) : \frac{\partial x_i(t)}{\partial t} = \lambda_i}$: ith characteristic
- $\boxed{\frac{\mathrm{d}w_i(x_i(t), t)}{\mathrm{d}t} \equiv 0}$: w_i Riemann Invariant (RI) on characteristic x_i

$\Rightarrow$ n different (linear) waves

$\Rightarrow$ n characteristic curves

$\Rightarrow$ n Riemann Invariants

Wave decomposition

$U = \mathrm{R} \cdot W$

- excite one characteristic wave: simple wave, with one single wave speed λ_i

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \left[\cdots \middle| R_i \middle| \cdots \right] \cdot \begin{bmatrix} \vdots \\ 0 \\ w_i(x - \lambda_i\, t) \\ 0 \\ \vdots \end{bmatrix}$$

$$\Rightarrow U(x - \lambda_i\, t) = R_i\, w_i(x - \lambda_i\, t) \quad : \quad \text{simple wave}$$

- excite several characteristic waves

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \left[R_1 \middle| \cdots \middle| R_n \right] \cdot \begin{bmatrix} w_1(x - \lambda_1\, t) \\ \vdots \\ w_n(x - \lambda_n\, t) \end{bmatrix}$$

$$\Rightarrow U(x, t) = \sum_{j=1}^{n} R_j\, w_j(x - \lambda_j\, t) \; : \quad \text{general wave}$$

$W = \mathsf{L} \cdot U$

- excite one conserved variable u_i

$$\begin{bmatrix} w_1 \\ \\ \vdots \\ \\ w_n \end{bmatrix} = \begin{bmatrix} L_1 \\ \hline \vdots \\ \hline L_n \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ 0 \\ u_i \\ 0 \\ \vdots \end{bmatrix}$$

$\Rightarrow \forall j \; : \; w_j = L_{j,i}\, u_i$: n characteristic waves are excited, each with speed λ_j

- excite several conserved variables

$$\begin{bmatrix} w_1 \\ \\ \vdots \\ \\ w_n \end{bmatrix} = \begin{bmatrix} L_1 \\ \hline \vdots \\ \hline L_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$\Rightarrow \forall j \; : \; w_j = L_j \cdot U$: general wave

A simple 2×2 example

$A = R \cdot \Lambda \cdot L$ with

- $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $R = \left[\begin{array}{c|c} R_1 & R_2 \end{array}\right] = \left[\begin{array}{c|c} 0.5 & 0.5 \\ 0.5 & -0.5 \end{array}\right]$
- $L = \left[\begin{array}{c} L_1 \\ \hline L_2 \end{array}\right] = R^{-1} = \left[\begin{array}{cc} 1 & 1 \\ \hline 1 & -1 \end{array}\right]$

$\Rightarrow$ excite w_1 (profile in the first characteristic wave), $w_2 \equiv 0$: simple wave

$$W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1(y_1 = x - \lambda_1\, t) \\ 0 \end{bmatrix}$$

$$U(x - \lambda_1\, t) = \left[\begin{array}{c|c} R_1 & R_2 \end{array}\right] \cdot \begin{bmatrix} w_1 \\ 0 \end{bmatrix} = R_1\, w_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} w_1(x - \lambda_1\, t)$$

$\Rightarrow$ excite w_1 and w_2 : not a simple wave

$$U = R_1\, w_1 + R_2\, w_2 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \; w_1(x - \lambda_1\, t) + \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix} \; w_2(x - \lambda_2\, t)$$

$\Rightarrow$ excite conservative variable $u_2(x,0)$, but take $u_1(x,0) \equiv 0$

$$W = \begin{bmatrix} L_1 \\ \hline L_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \equiv 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \hline 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_2$$

$\int u_1 \, dx$ and $\int u_2 \, dx$ are conserved !

Shocks

- integral conservation law:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

$$\Downarrow$$

$$\boxed{\overline{U}(t_1) - \overline{U}(t_0) + \overline{\overline{F}}(x_1) - \overline{\overline{F}}(x_0) = 0}$$

with $\overline{U}(t) = \int_{x_0}^{x_1} U(x,t)dx$ and $\overline{\overline{F}}(x) = \int_{t_0}^{t_1} F(U(x,t))dt$

- Rankine-Hugoniot relation:

$$\boxed{s(U_r - U_l) = F(U_r) - F(U_l) = \mathsf{A} \cdot (U_r - U_l)}$$

$s_i = \lambda_i$ and $U_r - U_l = R_i$

n wave families $\Rightarrow$ n shock waves

The Riemann problem

- Riemann problem:

$$U(x,0) = U_l \text{ } (x < 0)$$
$$U(x,0) = U_r \text{ } (x > 0)$$

- wave decomposition of initial state $W_l = \mathsf{L} \cdot U_l$, $W_r = \mathsf{L} \cdot U_r$
- characteristic waves propagate $w_i(y_i = x - \lambda_i\, t)$

$$w_i(y_i) = w_{l,i} \quad \text{if} \quad y_i = x - \lambda_i\, t < 0$$
$$w_i(y_i) = w_{r,i} \quad \text{if} \quad y_i = x - \lambda_i\, t > 0$$

