Hyperbolic theory of the "shallow water" magnetohydrodynamics equations

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(Received 27 December 2000; accepted 19 April 2001)

Recently the shallow water magnetohydrodynamic (SMHD) equations have been proposed for describing the dynamics of nearly incompressible conducting fluids for which the evolution is nearly two-dimensional (2D) with magnetohydrostatic equilibrium in the third direction. In the present paper the properties of the SMHD equations as a nonlinear system of hyperbolic conservation laws are described. Characteristics and Riemann invariants are studied for 1D unsteady and 2D steady flow. Simple wave solutions are derived, and the nonlinear character of the wave modes is investigated. The $\nabla\cdot(h\,\mathbf{B})=0$ constraint and its role in obtaining a regularized Galilean invariant conservation law form of the SMHD equations is discussed. Solutions of the Rankine–Hugoniot relations are classified and their properties are investigated. The derived properties of the wave modes are illustrated by 1D numerical simulation results of SMHD Riemann problems. A Roe-type linearization of the SMHD equations is given which can serve as a building block for accurate shock-capturing numerical schemes. The SMHD equations are presently being used in the study of the dynamics of layers in the solar interior, but they may also be applicable to problems involving the free surface flow of conducting fluids in laboratory and industrial environments. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1379045]

I. INTRODUCTION

Recently, the equations of "shallow water" magnetohydrodynamics (SMHD) have been proposed by Gilman[1] in the context of the study of the global dynamics of the solar tachocline, which is a thin layer in the solar interior at the base of the solar convection zone. The SMHD system is a magnetohydrodynamic (MHD) analog to the classical shallow water equations,[2–4] and describes the dynamics of nearly incompressible conducting fluids for which the evolution is nearly two-dimensional (2D) with magnetohydrostatic equilibrium in the third direction. Gilman[1] gives a brief phenomenological derivation of the SMHD equations, and sketches some of their general properties. To his and our best knowledge these equations had not been studied before. Presently the use of the SMHD equations in various applications is being explored. Dikpati and Gilman[5] have studied the prolateness of the solar tachocline using the SMHD equations.

In the present paper we derive the basic properties of SMHD as a nonlinear system of hyperbolic conservation laws. The SMHD system indeed belongs to the class of symmetric hyperbolic systems,[6,7] like the Euler equations and the full MHD equations.[6,8–11] An extensive general theory exists for symmetric hyperbolic systems, which describes their characteristic decomposition in wave modes, and the properties of discontinuous solutions including shocks. We apply this theory to the recently proposed SMHD equations.

Many modern numerical schemes for hyperbolic conservation laws, including, e.g., the Roe scheme,[34] heavily rely on decomposition of the dynamics into characteristic wave modes,[3,4,12] and the theory derived in this paper can serve as the basis for formulating schemes of this type for the SMHD equations. In the Appendix of the present paper a Roe-type linearization[34] of the SMHD equations is given which can serve as a building block for accurate shock-capturing numerical schemes.

The derived properties of the wave modes are illustrated by one-dimensional (1D) numerical simulation results of SMHD Riemann problems. In a rotating system some of the properties of the SMHD wave modes are changed as compared to the system without rotation. The properties of SMHD waves in a rotating system are discussed in Schecter *et al.*[13] Throughout the paper the SMHD system is compared to the full MHD system, and similarities and differences in the theoretical properties[6–11,14–18] are pointed out.

II. GOVERNING EQUATIONS

A. Primitive form of the SMHD equations

The equations of SMHD (Ref. 1) describing the temporal evolution of the primitive variables $\mathbf{V}=(h,v_x,v_y,B_x,B_y)^T$ are given by

$$\begin{aligned}
&\frac{\partial h}{\partial t}+\nabla\cdot(h\,\mathbf{v})=0,\\
&\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}-(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}+g\,\boldsymbol{\nabla}h=0,\\
&\frac{\partial \mathbf{B}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{B}-(\mathbf{B}\cdot\nabla)\mathbf{v}=0,\\
&\nabla\cdot(h\,\mathbf{B})=0.
\end{aligned}\tag{1}$$

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Here h is the height of the conducting fluid, v_x and v_y are the components of the fluid velocity vector $\mathbf{v}$, and B_x and B_y are the components of the magnetic field vector $\mathbf{B}$. These five state variables are functions of time t and spatial coordinates x and y. The parameter g is the magnitude of the gravitational acceleration. The evolution equations are supplemented with a divergence constraint $\nabla\cdot(h\mathbf{B})=0$. The evolution equations are such that this constraint remains automatically satisfied at later times if it is satisfied at an initial time.

As is shown by Gilman,[1] the SMHD equations (1) can be derived from the full MHD equations by integration in the z direction under the assumptions of incompressibility, 2D variation of the flow variables, and magnetohydrostatic equilibrium in the z direction,

$$\frac{\partial}{\partial z}\left(p+\frac{B^2}{2}\right)=\rho g,$$
$$p+\frac{B^2}{2}=\rho g z, \tag{2}$$
$$\int_0^h\left(p+\frac{B^2}{2}\right)dz=\rho g\frac{h^2}{2}.$$

The derivation of SMHD from MHD is analogous to the derivation of the shallow water equations from the Euler equations.[3]

B. Conservation law form of the SMHD equations

Using the $\nabla\cdot(h\mathbf{B})=0$ constraint, the SMHD equations (1) can be recast in the form of a system of conservation laws, as a function of the conserved variables $\mathbf{U}=(h,m_x,m_y,\phi_x,\phi_y)^T=(h,h\,v_x,h\,v_y,h\,B_x,h\,B_y)^T$,

$$\frac{\partial}{\partial t}\begin{bmatrix}h\\ h\mathbf{v}\\ h\mathbf{B}\end{bmatrix}+\nabla\cdot\begin{bmatrix}h\mathbf{v}\\ h\mathbf{v}\mathbf{v}-h\mathbf{B}\mathbf{B}+\mathsf{I}(gh^2/2)\\ h\mathbf{v}\mathbf{B}-h\mathbf{B}\mathbf{v}\end{bmatrix}=0. \tag{3}$$

Tensor notation is used here, with for instance the i,jth component of $\mathbf{Bv}$ given by $(\mathbf{Bv})_{i,j}=v_i B_j$. This form of the equations explicitly shows that the temporal variation of the conserved quantities is balanced by the divergence of a flux.