- reconstruct conserved variables

$$U = \mathsf{R} \cdot W = \sum_{i=1}^{n} R_i\, w_i(y_i = x - \lambda_i\, t)$$
$$= \sum_j^{y_j<0} R_j\, w_{l,j} + \sum_k^{y_k>0} R_k\, w_{r,k}$$

$\Rightarrow$ n shocks propagating with speed λ_i

$\Rightarrow$ initial discontinuity $U_r - U_l$ splits up in n discontinuities

$$U_r - U_l = \mathsf{R} \cdot (W_r - W_l) = \sum_{i=1}^{n} R_i \left(w_{r,i} - w_{l,i}\right)$$

that propagate with speed λ_i

$\Rightarrow$ jump over shock i equals $R_i \left(w_{r,i} - w_{l,i}\right) = R_i \left(\mathsf{L} \left(U_r - U_l\right)\right)_i$

1.4.2 The linear wave equation

Classification of linear scalar 2nd order PDEs with two independent variables

$$\boxed{a\, u_{tt} + b\, u_{xt} + c\, u_{xx} + d\, u_t + e\, u_x = 0}$$

for $u(x,t)$, with $u_x = \frac{\partial u}{\partial x}$, $u_t = \frac{\partial u}{\partial t}$, $a \neq 0$

- write as first order system

define $v = u_t$ and $w = u_x$

$$\Rightarrow u_{tt} = v_t,\, u_{xt} = v_x,\, u_{xx} = w_x$$

then $v_t = u_{tt} = -(b\, u_{xt} + c\, u_{xx} + d\, u_t + e\, u_x)/a$

$$= -(b\, v_x + c\, w_x + d\, v + e\, w)/a$$

$$w_t = u_{xt} = v_x$$

$$\Rightarrow \frac{\partial U}{\partial t} + \mathsf{A} \cdot \frac{\partial U}{\partial x} + \mathsf{B} \cdot U = 0$$

with $U = \begin{bmatrix} v \\ w \end{bmatrix}$, $\mathsf{A} = \begin{bmatrix} b/a & c/a \\ -1 & 0 \end{bmatrix}$, $\mathsf{B} = \begin{bmatrix} d/a & e/a \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ eigenstructure A: $\quad \lambda = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow$ the 2nd order equation is

- hyperbolic $\Leftrightarrow b^2 - 4ac > 0 \Leftrightarrow \lambda$ real, complete eigenvector set
- parabolic $\Leftrightarrow b^2 - 4ac = 0 \Leftrightarrow \lambda$ real, but incomplete eigenvector set
- elliptic $\Leftrightarrow b^2 - 4ac < 0 \Leftrightarrow \lambda$ complex

The wave equation

$$\boxed{u_{tt} - \alpha^2 \, u_{xx} = 0}$$

- $a = 1, c = -\alpha^2 \quad \Rightarrow \quad \mathsf{A} = \begin{bmatrix} 0 & -\alpha^2 \\ -1 & 0 \end{bmatrix}$

- $\lambda_1 = \alpha, \lambda_2 = -\alpha$ real

  $\mathsf{R} = \left[\begin{array}{c|c} 1 & -1/\alpha \\ 1 & 1/\alpha \end{array}\right] \qquad \mathsf{L} = \mathsf{R}^{-1} = \left[\begin{array}{cc} 0.5 & -0.5\,\alpha \\ \hline 0.5 & 0.5\,\alpha \end{array}\right]$

  $\Rightarrow$ hyperbolic

- remark: $(\frac{\partial}{\partial t} + \alpha\,\frac{\partial}{\partial x})\,(\frac{\partial}{\partial t} - \alpha\,\frac{\partial}{\partial x})u = 0$

  $\Rightarrow u(x,t) = f(x + \alpha\, t) + g(x - \alpha\, t)$

  (initial conditions: $u(x,0), u_t(x,0)$)

The heat equation

$$\boxed{u_t - u_{xx} = 0}$$

$$\boxed{a^{‘}\, u_{xx} + b^{‘}\, u_{xt} + c^{‘}\, u_{tt} + d^{‘}\, u_x + e^{‘}\, u_t = 0}$$

- $a^{‘} = -1, e^{‘} = 1 \quad \Rightarrow \quad \mathsf{A} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$

- $\lambda_{1,2} = 0$ real

  $R_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, but no linearly independent R_2

  $\Rightarrow$ parabolic ($b^{‘2} - 4a^{‘}c^{‘} = 0$)

The Poisson equation

$$\boxed{u_{xx} + u_{yy} = 0}$$

$$\boxed{a^{‘}\, u_{xx} + b^{‘}\, u_{xy} + c^{‘}\, u_{yy} + d^{‘}\, u_x + e^{‘}\, u_y = 0}$$

- $a^{‘} = 1, c^{‘} = 1 \quad \Rightarrow \quad \mathsf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

- $\lambda_{1,2} = \pm\sqrt{-4} = \pm 2\, i$ complex

  $\Rightarrow$ elliptic ($b^{‘2} - 4a^{‘}c^{‘} < 0$)

1.4.3 Nonlinear systems

Nonlinear hyperbolic systems

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

$$\Downarrow$$

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial U}\frac{\partial U}{\partial x} = 0$$

$$\Downarrow \frac{\partial F(U)}{\partial U} = \mathsf{A}(U)$$

$$\boxed{\frac{\partial U}{\partial t} + \mathsf{A}(U) \cdot \frac{\partial U}{\partial x} = 0 \quad (1)}$$

with $U(x,t) = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in R^n$

and $F(U) = \begin{bmatrix} f_1(U) \\ \vdots \\ f_n(U) \end{bmatrix}$

with Jacobian matrix $\mathsf{A}(U) = \dfrac{\partial F(U)}{\partial U} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \ldots & \frac{\partial f_1}{\partial u_n} \\ \frac{\partial f_2}{\partial u_1} & \ldots & & \\ \vdots & & & \\ \frac{\partial f_n}{\partial u_1} & \ldots & \ldots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}$

$$\boxed{\frac{\partial U}{\partial t} + \mathsf{A}(U) \cdot \frac{\partial U}{\partial x} = 0 \quad (1)}$$

(1) is a hyperbolic system of equations

$\Updownarrow$

$\mathsf{A}(U)$ has n real eigenvalues and a complete set of eigenvectors $\forall U$

$\Updownarrow$

the system has n real characteristic curves

$\Rightarrow \mathsf{A}(U) = \mathsf{R}(U) \cdot \Lambda(U) \cdot \mathsf{L}(U)$

Characteristic variables

$$\frac{\partial U}{\partial t} + \mathsf{R}(U) \cdot \Lambda(U) \cdot \mathsf{L}(U) \cdot \frac{\partial U}{\partial x} = 0$$

$$\frac{\mathsf{L}(U) \cdot \partial U}{\partial t} + \Lambda(U) \cdot \frac{\mathsf{L}(U) \cdot \partial U}{\partial x} = 0$$

define (formally) $\boxed{\partial W = \mathsf{L}(U) \cdot \partial U : \quad \text{characteristic variables } \partial W}$

$$\boxed{\frac{\partial W}{\partial t} + \Lambda(U) \cdot \frac{\partial W}{\partial x} = 0}$$

- n wave modes with wave speeds λ_i
- the definition of the characteristic variables ∂W only on the differential, local level

  $\Rightarrow$ the equations (and thus the wave modes) do generally not globally decouple due to the nonlinearity (see below)

Characteristics and Riemann Invariants

$$\frac{\mathsf{L}(U)\cdot\partial U}{\partial t}+\Lambda(U)\cdot\frac{\mathsf{L}(U)\cdot\partial U}{\partial x}=0$$

$\forall i$: (n characteristic fields)

- $\boxed{x_i(t) : \dfrac{\partial x_i(t)}{\partial t}=\lambda_i(U)}$: ith characteristic

- if $\exists\, w_i(U)\ :\ \partial w_i(U)\approx L_i\cdot\partial U$

$\Rightarrow \boxed{\dfrac{\mathrm{d}w_i(x_i(t),t)}{\mathrm{d}t}\equiv 0}$: w_i Riemann Invariant (RI) on characteristic x_i

$\Rightarrow$ n different (nonlinear) waves, n characteristic curves

$\Rightarrow$ less than n Riemann Invariants (wave information for wave i only locally determined)

$\Rightarrow$ characteristics are not straight lines !!

Shocks

- Rankine-Hugoniot relation:

$$\boxed{s(U_r - U_l) = F(U_r) - F(U_l)}$$

  - nonlinear
  - small-amplitude shocks: linearize

  $s(U_r - U_l) = \mathsf{A}(U_l)\,(U_r - U_l)$

  given U_l, there are n small-amplitude shocks, with speeds λ_i, and $U_r - U_l = R_i$

  - in general: n shock curves in phase plane (see Leveque)

The Riemann problem

- the initial jump $U_r - U_l$ splits up in n waves
- in general: some shocks (compressive), some rarefactions (like Burgers)
- (see Leveque)

1.4.4 The Euler equations

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0}$$

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho\, e \end{bmatrix} \qquad F(U) = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p \\ \rho v_x v_y \\ (\rho\, e + p) v_x \end{bmatrix}$$

- Euler = dissipationless hydrodynamics or gasdynamics, compressible

- 4 equations in 4 unknowns: ρ, v_x, v_y, p

  with $\rho\, e = \rho\, \epsilon(\rho, p) + \frac{1}{2}\rho v^2 = E$

  e: specific total energy (J/kg)

  $E = \rho\, e$: volumetric total energy (J/m^3), conserved quantity

  $\epsilon(\rho, p)$: specific internal energy (J/kg), equation of state (EOS)