C. Application to physical plasmas

The SMHD equations have been proposed by Gilman[1] in the context of the study of the global dynamics of the solar tachocline. The tachocline is a thin layer of the solar interior, making the transition between the convection zone, which rotates in a differential way, and the uniformly rotating radiative interior.[19] Helioseismic observations show that the tachocline has a thickness between 2% and 5% of the solar radius. It is widely believed that a toroidal field of at least 10^5 G, generated by action of the solar dynamo, is present in this layer. This field may well be the source for sunspot fields. It is believed that the formation of sunspots in the 11-year solar cycle is related to the instability of this strong toroidal magnetic field in the differentially rotating tachocline. Gilman[1] and Schecter *et al.*[13] argue that the SMHD equations are a good model to study the stability of tachocline differential rotation and the propagation of waves in the tachocline layer. Indeed, the tachocline is believed to be stably stratified, and consequently its global disturbances should be nearly hydrostatic [Eq. (2)], since their horizontal scale is large compared with the thickness of the tachocline.[1]

The shallow water approximation has been exploited to great advantage for studying the global dynamics of the Earth's atmosphere and oceans.[20] Because of its stable stratification, global differential rotation imposed by the convection zone above, and its thin vertical extent, the solar tachocline is similar to both the ocean and the atmosphere.[1] In the solar tachocline, but also in many other astrophysical processes, magnetic fields play an important dynamical role in addition to the hydrodynamic processes that are active. Gilman[1] anticipates that the SMHD equations may play a role for tachocline-type astophysical problems that is similar to the role played by the shallow water equations for describing the Earth's atmosphere and oceans. As a first application, Dikpati and Gilman[5] have studied the prolateness of the solar tachocline using the SMHD equations, and the same authors are presently studying the magnetohydrodynamic instability of the tachocline differential rotation using the SMHD equations. Many astrophysical systems are rotating. The properties of SMHD waves in a rotating system are discussed in Schecter *et al.*[13]

The SMHD equations may also be applicable to problems involving the free surface flow[4] of conducting fluids in laboratory and industrial environments. It is easy to imagine how a bow shock may form in the flow of a supersonic conducting shallow fluid around a conducting obstacle, or how flow of a conducting shallow fluid around a corner may result in a stationary simple wave expansion similar to the Prandtl-Meyer fan of gas dynamics.[21] The SMHD equations and the hyperbolic theory to be described in this paper may apply to such experimental configurations. Describing such shallow free surface flows by the reduced set of SMHD equations has obvious advantages in terms of computational cost and model complexity over solving the full 3D MHD equations with a gravitational term in a coupled model with explicit description of the free surface interface.

III. HYPERBOLIC THEORY OF THE 1D TIME-DEPENDENT SMHD SYSTEM

A. SMHD as a hyperbolic system

A first order nonlinear system of n coupled equations in quasi-linear form with two independent variables x and t is described by

$$\frac{\partial\mathbf{V}}{\partial t}+\mathsf{A}(\mathbf{V})\cdot\frac{\partial\mathbf{V}}{\partial x}=0, \tag{4}$$

and is hyperbolic in state $\mathbf{V}$ if a complete eigenvector decomposition of $\mathsf{A}(\mathbf{V})$ exists and all the eigenvalues of $\mathsf{A}(\mathbf{V})$ are real. The eigenvalues determine the slopes of the characteristic curves in xt space.

We write Eq. (1), with $\partial/\partial y\equiv 0$, in this quasilinear form and obtain

$$\mathsf{A}=\begin{bmatrix} v_x & h & 0 & 0 & 0 \\ g & v_x & 0 & -B_x & 0 \\ 0 & 0 & v_x & 0 & -B_x \\ 0 & -B_x & 0 & v_x & 0 \\ 0 & 0 & -B_x & 0 & v_x \end{bmatrix}. \tag{5}$$

We remark that the conservation law form Eq. (3) can also be written in quasilinear form, with matrix A_C given by the Jacobian matrix of the conservative flux.

The eigenvalues of A in Eq. (5) are

$$\lambda_{1,2}=v_x \mp c_{ax}, \quad \lambda_{3,4}=v_x \mp c_{gx}, \quad \lambda_5=v_x, \tag{6}$$

with

$$c_{ax}=B_x, \tag{7}$$

$$c_{gx}=\sqrt{B_x^2+g\,h}. \tag{8}$$

The rows $\mathbf{L}_i$ of matrix

$$\mathsf{L}=\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ -g & \sqrt{B_x^2+g\,h} & 0 & B_x & 0 \\ -g & -\sqrt{B_x^2+g\,h} & 0 & B_x & 0 \\ B_x & 0 & 0 & h & 0 \end{bmatrix} \tag{9}$$

are left eigenvectors of A, and the columns $\mathbf{R}_i$ of matrix

$$\mathsf{R}=\begin{bmatrix} 0 & 0 & -h & -h & B_x \\ 0 & 0 & \sqrt{B_x^2+g\,h} & -\sqrt{B_x^2+g\,h} & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & B_x & B_x & g \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

are right eigenvectors. The eigenvectors are linearly independent. The eigenvectors $\mathbf{L}_i$ and $\mathbf{R}_i$ can be normalized such that $\mathsf{L}_n\cdot\mathsf{R}_n=\mathsf{I}$, with normalization factors $1/\sqrt{2}$ for $i=1,2$, $1/\sqrt{2\,(B_x^2+g\,h)}$ for $i=3,4$, and $1/\sqrt{B_x^2+g\,h}$ for $i=5$.

The SMHD system has thus five wave modes, and as all the eigenvalues of A are real and the eigenvectors are linearly independent, the system is hyperbolic for all states $\mathbf{V}$. The eigenvalues do not coincide, except in the void state $(h=0)$. Therefore the SMHD system is strictly hyperbolic, like the Euler and shallow water systems, but unlike the MHD system, which is non-strictly hyperbolic.[10,11,17,18,22]

B. Properties of the SMHD wave modes

Characteristic variables $\mathbf{W}$ can now be defined by

$$\partial\mathbf{W}=\mathsf{L}_n\cdot\partial\mathbf{V},$$
$$\partial\mathbf{V}=\mathsf{R}_n\cdot\partial\mathbf{W}. \tag{11}$$

The magnitudes of the perturbations in the characteristic variables $\mathbf{W}$ correspond to the strength of the respective wave modes. A given perturbation of the primitive variables $\mathbf{V}$ corresponds to an excitation of the characteristic wave modes $\mathbf{W}$ in a way which is determined by the matrix of left eigenvectors L. A given perturbation ∂w_i in characteristic wave mode i corresponds to a perturbation of the primitive vector $\mathbf{V}$ given by $\partial\mathbf{V}=\mathbf{R}_i\,\partial w_i$.

Wave modes 1 and 2 are called Alfvén waves due to their close resemblance to the MHD Alfvén waves. As can be seen from inspecting $\mathbf{R}_1$ and $\mathbf{R}_2$, the Alfvén waves carry only perturbations in v_y and B_y, the vector components perpendicular to the direction of propagation.

Wave modes 3 and 4 are gravity waves that are modified by magnetic forces, and are called magnetogravity waves. They carry perturbations in the longitudinal field components, and in the height.

Wave mode 5 is a spurious mode which cannot carry any wave strength. This can be seen as follows. The $\nabla\cdot(h\mathbf{B})=0$ constraint translates to the condition that $\partial(hB_x)=0$ for any wave perturbation. If mode 5 would have a perturbation strength ∂w_5, then corresponding perturbations in h and B_x would be $\partial h=B_x\,\partial w_5$ and $\partial B_x=g\,\partial w_5$. This would mean that $\partial(h\,B_x)=(g\,h+B_x^2)\,\partial w_5$, which means that the divergence constraint is satisfied only for vanishing wave strength, $\partial(h\,B_x)=0 \Leftrightarrow \partial w_5=0$. Wave mode 5 is thus a spurious mode. It can easily be seen that the other modes satisfy the divergence constraint. Mode 5 is completely analogous to the 8th MHD wave mode, the so-called divergence wave,[23,24] which in MHD too is forbidden to carry any wave strength due to the divergence constraint. As in MHD, the spurious wave can be eliminated by considering a reduced system, for instance by eliminating B_x as a variable using the fact that hB_x is constant in space and time for 1D flow. We did not do this in this section, because we want our formalism to be generally applicable to 2D flows and numerical schemes, in which there is no preferred direction.

In Eq. (6), c_{ax} and c_{gx} are the phase speeds of the SMHD Alfvén and magnetogravity waves, respectively, which both depend on the direction x—they are *anisotropic*. These wave speeds satisfy the property

$$c_{gx}\geqslant|c_{ax}|, \tag{12}$$

for any direction x.

C. Phase and group velocities

The group velocities $\mathbf{v}_{\rm gr}$ of the SMHD wave modes can be derived from the phase velocities $\mathbf{v}_{\rm ph}$ (which are oriented along the propagation direction $\mathbf{k}$) using the following relations:

$$v_{\rm ph}=\frac{\omega}{k},$$
$$\mathbf{v}_{\rm ph}=v_{\rm ph}\frac{\mathbf{k}}{k}, \tag{13}$$
$$\mathbf{v}_{\rm gr}=\left(\frac{\partial\omega}{\partial k_x},\frac{\partial\omega}{\partial k_y}\right).$$

Without loss of generality we derive expressions for the group velocities for the case that the magnetic field is aligned with the x-axis, $\mathbf{B}=(B_x,0)$. For the Alfvén mode this leads to

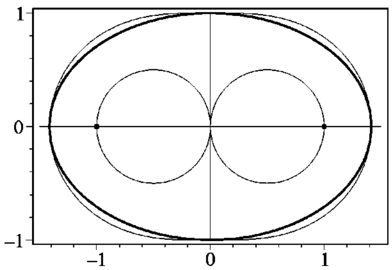

FIG. 1. Wave normal diagrams (with phase velocities, thin) and ray surface diagrams (with group velocities, thick) for the SMHD Alfvén and magnetogravity modes.

$$
\begin{aligned}
v_{\mathrm{ph},a} &= \mp\, B_x \frac{k_x}{k},\\
\mathbf{v}_{\mathrm{gr},a} &= (\mp\, B_x, 0),
\end{aligned}
\tag{14}
$$

while for the magnetogravity wave mode we find

$$
\begin{aligned}
v_{\mathrm{ph},g} &= \mp \sqrt{\left(B_x \frac{k_x}{k}\right)^2 + g\,h},\\
\mathbf{v}_{\mathrm{gr},g} &= \mp\left(\frac{k_x\,(B_x^2+g\,h)}{\sqrt{(B_x^2+g\,h)\,k_x^2+g\,h\,k_y^2}}, \frac{k_y\,g\,h}{\sqrt{(B_x^2+g\,h)\,k_x^2+g\,h\,k_y^2}}\right).
\end{aligned}
\tag{15}
$$

In Fig. 1 the phase velocity vectors (thin) and group velocity vectors (thick) of the Alfvén and magnetogravity modes are plotted for $\mathbf{k}$ going through the positions of a rotating unit vector (for $h=1$, $\mathbf{v}=0$, $B_x=1$, $B_y=0$, and $g=1$). The phase velocity diagrams (thin) are also called wave normal diagrams, and the group velocity diagrams (thick) are called ray surface diagrams. The Alfvén wave has wave normal diagrams of circular shape, and the ray surface diagrams degenerate into points in the direction of the magnetic field B_x. This is entirely analogous to the MHD Alfvén wave. For the magnetogravity mode the wave normal and ray surface diagrams have shapes similar to each other, analogous to the case of the fast MHD wave. For Euler sound waves the two diagrams would coincide because sound waves are isotropic. For propagation along the magnetic field the magnetogravity wave speeds equal $\sqrt{B_x^2+g\,h}$, but for propagation perpendicular to the magnetic field the magnetogravity waves degenerate into classical gravity waves and their speed drops to the gravity wave speed $\sqrt{g\,h}$.