- perfect (ideal) gas EOS: $\epsilon(\rho, p) = \dfrac{p}{\rho\,(\gamma - 1)}$

  $$\Rightarrow \rho\, e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2$$

- $\gamma = \dfrac{f+2}{f}$ adiabatic constant, with f =degrees of freedom

  $\gamma = 7/5 = 1.4$ for air: diatomic gas: $f = 3$ translational + 2 rotational

  $\gamma = 5/3 = 1.667$ for hydrogen: monatomic gas: $f = 3$ translational

Conservative and primitive variables

- vector of conservative variables:

$$U = \begin{bmatrix} \rho \\ m_x \\ m_y \\ E \end{bmatrix} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho\, e = \frac{p}{(\gamma-1)} + \frac{1}{2}\rho v^2 \end{bmatrix} \text{ in } \frac{\partial U}{\partial t} + \mathsf{A}(U)_U \cdot \frac{\partial U}{\partial x} = 0$$

- vector of primitive variables: $V = \begin{bmatrix} \rho \\ v_x \\ v_y \\ p \end{bmatrix}$

- transformation: $\partial U = \dfrac{\partial U}{\partial V} \cdot \partial V$

$$\frac{\partial U}{\partial V} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ v_x & \rho & 0 & 0 \\ v_y & 0 & 1 & 0 \\ v^2/2 & \rho\, v_x & \rho\, v_y & 1/(\gamma - 1) \end{bmatrix}$$

$$\frac{\partial V}{\partial U} = \frac{\partial U}{\partial V}^{-1}$$

$$\Rightarrow \frac{\partial U}{\partial V} \cdot \frac{\partial V}{\partial t} + \mathsf{A}(U)_U \cdot \frac{\partial U}{\partial V} \cdot \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial U} \cdot \mathsf{A}(U)_U \cdot \frac{\partial U}{\partial V} \cdot \frac{\partial V}{\partial x} = 0$$

define $\mathsf{A}(V)_V = \dfrac{\partial V}{\partial U} \cdot \mathsf{A}(U)_U \cdot \dfrac{\partial U}{\partial V}$

with property $\lambda(\mathsf{A}(V)_V) = \lambda(\mathsf{A}(U)_U)$ (similarity transformation)

$$\Rightarrow \frac{\partial V}{\partial t} + \mathsf{A}(V)_V \cdot \frac{\partial V}{\partial x} = 0$$

Hyperbolic system

- $\dfrac{\partial V}{\partial t} + \mathsf{A}(V)_V \cdot \dfrac{\partial V}{\partial x} = 0$

  with $V = \begin{bmatrix} \rho \\ v_x \\ v_y \\ p \end{bmatrix}$ and $\mathsf{A}_V = \begin{bmatrix} v_x & \rho & 0 & 0 \\ 0 & v_x & 0 & 1/\rho \\ 0 & 0 & v_x & 0 \\ 0 & c^2\,\rho & 0 & v_x \end{bmatrix}$

  $c = \sqrt{\dfrac{\gamma p}{\rho}}$

  $\Rightarrow$

  $\lambda_1 = v_x$: entropy wave

  $\lambda_2 = v_x$: shear wave

  $\lambda_3 = v_x + c$: sound wave, right traveling

  $\lambda_4 = v_x - c$: sound wave, left traveling

$$\Rightarrow \mathsf{R} = \left[\begin{array}{c|c|c|c} 1 & 0 & \rho & \rho \\ 0 & 0 & c & -c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho\, c^2 & \rho\, c^2 \end{array}\right]$$

$$\Rightarrow \mathsf{L} = \left[\begin{array}{cccc} 1 & 0 & 0 & -1/c^2 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1/(2c) & 0 & 1/(2\rho c^2) \\ \hline 0 & -1/(2c) & 0 & 1/(2\rho c^2) \end{array}\right]$$

$\Rightarrow$ hyperbolic system

Characteristics and Riemann Invariants

$$\frac{\partial V}{\partial t} + \mathsf{R}(V) \cdot \Lambda(V) \cdot \mathsf{L}(V) \cdot \frac{\partial V}{\partial x} = 0$$

$$\frac{\mathsf{L}(V) \cdot \partial V}{\partial t} + \Lambda(V) \cdot \frac{\mathsf{L}(V) \cdot \partial V}{\partial x} = 0$$

$\forall i$: (n characteristic fields)

- $\boxed{x_i(t) : \frac{\partial x_i(t)}{\partial t} = \lambda_i(V)}$: ith characteristic

- if $\exists\, w_i(V) \ : \ L_i(V) \cdot \partial V = \alpha(V)\, \partial w_i(V)$

$$\Rightarrow L_i(V) \cdot \frac{\partial V}{\partial t} + \lambda_i(V) L_i(V) \cdot \frac{\partial V}{\partial x} = 0$$

$$\alpha(V)\, \frac{\partial w_i(V)}{\partial t} + \lambda_i(V) \alpha(V)\, \frac{\partial w_i(V)}{\partial x} = 0$$

$$\Rightarrow \frac{\mathrm{d}w_i(x_i(t),t)}{\mathrm{d}t} = 0 \ : \quad w_i \text{ Riemann Invariant (RI) on characteristic } x_i$$