D. Riemann invariants

From the left eigenvectors Riemann invariants can be derived. Riemann invariants are constant on their associated characteristic curves. There can thus be at most n Riemann invariants for an $n\times n$ hyperbolic system. A Riemann invariant χ_i exists for mode i if functions χ_i and α_i can can be found, both functions of the components of $\mathbf{V}$, such that $\mathbf{L}_i\cdot\partial\mathbf{V}=\mathbf{L}_i\cdot(\partial h,\partial v_x,\partial v_y,\partial B_x,\partial B_y)^T=\alpha_i\,\partial\chi_i$. α_i is called the integrating factor. Knowledge of Riemann invariants is useful for various purposes. Riemann invariants can for instance be used to measure grid convergence[25] for flows for which the analytical solution is not fully known.

For modes 1, 2, and 5 the following Riemann invariants can be found:

$$
\begin{aligned}
\mathbf{L}_1\cdot\partial\mathbf{V} &= \partial(u_y+B_y),\\
\mathbf{L}_2\cdot\partial\mathbf{V} &= \partial(-u_y+B_y),\\
\mathbf{L}_5\cdot\partial\mathbf{V} &= B_x\,\partial h + h\,\partial B_x = \partial(h\,B_x).
\end{aligned}
\tag{16}
$$

For modes 3 and 4 the expressions seem complicated and at first sight Riemann invariants cannot be derived. However, the expressions can be rewritten [Eq. (17)] such that they contain only two variables—hB_x is a constant—and a general theorem says that in this case an integrating factor can be found, such that we have proven that the Riemann invariants exist for modes 3 and 4,

$$
\begin{aligned}
\mathbf{L}_3\cdot\partial\mathbf{V} &= -g\,\partial h + \sqrt{B_x^2+g\,h}\,\partial u + B_x\partial B_x\\
&\sim -\sqrt{(hB_x)^2+g\,h^3}\,\partial h + h^2\partial u,\\
\mathbf{L}_4\cdot\partial\mathbf{V} &= -g\,\partial h - \sqrt{B_x^2+g\,h}\,\partial u + B_x\partial B_x\\
&\sim -\sqrt{(hB_x)^2+g\,h^3}\,\partial h - h^2\partial u.
\end{aligned}
\tag{17}
$$

E. Linearity properties of the SMHD wave modes

Wave solutions associated with genuinely nonlinear wave modes[14] steepen or rarefy. Steepening or rarefaction for a genuinely nonlinear wave mode i occurs because the wave speed varies in phase space along the integral curve of the right eigenvector associated with the mode ($\nabla\lambda_i\cdot\mathbf{R}_i\neq 0$, with ∇ the gradient in $\mathbf{V}$-space). Linearly degenerate waves, for which $\nabla\lambda_i\cdot\mathbf{R}_i\equiv 0$, do not steepen nor rarify, such that wave profiles of arbitrary amplitude are convected as a passive scalar with the (constant) wave speed.

SMHD wave modes 1, 2, and 5 are linearly degenerate waves, as $\nabla\lambda_1\cdot\mathbf{R}_1\equiv 0$, $\nabla\lambda_2\cdot\mathbf{R}_2\equiv 0$ and $\nabla\lambda_5\cdot\mathbf{R}_5\equiv 0$. The magnetogravity modes, however, are genuinely nonlinear, as $\nabla\lambda_3\cdot\mathbf{R}_3=3\,g\,h/(2\sqrt{B_x^2+g\,h})\neq 0$ and $\nabla\lambda_4\cdot\mathbf{R}_4=-3\,g\,h/(2\sqrt{B_x^2+g\,h})\neq 0$. Compressional magnetogravity modes can steepen into shocks (see below). A formula for the steepening time of compressional magnetogravity SMHD waves is derived in Ref. 13.

It is important to note that $\nabla\lambda_3\cdot\mathbf{R}_3$ and $\nabla\lambda_4\cdot\mathbf{R}_4$ do not change sign, which means that modes 3 and 4 are convex modes, such that compound shocks do not occur.[3] This is in contrast to the fast and slow MHD waves, which are nonconvex,[10,11,26] such that compound shocks occur in MHD flows.

F. Simple SMHD waves

Simple waves[14] are flow solutions that only depend on one independent variable ξ, $\mathbf{V}(x,t)=\mathbf{V}(\xi(x,t))$. The function $\xi(x,t)$ defines curves $\xi(x,t)=c$, on which $\mathbf{V}(x,t)$ is constant. It can easily be seen, by using the conservation law, that the slope of those curves $\xi(x,t)=c$ has to be an eigenvalue of the flux Jacobian, $dx/dt=\lambda_i(\mathbf{V})$, and that the derivative of the simple wave state vector has to be proportional to the associated right eigenvector, $d\mathbf{V}/d\xi \sim \mathbf{R}_i$. As the state $\mathbf{V}$ is constant on those curves, and their slope depends only on the state, the curves $\xi(x,t)=c$ are straight lines and are actually the i-characteristic curves in the xt plane. In an i-simple wave the i-characteristics are thus straight lines.

A special class of simple waves are the so-called centered simple waves,[3] for which the parameter ξ is given by $\xi = x/t$. The slope of the straight characteristics is given by $dx/dt = x/t = \xi$, such that those characteristics originate in one central point. A Riemann problem is an initial value problem in which initially a discontinuity separates constant left and right states (see below). Centered simple waves and propagating shocks are the two building blocks of Riemann problem solutions.

Simple waves can also be characterized by the variation of flow variables as a function of one of the primitive variables, i.e., we can take $\xi = V_i(x,t)$. In particular we make the choice $\xi = h(x,t)$ below.

It can easily be proven that for simple waves associated with linearly degenerate modes i, $\partial\lambda_i/\partial t \equiv 0$ and $\partial\lambda_i/\partial x \equiv 0$. This means that the i-characteristics are all parallel in the simple wave, and that any profile $\sim R_i$ is a simple wave, without any steepening or rarefying occurring.

For the genuinely nonlinear magnetogravity wave mode 4, simple waves satisfy

$$\frac{\partial}{\partial \xi}\begin{bmatrix} h \\ v_x \\ v_y \\ B_x \\ B_y \end{bmatrix} = \alpha \begin{bmatrix} -h \\ -\sqrt{B_x^2 + g\,h} \\ 0 \\ B_x \\ 0 \end{bmatrix}. \quad (18)$$

This shows that v_y and B_y remain constant throughout the simple wave. The other primitive variables v_x and B_x can be expressed in terms of h by using the divergence constraint $\partial(h\,B_x)/\partial\xi \equiv 0$, and the expression

$$v_x = \int \frac{\sqrt{(h\,B_x)^2 + g\,h^3}}{h^2}\,dh. \quad (19)$$

This offers a closed analytical solution for a SMHD simple magnetogravity wave, as the integral can be expressed as a function of elliptic integrals.

For the special case of a centered[3] simple magnetogravity wave the analytical solution is completed by the following relation between the parameter ξ and the state variable h:

$$\xi = \int \frac{3\,g\,h}{\sqrt{(h\,B_x)^2 + g\,h^3}}\,dh. \quad (20)$$

IV. HYPERBOLIC THEORY OF THE STATIONARY 2D SMHD SYSTEM

A general quasilinear time-dependent system of n first order partial differential equations in two space dimensions is described by

$$\frac{\partial \mathbf{V}}{\partial t} + \mathsf{A}(\mathbf{V})\cdot\frac{\partial \mathbf{V}}{\partial x} + \mathsf{B}(\mathbf{V})\cdot\frac{\partial \mathbf{V}}{\partial y} = 0, \quad (21)$$

with $\mathbf{V}$ a vector of dependent variables and A and B $n\times n$ matrices. This system reduces to a system with two independent variables for the case of a 1D time-dependent flow ($\partial/\partial y \equiv 0$) and for the case of a 2D steady flow ($\partial/\partial t \equiv 0$). In the first case, the last term of Eq. (21) vanishes, and the procedure which was outlined above can be used to study the characteristic structure of the equations. In the second case, the first term of Eq. (21) vanishes, and one can examine the characteristic properties of the equation,

$$\frac{\partial \mathbf{V}}{\partial x} + \mathsf{A}^{-1}\cdot\mathsf{B}\cdot\frac{\partial \mathbf{V}}{\partial y} = 0, \quad (22)$$

leading to characteristic analysis in the xy plane. The eigenvalues and left eigenvectors of the matrix $\mathsf{C}\equiv\mathsf{A}^{-1}\cdot\mathsf{B}$ again determine the type of the system, the characteristic directions in the xy plane, and the Riemann invariants.[6–8,11]

For the SMHD equations, the eigenvalues of matrix C are given by

$$\lambda_{1,2} = \frac{B_y \mp v_y}{B_x \mp v_x},$$

$$\lambda_{3,4} = \frac{B_x\,B_y - v_x\,v_y}{(B_x^2 - v_x^2 + g\,h)} \mp \frac{\sqrt{(B_xB_y - v_xv_y)^2 - (B_x^2 - v_x^2 + gh)(B_y^2 - v_y^2 + gh)}}{(B_x^2 - v_x^2 + g\,h)}, \quad (23)$$

$$\lambda_5 = \frac{v_y}{v_x}.$$

These eigenvalues, when they are real, determine the slopes of the characteristics in the xy plane. The first two eigenvalues are always real, so the associated characteristic fields are hyperbolic. The eigenvalues $\lambda_{1,2}$ are associated with the Alfvén modes, and they determine characteristic directions such that perpendicularly to the characteristics $v_\perp = c_{a\perp}$.

Eigenvalues $\lambda_{3,4}$ are associated with the magnetogravity modes, and are such that $v_\perp = c_{g\perp}$ perpendicularly to the characteristics. These characteristic fields can be hyperbolic or elliptic, depending on the state. When the velocity vector lies outside the ray surface diagram (Fig. 1), the characteristic fields are hyperbolic, and xy characteristics can be determined by drawing tangents to the ray surface.[6–8] Like in the MHD case, simple analytic conditions for hyperbolicity can be found for the case of steady field–aligned flow, for which $\mathbf{v} = \alpha\,\mathbf{B}$. In this case eigenvalues $\lambda_{3,4}$ are real when $v_x^2 + v_y^2 > B_x^2 + B_y^2 + g\,h$, which is a condition analogous to the supersonic condition for hyperbolicity in the case of the Euler equations.

Mode 5 is always hyperbolic, and its characteristics are the streamlines of the steady flow. The associated left eigenvector is given by $\mathbf{L}_5 = (B_y\,v_x - B_x\,v_y, h\,B_y, -h\,B_x, -h\,v_y, h\,v_x)$, and the associated Riemann invariant is found by considering $L_5\cdot\partial\mathbf{V} = \partial(h(B_y\,v_x - B_x\,v_y))$. Inspection of the steady equations shows that this Riemann invariant, which is proportional to the z component of the electric field E_z, is actually a global invariant over the whole xy domain. It can be used to eliminate one state variable from the steady equations, as in the 2D MHD case. It also follows that in steady flow, if the magnetic and velocity fields are aligned in one point of the flow field ($E_z=0$), then they have to be aligned everywhere. This is a manifestation of the ''frozen-in'' property of ideal SMHD flows, analogous to the same property of full ideal MHD.

Expressions for the left and right eigenvectors are complicated. We did not find other Riemann invariants. Expressions for stationary simple waves could in principle be derived from the right eigenvectors,[9] but we did not attempt this complicated exercise.

V. DISCONTINUOUS SOLUTIONS

Systems of hyperbolic conservation laws allow for discontinuous solutions, which have to satisfy the Rankine–Hugoniot jump relations.[3,6,7] Discontinuous solutions of the SMHD equations can be classified as follows.

We consider discontinuous solutions with variation in the x direction. The SMHD flux function in the x direction is given by

$$F_x=\begin{bmatrix} h v_x \\ h v_x^2+g h^2/2-h B_x^2 \\ h v_x v_y-h B_x B_y \\ 0 \\ h B_y v_x-h B_x v_y \end{bmatrix}. \tag{24}$$

The Rankine–Hugoniot condition in the coordinate frame that moves with the shock is given by $F_{x1}=F_{x2}$, with indices 1 and 2 denoting the states on the two sides of the discontinuity. For SMHD the divergence constraint leads to the additional condition $(hB_x)_1=(hB_x)_2$.

For convenience we introduce a new set of state variables m,ϕ,τ,v_y,B_y related to the primitive variables by

$$\begin{aligned} m&=h v_x, \\ \phi&=h B_x, \\ \tau&=\frac{1}{h}. \end{aligned} \tag{25}$$

The jump relations then read

$$\begin{aligned} &\langle m\rangle=0, \\ &\langle \phi\rangle=0, \\ &m^2\langle\tau\rangle+\frac{g}{2}\left\langle\frac{1}{\tau^2}\right\rangle-\phi^2\langle\tau\rangle=0, \\ &m\langle v_y\rangle-\phi\langle B_y\rangle=0, \\ &m\langle B_y\rangle-\phi\langle v_y\rangle=0, \end{aligned} \tag{26}$$

with the notational convention $\langle m\rangle=m_2-m_1$, etc.