- $i = 1$: find $w_1(V)$ such that

$$L_1(V) \cdot \partial V = \partial\rho - \frac{1}{c^2}\partial p = \alpha(V)\,\partial w_1(V)$$

choose $w_1 = s = \dfrac{p}{\rho^\gamma}$

$$\Rightarrow \partial w_1 = -\frac{\gamma p}{\rho^{\gamma+1}}\partial\rho + \frac{\partial p}{\rho^\gamma} = \frac{-c^2}{\rho^\gamma}\left(\partial\rho - \frac{1}{c^2}\partial p\right)$$

$$\Rightarrow L_1(V) \cdot \partial V = \partial\rho - \frac{1}{c^2}\partial p = \alpha(V)\,\partial w_1(V)$$

with $\alpha(V) = \dfrac{-\rho^\gamma}{c^2}$ and $w_1 = s = \dfrac{p}{\rho^\gamma}$

$\Rightarrow$ s is a Riemann Invariant on $x_1(t)$ with $\dfrac{\partial x_1(t)}{\partial t} = v_x$

the entropy of a fluid element is conserved on its path

- $i = 2$: find $w_2(V)$ such that

  $$L_2(V) \cdot \partial V = \partial v_y = \alpha(V)\, \partial w_2(V)$$

  choose $w_2 = v_y$ and $\alpha(V) = 1$ (trivial)

$\Rightarrow$ v_y is a Riemann Invariant on $x_2(t)$ with $\dfrac{\partial x_1(t)}{\partial t} = v_x$

- $i = 3$: find $w_3(V)$ such that

  $$L_3(V) \cdot \partial V = \frac{1}{2c}\partial v_x + \frac{1}{2\rho c^2}\partial p = \alpha(V)\, \partial w_3(V)$$

  no solution, RI does not exist

- $i = 4$: no RI

Linear waves

$$\frac{\partial V}{\partial t} + A_V(V) \cdot \frac{\partial V}{\partial x} = 0$$

- assume $V(x,t) = V_0 + V_1(x,t)$

  with V_0 constant background, $V_1(x,t)$ small perturbation

- linearize: $A_V(V) \approx A_V(V_0) \;\Rightarrow\; \dfrac{\partial V_1}{\partial t} + A_V(V_0)\,\dfrac{\partial V_1}{\partial x} \approx 0$

  $\mathsf{A}_V(V_0) = \mathsf{R}(V_0) \cdot \Lambda(V_0) \cdot \mathsf{L}(V_0)$

- define $W_1 = \mathsf{L}(V_0) \cdot V_1$: characteristic variables W_1

$$\Rightarrow \frac{\partial W_1}{\partial t} + \Lambda(V_0) \cdot \frac{\partial W_1}{\partial x} = 0$$

$\Rightarrow$ n scalar linear advection equations

$$\mathsf{R} = \left[\begin{array}{c|c|c|c} 1 & 0 & \rho & \rho \\ 0 & 0 & c & -c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho\, c^2 & \rho\, c^2 \end{array}\right] \text{ and } \mathsf{L} = \left[\begin{array}{cccc} 1 & 0 & 0 & -1/c^2 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1/(2c) & 0 & 1/(2\rho c^2) \\ \hline 0 & -1/(2c) & 0 & 1/(2\rho c^2) \end{array}\right] \text{ with } V = \left[\begin{array}{c} \rho \\ v_x \\ v_y \\ p \end{array}\right]$$

$\lambda_1 = v_x$: entropy wave $\quad$ $\lambda_3 = v_x + c$: sound wave, right traveling

$\lambda_2 = v_x$: shear wave $\quad$ $\lambda_4 = v_x - c$: sound wave, left traveling

- $V_1 = \mathsf{R}(V_0) \cdot W_1$: properties of Euler waves
  - entropy: $\Delta\rho$
  - shear: Δv_y
  - sound: $\Delta\rho, \pm\Delta v_x, \Delta p$
- $W_1 = \mathsf{L}(V_0) \cdot V_1$: waves generated by perturbation of the primitive variables
- large-amplitude = nonlinear

  $\Rightarrow$ waves interact, steepen into shocks, rarefactions, wave coupling, . . .