We classify the solutions of these conditions in two main categories.

(A) $\phi^2=m^2$:

In this case $\langle\tau\rangle=0$, such that $\langle h\rangle=0$, $\langle v_x\rangle=0$ and $\langle B_x\rangle=0$. Then it follows that on both sides of the discontinuity $v_x=\pm B_x$. This means that this type of discontinuity propagates with the Alfvén velocity. One of the jumps $\langle v_y\rangle$ or $\langle B_y\rangle$ can be chosen arbitrarily, with the other jump following from $\langle v_y\rangle=\pm\langle B_y\rangle$. The jump is thus proportional to the right eigenvectors of the Alfvén modes. We call this type of discontinuity an Alfvén discontinuity. In fact, this Alfvén discontinuity is just a special case of the simple wave solution associated with the Alfvén wave. Because this wave is linearly degenerate, any profile proportional to the right eigenvectors is advected as a simple wave without steepening. The Alfvén discontinuity is just such a profile that happens to be discontinuous. The characteristics are parallel to this discontinuity on the two sides.

(B) $\phi^2\neq m^2$:

In this case the discontinuities have to satisfy $m^2\langle\tau\rangle+g/2\langle 1/\tau^2\rangle-\phi^2\langle\tau\rangle=0$. It follows that $\langle h\rangle\neq 0$, $\langle v_x\rangle\neq 0$ and $\langle B_x\rangle\neq 0$, while $\langle v_y\rangle=0$ and $\langle B_y\rangle=0$. These discontinutities are associated with the genuinely nonlinear magneto-gravity modes, and can be formed through steepening. Therefore we call them shocks. The characteristics are not parallel to the shocks on the two sides. Shocks are stable when the characteristics enter into the shocks.

We can consider two limiting cases:

(B1) $m=0$ and $\phi\neq 0$:

In this case $v_x=0$ on the two sides of the shock, such that there is no mass flow through the shock.

(B2) $m\neq 0$ and $\phi=0$:

In this case $B_x=0$ on the two sides. The magnetic field does not change through the shock, so we call it a hydrodynamic shock.

We further investigate the shock solutions. For convenience we define

$$\begin{aligned} \alpha&=\frac{g\,h}{2}+\sqrt{\left(\frac{g\,h}{2}\right)^2+g\,h\,\beta}, \\ \beta&=2\,(v_x^2-B_x^2). \end{aligned} \tag{27}$$

The state variables on side 2 can then be expressed in terms of the variables on side 1 as

$$\begin{aligned} \frac{h_2}{h_1}&=\frac{\beta_1}{\alpha_1}, \\ \frac{v_{x2}}{v_{x1}}&=\frac{\alpha_1}{\beta_1}, \\ \frac{B_{x2}}{B_{x1}}&=\frac{\alpha_1}{\beta_1}, \\ c_{gx2}^2&=c_{gx1}^2\frac{\alpha_1^2}{\beta_1^2}+g\,h_1\left(\frac{\beta_1}{\alpha_1}-\frac{\alpha_1^2}{\beta_1^2}\right). \end{aligned} \tag{28}$$

Given a state 1 with height h_1, we now investigate if a state 2 with positive height h_2 can be found such that the jump relations are satisfied. We distinguish two cases.

(1) $m^2>\phi^2$:

In this case one positive root exists for h_2.

(2) $m^2<\phi^2$:

In this case no positive roots exists for h_2.

We conclude that on both sides of a shock $m^2>\phi^2$. This means that the flow is super-Alfvénic on both sides.

Now we investigate the relation of the fluid speed and the magnetogravity wave speed on the two sides of the shock. If the fluid speed is greater than the wave speed, we call the state supersonic. Let us assume that on one side of the shock, say in state 1, the state is subsonic. We want to prove that the flow is then necessarily supersonic on the other side (state 2).

It can easily be derived that the subsonic condition $v_{x1} < \sqrt{B_{x1}^2 + g\, h_1}$ in state 1 is equivalent to $2\, g\, h_1 > \beta_1$, or also to $\alpha_1 > \beta_1$. We investigate whether $v_{x2} > \sqrt{B_{x2}^2 + g\, h_2}$. By the equivalences given above this condition corresponds to $2\, g\, h_1\, \beta_1^2/\alpha_1^3 < 1$. This inequality can be proved using $2\, g\, h_1 > \beta_1$ and the expression for α that was given in Eq. (27).

We conclude that for an SMHD shock the state is subsonic on one side, and supersonic on the other side. For small-amplitude shocks this has been proven in general by Lax[14] for any strictly hyperbolic system. Our derivation proves that this property holds for SMHD shocks with arbitrary amplitude. The states on both sides of the shock have to be super-Alfvénic. This means that exactly one family of characteristics converges into stable SMHD shocks. The shocks are thus of the classical Lax-type.[14] So-called overcompressive shocks, in which more characteristics converge, do not occur in the SMHD system. This is contrary to the case of full MHD, for which the so-called intermediate shocks are overcompressive shocks, which are unstable in the ideal case.[8–10,17,18,27]

Together with the analytical solution for SMHD simple waves that was derived above, the explicit solutions of the SMHD Rankine–Hugoniot relations that were presented here [including Eq. (28)] form the building blocks for the full analytical solution of all SMHD Riemann problems.