1.4.5 The MHD equations

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0}$$

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ \rho\, e \end{bmatrix} \qquad F(U) = \begin{bmatrix} \rho\, v_x \\ \rho\, v_x^2 + p + B^2/2 - B_x^2 \\ \rho\, v_x v_y - B_x B_y \\ \rho\, v_x v_z - B_x B_z \\ 0 \\ B_y v_x - B_x v_y \\ B_z v_x - B_x v_z \\ (\rho\, e + p + B^2/2) v_x - B_x\, (\vec{v} \cdot \vec{B}) \end{bmatrix}$$

- MHD = dissipationless magnetohydrodynamics or magnetogasdynamics, compressible

- 8 equations in 8 unknowns: $\rho, v_x, v_y, v_z, B_x, B_y, B_z, p$

  with $\rho\, e = \rho\, \epsilon(\rho, p) + \frac{1}{2}\rho v^2 + \frac{1}{2}B^2 = E$

  e: specific total energy (J/kg)

  $E = \rho\, e$: volumetric total energy (J/m^3), conserved quantity

  $\epsilon(\rho, p)$: specific internal energy (J/kg), equation of state (EOS)

- perfect (ideal) gas EOS: $\epsilon(\rho, p) = \dfrac{p}{\rho\,(\gamma - 1)}$

  $$\Rightarrow \rho\, e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2 + \frac{1}{2}B^2$$

- $\gamma = 5/3 = 1.667$ for monatomic plasmas

- $\nabla \cdot \vec{B} = \dfrac{\partial B_x(x,t)}{\partial x} \equiv 0$ is a constraint

  $\Rightarrow B_x$ constant in space (in 1D) = initial condition

  then it follows from the equations that $\dfrac{\partial B_x(x,t)}{\partial t} \equiv 0$: the constraint is preserved by the evolution equations

  we can leave B_x out as a variable in 1D

$\Rightarrow$ in general: the $\nabla \cdot \vec{B}$ constraint only needs to be specified as an initial condition (on the analytical level, numerically: more tricky, see later!)

‘Physical’ form of the MHD equations

- mass continuity equation: $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$
- Newton’s law of motion: $\rho \dfrac{d\vec{v}}{dt} = -\nabla p + (\nabla \times \vec{B}) \times \vec{B})$

  Ampère’s law $\vec{J} = \nabla \times \vec{B}$ (units such that $\mu = 1$)
- vector induction equation: $\dfrac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}$

  derives directly from one of Maxwell’s equations, namely, Faraday’s law of induction, and from Ohm’s law for an ideal plasma: $\vec{E} = -\vec{v} \times \vec{B}$ with $\vec{E}$ the electric field
- evolution of the pressure (perfect (ideal) gas): $\dfrac{\partial p}{\partial t} + (\vec{v} \cdot \nabla)p + \gamma p \nabla \cdot \vec{v} = 0$

  we can rewrite as $\dfrac{ds}{dt} = 0$ with the (specific) entropy $s = \dfrac{p}{\rho^\gamma}$

- magnetic monopoles do not exist in nature: $\nabla \cdot \vec{B} = 0$
- conservation of magnetic flux in time: $\Phi = \int_S \vec{A} \cdot \vec{n} dS$

$$\frac{d\Phi}{dt} = \int_S (\frac{\partial \vec{A}}{\partial t} + \vec{u}(\nabla \cdot \vec{A}) + \nabla \times (\vec{A} \times \vec{u})) \cdot \vec{n} dS$$

total time derivative of the flux Φ_B of the magnetic field $\vec{B}$ through a surface moving with the plasma speed $\vec{v}$

$$\frac{d\Phi_B}{dt} = \int_S (\frac{\partial \vec{B}}{\partial t} + \vec{v}(\nabla \cdot \vec{B}) + \nabla \times (\vec{B} \times \vec{v})) \cdot \vec{n} dS$$

using the induction equation: $\boxed{\frac{d\Phi_B}{dt} = 0}$

the magnetic field is frozen into the plasma for ideal MHD, or fluid elements which reside on a common field line at one time, remain on this magnetic field line at all times

Conservative and primitive variables

- vector of conservative variables:

$$U = \begin{bmatrix} \rho \\ m_x \\ m_y \\ m_z \\ B_x \\ B_y \\ B_z \\ E \end{bmatrix} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ \rho\, e = \frac{p}{(\gamma-1)} + \frac{1}{2}\rho v^2 + \frac{1}{2}B^2 \end{bmatrix} \text{ in } \frac{\partial U}{\partial t} + \mathsf{A}(U)_U \cdot \frac{\partial U}{\partial x} = 0$$

- vector of primitive variables: $V = \begin{bmatrix} \rho \\ v_x \\ v_y \\ v_z \\ B_x \\ B_y \\ B_z \\ p \end{bmatrix}$

- transformation: $\partial U = \dfrac{\partial U}{\partial V} \cdot \partial V$

$$\Rightarrow \frac{\partial V}{\partial t} + \mathsf{A}(V)_V \cdot \frac{\partial V}{\partial x} = 0$$

- Hyperbolic system:

$$\text{with } \mathsf{A}_V = \begin{bmatrix} v_x & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_x & 0 & 0 & 0 & B_y/\rho & B_z/\rho & 1/\rho \\ 0 & 0 & v_x & 0 & 0 & -B_x/\rho & 0 & 0 \\ 0 & 0 & 0 & v_x & 0 & 0 & -B_x/\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_y & -B_x & 0 & 0 & v_x & 0 & 0 \\ 0 & B_z & 0 & -B_x & 0 & 0 & v_x & 0 \\ 0 & c^2\rho & 0 & 0 & 0 & 0 & 0 & v_x \end{bmatrix} \qquad \left(c = \sqrt{\frac{\gamma p}{\rho}}\right)$$

$\lambda_1 = v_x + c_{fx}$: fast wave, right

$\lambda_2 = v_x - c_{fx}$: fast wave, left

$\lambda_3 = v_x + c_{Ax}$: Alfvén wave, right

$\lambda_4 = v_x - c_{Ax}$: Alfvén wave, left

$\lambda_5 = v_x + c_{sx}$: slow wave, right

$\lambda_6 = v_x - c_{sx}$: slow wave, left

$\lambda_7 = v_x$: entropy wave

$\lambda_8 = 0$: not Galilean invariant!!

with

$$c_{fx}^2 = \frac{1}{2}\left(\frac{\gamma p + B^2}{\rho} + \sqrt{\left(\frac{\gamma p + B^2}{\rho}\right)^2 - 4\frac{\gamma p B_x^2}{\rho^2}}\right)$$

$$c_{Ax}^2 = \frac{B_x^2}{\rho}$$

$$c_{sx}^2 = \frac{1}{2}\left(\frac{\gamma p + B^2}{\rho} - \sqrt{\left(\frac{\gamma p + B^2}{\rho}\right)^2 - 4\frac{\gamma p B_x^2}{\rho^2}}\right)$$

wave speeds anisotropic!! (depending on angle between propagation direction x and local magnetic field $\vec{B}$)

- a way to restore Galilean invariance: add a source term S

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = S}$$

$$U = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ B_x \\ B_y \\ B_z \\ \rho\, e \end{bmatrix} \qquad F(U) = \begin{bmatrix} \rho\, v_x \\ \rho\, v_x^2 + p + B^2/2 - B_x^2 \\ \rho\, v_x v_y - B_x B_y \\ \rho\, v_x v_z - B_x B_z \\ 0 \\ B_y v_x - B_x v_y \\ B_z v_x - B_x v_z \\ (\rho\, e + p + B^2/2) v_x - B_x\, (\vec{v} \cdot \vec{B}) \end{bmatrix}$$

$$S = -\begin{bmatrix} 0 \\ B_x \\ B_y \\ B_z \\ v_x \\ v_y \\ v_z \\ \vec{v} \cdot \vec{B} \end{bmatrix} \nabla \cdot \vec{B}$$

$\Rightarrow$ source term S has no effect if $\nabla \cdot \vec{B} \equiv 0$ (which should be fulfilled on the analytical level), but regularizes the equations (Galilean invariant, complete set of eigenvectors)

useful when

- $\nabla \cdot \vec{B} = 0$ is not exactly satisfied, e.g. numerically
- entropy symmetrization of the MHD equations

- Hyperbolic system:

$$A`_V = \begin{bmatrix} v_x & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_x & 0 & 0 & 0 & B_y/\rho & B_z/\rho & 1/\rho \\ 0 & 0 & v_x & 0 & 0 & -B_x/\rho & 0 & 0 \\ 0 & 0 & 0 & v_x & 0 & 0 & -B_x/\rho & 0 \\ 0 & 0 & 0 & 0 & v_x & 0 & 0 & 0 \\ 0 & B_y & -B_x & 0 & 0 & v_x & 0 & 0 \\ 0 & B_z & 0 & -B_x & 0 & 0 & v_x & 0 \\ 0 & c^2\rho & 0 & 0 & 0 & 0 & 0 & v_x \end{bmatrix}$$

$\lambda_i, \; i = 1..7$ remain unchanged

$\lambda_8 = v_x$: Galilean invariant!!

complete set of eigenvectors

$\Rightarrow$ use this form of the equations from now on

$\Rightarrow$ hyperbolic system, but: non-strictly hyperbolic, because wave speeds can coincide

- define $\mu_f = \dfrac{c_f}{c_f^2 - c_A^2}$ and $\mu_s = \dfrac{c_s}{c_s^2 - c_A^2}$