VI. GALILEAN INVARIANT CONSERVATION LAW FORM WITH SOURCE TERM

Analogous to the case of full MHD,[10,15,23,24,28] the flux given in Eq. (24), which derives from the strict conservation law form Eq. (3), leads to a singular Jacobian matrix with eigenvalue $\lambda_5 = 0$. For various reasons, to be discussed in more detail below, this is undesirable. Like in MHD, a regular Jacobian matrix can be obtained for the SMHD equations in conservative variables by including a source term proportional to the divergence constraint,

$$\frac{\partial}{\partial t}\begin{bmatrix} h \\ h\mathbf{v} \\ h\mathbf{B} \end{bmatrix} + \nabla\cdot\begin{bmatrix} h\mathbf{v} \\ h\mathbf{v}\mathbf{v} - h\,\mathbf{B}\mathbf{B} + \mathrm{I}(gh^2/2) \\ h\mathbf{B}\mathbf{v} - h\mathbf{v}\mathbf{B} \end{bmatrix} = -\begin{bmatrix} 0 \\ \mathbf{B} \\ \mathbf{v} \end{bmatrix}\nabla\cdot(h\mathbf{B}). \tag{29}$$

The source term, which in the MHD case is called the Powell source term,[23,24] is proportional to $\nabla\cdot(h\mathbf{B})$. The source term regularizes the flux Jacobian, such that $\lambda_5 = v_x$, without changing the eigenvectors.

Adding the regularizing source term has several advantages.

First, it makes the equations Galilean invariant, also when the divergence constraint is not exactly satisfied. Indeed, for Galilean invariance all eigenvalues should be transformed when transforming to a new reference frame. Clearly, without the source term, $\lambda_5 = 0$ does not satisfy this requirement, while with the source term $\lambda_5 = v_x$ is Galilean invariant. Of course this only matters when the divergence constraint is not exactly satisfied, because otherwise the wave strength carried by the fifth mode vanishes anyway, as we have shown above. It is interesting to note that the occurrence of this problem for MHD is related to the fact that MHD attempts to combine the Lorenz invariant Maxwell equations with the Galilean invariant Euler equations. Relativistic physics thus needs to be reconciled with nonrelativistic physics, and this contributes to the difficulties discussed here.

Second, for the MHD case the source term form is the unique form of the equations that is symmetrizable[15,28] (without using the divergence constraint in the symmetrization process). This may also be true for the SMHD equations. Study of the SMHD symmetrizability properties remains a topic for further investigation.

Third, in many conservative, shock-capturing numerical schemes, the divergence constraint is only satisfied up to a discretization error. Inclusion of the source term assures that small $\nabla\cdot(h\mathbf{B})$ errors are consistently accounted for in a Galilean invariant way. This stabilizes the numerical schemes as inaccuracies are not accumulated but are advected with the flow. Indeed, from the conservation law form Eq. (3) one can derive

$$\frac{\partial}{\partial t}(\nabla\cdot(h\mathbf{B})) = 0, \tag{30}$$

while for the source term form one obtains

$$\frac{\partial}{\partial t}\left(\frac{\nabla\cdot(h\mathbf{B})}{h}\right) + (\mathbf{v}\cdot\nabla)\left(\frac{\nabla\cdot(h\mathbf{B})}{h}\right) = 0. \tag{31}$$

The latter equality means that the quantity $\nabla\cdot(h\mathbf{B})/h$ is advected by the flow as a passive scalar. The source term approach is a highly attractive alternative to the use of an extra artificial $\nabla\cdot(h\mathbf{B})$ correction in every time step obtained via solution of an elliptic equation, because it consumes much less computing time and because it cures the $\nabla\cdot(h\mathbf{B})$ problems in a way which is more natural for the hyperbolic system, as pointed out by Powell.[23] For MHD this approach has been tested on many model problems[10,23–25] and it seems that it works properly in many cases.

Use of staggered schemes in which $\nabla\cdot(h\mathbf{B}) = 0$ is automatically conserved is an alternative way to obtain stable schemes which maintain the divergence constraint.[29] Recently, several new variants of this approach have been proposed for MHD.[30–33] It has often been argued that the source term approach may not lead to reliable solutions for all types of MHD flows (see, e.g., Ref. 30). Toth[33] gives an explicit example of a 2D rotated MHD Riemann problem for which the source term approach fails to produce a valid solution.

VII. 1D SMHD RIEMANN PROBLEMS

Many modern numerical schemes for hyperbolic conservation laws—including, e.g., the Roe scheme[34]—heavily rely on decomposition of the dynamics into characteristic wave

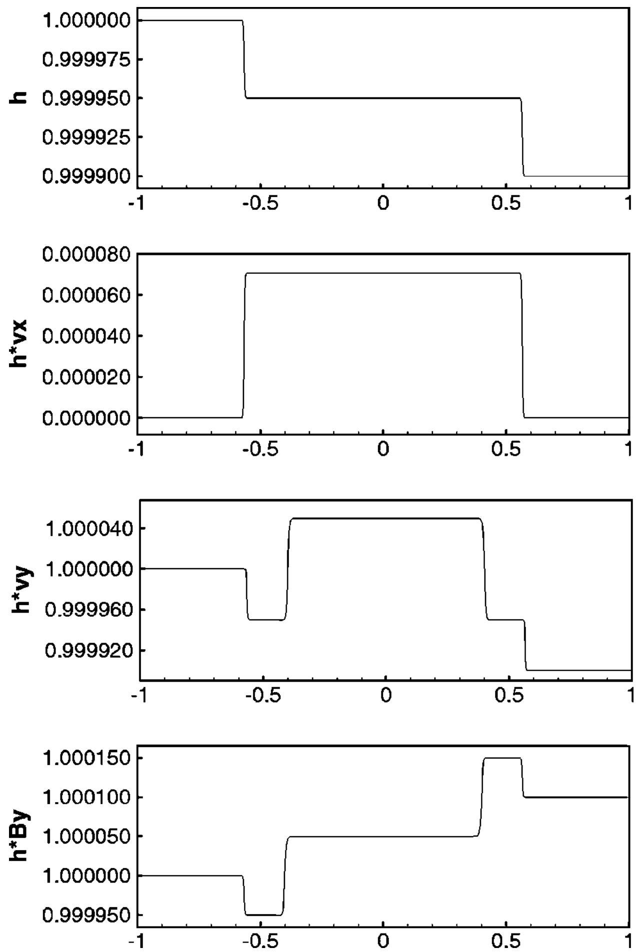

FIG. 2. Weak Riemann problem. The conserved variables are plotted.