$$\Rightarrow \mathsf{R}_{1-4} = \left[\begin{array}{c|c|c|c} \rho & \rho & 0 & 0 \\ \mu_f\,(c_f^2 - c_A^2) & -\mu_f\,(c_f^2 - c_A^2) & 0 & 0 \\ -\mu_f\,(B_x B_y/\rho) & \mu_f\,(B_x B_y/\rho) & -B_z & B_z \\ -\mu_f\,(B_x B_z/\rho) & \mu_f\,(B_x B_z/\rho) & B_y & -B_y \\ 0 & 0 & 0 & 0 \\ \mu_f c_f\,(B_y/\sqrt{\rho}) & \mu_f c_f\,(B_y/\sqrt{\rho}) & -\sqrt{\rho}B_z & -\sqrt{\rho}B_z \\ \mu_f c_f\,(B_z/\sqrt{\rho}) & \mu_f c_f\,(B_z/\sqrt{\rho}) & -\sqrt{\rho}B_y & -\sqrt{\rho}B_y \\ \rho c^2 & \rho c^2 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \mathsf{R}_{5-8} = \left[\begin{array}{c|c|c|c} \rho & \rho & 1 & 0 \\ \mu_s\,(c_s^2 - c_A^2) & \mu_s(c_s^2 - c_A^2) & 0 & 0 \\ -\mu_s\,(B_x B_y/\rho) & \mu_s(B_x B_y/\rho) & 0 & 0 \\ -\mu_s\,(B_x B_z/\rho) & \mu_s(B_x B_z/\rho) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \mu_s c_s\,(B_y/\sqrt{\rho}) & \mu_s c_s\,(B_y/\sqrt{\rho}) & 0 & 0 \\ \mu_s c_s\,(B_z/\sqrt{\rho}) & \mu_s c_s\,(B_z/\sqrt{\rho}) & 0 & 0 \\ \rho c^2 & \rho c^2 & 0 & 0 \end{array}\right]$$

Characteristics and Riemann Invariants

$\forall i$: (n characteristic fields)

- $\boxed{x_i(t) : \dfrac{\partial x_i(t)}{\partial t} = \lambda_i(V)}$: ith characteristic

- if $\exists\, w_i(V)$: $L_i(V) \cdot \partial V = \alpha(V)\, \partial w_i(V)$

$\Rightarrow \dfrac{\mathrm{d} w_i(x_i(t), t)}{\mathrm{d}t} = 0$: w_i Riemann Invariant (RI) on characteristic x_i

- $i = 7$: find $w_7(V)$ such that

$$L_7(V) \cdot \partial V = \partial \rho - \frac{1}{c^2} \partial p = \alpha(V)\, \partial w_7(V)$$

$\Rightarrow$ s is a Riemann Invariant on $x_7(t)$ with $\dfrac{\partial x_7(t)}{\partial t} = v_x$

the entropy of a fluid element is conserved on its path

- $i = 8$: find $w_8(V)$ such that

  $$L_8(V) \cdot \partial V = \partial B_x = \alpha(V)\,\partial w_8(V)$$

$\Rightarrow B_x$ is a Riemann Invariant on $x_8(t)$ with $\dfrac{\partial x_8(t)}{\partial t} = v_x$

- $i = 1, 2, 3, 4, 5, 6$: find $w_i(V)$ such that

  $$L_i(V) \cdot \partial V = \alpha(V)\,\partial w_i(V)$$

  no solution, RI does not exist

$$
\mathsf{R}_{1-4} = \left[\begin{array}{c|c|c|c}
\rho & \rho & 0 & 0 \\
\mu_f\,(c_f^2 - c_A^2) & -\mu_f\,(c_f^2 - c_A^2) & 0 & 0 \\
-\mu_f\,(B_x B_y/\rho) & \mu_f\,(B_x B_y/\rho) & 0 & 0 \\
0 & 0 & B_y & -B_y \\
0 & 0 & 0 & 0 \\
\mu_f c_f\,(B_y/\sqrt{\rho}) & \mu_f c_f\,(B_y/\sqrt{\rho}) & 0 & 0 \\
0 & 0 & -\sqrt{\rho}B_y & -\sqrt{\rho}B_y \\
\rho c^2 & \rho c^2 & 0 & 0
\end{array}\right]
$$

$$
\mathsf{R}_{5-8} = \left[\begin{array}{c|c|c|c}
\rho & \rho & 1 & 0 \\
\mu_s\,(c_s^2 - c_A^2) & \mu_s(c_s^2 - c_A^2) & 0 & 0 \\
-\mu_s\,(B_x B_y/\rho) & \mu_s(B_x B_y/\rho) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\mu_s c_s\,(B_y/\sqrt{\rho}) & \mu_s c_s\,(B_y/\sqrt{\rho}) & 0 & 0 \\
0 & 0 & 0 & 0 \\
\rho c^2 & \rho c^2 & 0 & 0
\end{array}\right]
$$

- $V_1 = \mathrm{R}(V_0) \cdot W_1$: properties of MHD waves, take $B_{z0} = 0$

  - fast (1,2): $\Delta\rho, \pm\Delta v_x, \pm\Delta v_y, \Delta B_y, \Delta p$

    compressive, Δp in phase with $\Delta B (= \Delta B_y)$

  - slow (5,6): $\Delta\rho, \pm\Delta v_x, \pm\Delta v_y, \Delta B_y, \Delta p$

    compressive, Δp in anti-phase with $\Delta B (= \Delta B_y)$

  - Alfvén (3,4): $\pm\Delta v_z, \Delta B_z$

    non-compressive, non-planar

  - entropy (7): $\Delta\rho$

  - $\nabla \cdot \vec{B}$ wave (8): $\Delta B_x \sim \nabla \cdot \vec{B}$

- large-amplitude = nonlinear

  $\Rightarrow$ waves interact, steepen into shocks, rarefactions, wave coupling, . . .