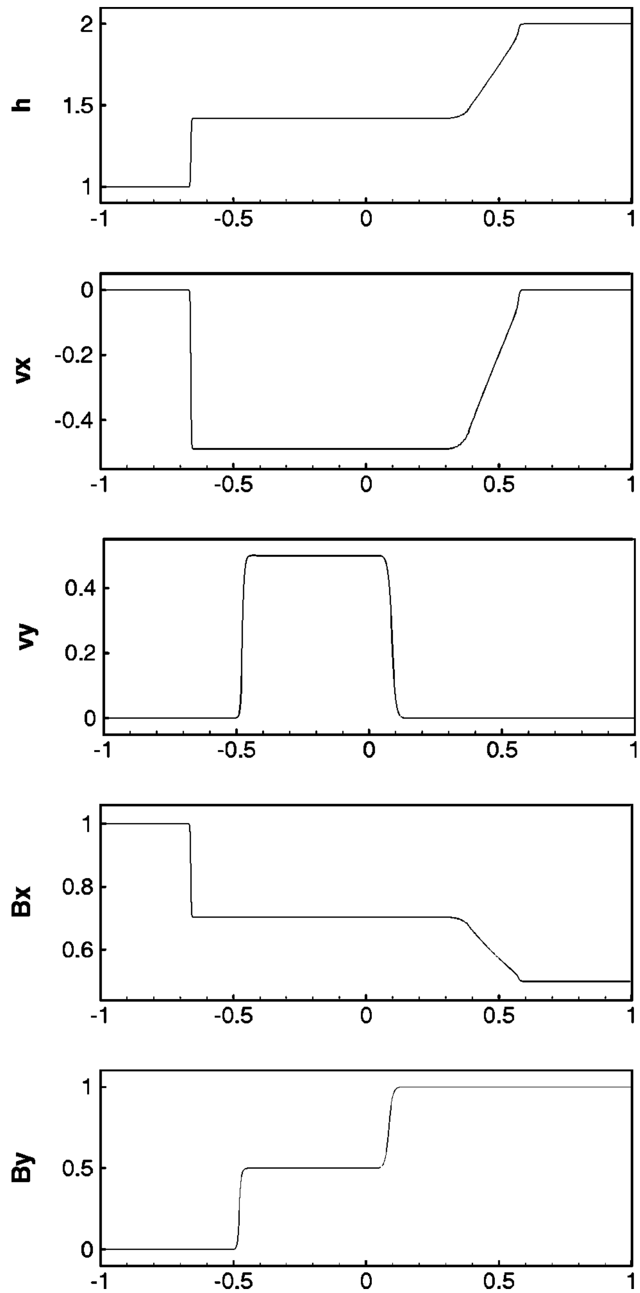

FIG. 3. Strong Riemann problem. The primitive variables are plotted.

modes,[3,4,12] and the theory derived in this paper can serve as the basis for formulating schemes of this type for the SMHD equations. In the Appendix of the present paper a Roe-type linearization[34] of the SMHD equations is given which can serve as a building block for accurate shock-capturing numerical schemes.

Numerical simulation studies of 1D and 2D SMHD flows using various characteristic decomposition-based numerical schemes are presently under way, and will be presented elsewhere.[35] In the present section we describe 1D Riemann problem simulations that illustrate the behavior of simple wave solutions and discontinuities associated with the linearly degenerate and genuinely nonlinear SMHD wave modes. The numerical solutions are obtained with a simple second-order Lax–Friedrichs scheme.[14,25,36,37] The gravitational acceleration constant is taken $g=1$ for all the simulations.

A. Weak Riemann problem

In Fig. 2 we plot the solution of a Riemann problem with the initial primitive state vector given by $\mathbf{V}_l^T=(1,0,1,1,1)$ for the left half of the simulation domain, and by $\mathbf{V}_r^T=\mathbf{V}_l^T+10^{-4}(-1,0,0,(1-10^{-4})^{-1},2)$ for the right half of the simulation domain. Two magnetogravity waves and two Alfvén waves originate from the locus of the initial discontinuity and propagate away in the two directions. The magnetogravity waves carry perturbations only in h and $h\,v_x$, while the Alfvén waves only perturb v_y and $h\,B_y$. Wave mode 5 does not show up in the solution, which confirms our analysis that wave mode 5 is spurious and cannot arise due to the $\nabla(h\mathbf{B})$ constraint. The magnetogravity waves are faster than the Alfvén waves. The left-going magnetogravity wave is a rarefaction, and the right-going wave is a shock, but due to the small amplitude of the perturbation the response is almost linear and the rarefaction has not had enough time yet to spread out noticeably. Another manifestation of the near-linearity of the response is that the left-going and right-going waves travel with almost equal speeds, such that the Riemann problem solution is almost symmetrical.

B. Strong Riemann problem

In Fig. 3 we plot the solution of a Riemann problem with the initial primitive state vector given by $\mathbf{V}_l^T=(1,0,0,1,0)$ for the left half of the simulation domain, and by $\mathbf{V}_r^T=(2,0,0,0.5,1)$ for the right half of the simulation domain. Again two magnetogravity waves and two Alfvén waves originate from the locus of the initial discontinuity and propagate away in the two directions. The magnetogravity waves are faster than the Alfvén waves. This time the right-going magnetogravity wave is a rarefaction, and the left-going wave is a shock. The amplitude of the perturbation is large enough for the rarefaction to spread out noticeably, and for the left-going and right-going waves to travel with different speeds, such that the Riemann problem solution is far from symmetrical.

Simulations are carried out with a second order Lax–Friedrichs scheme on a grid of 1000 equidistant points. Both shocks and Alfvén discontinuities are reasonably well resolved.

VIII. CONCLUSIONS

The hyperbolic properties of the newly proposed SMHD equations have been derived. Characteristic analysis has been performed for both the unsteady xt case and the steady xy case. An analytical solution for SMHD simple waves has been derived. Solutions of the Rankine–Hugoniot relations have been classified and their properties have been investigated. Thus the building blocks for the full analytical solution of all SMHD Riemann problems have been given. 1D numerical simulation results have been presented of SMHD Riemann problems that illustrate the wave properties of the hyperbolic SMHD modes.

The theory presented in this paper provides a basic description of nonlinear waves in physical systems that are described by the SMHD equations. The influence of rotation on the properties of SMHD waves is discussed in Schecter *et al.*[13] The theory derived in the present paper can serve as the basis for formulating modern numerical schemes for the SMHD equations that rely on the characteristic deomposition of hyperbolic systems. A Roe-type linearization[34] of the SMHD equations is given in the Appendix which can serve as a building block for accurate shock-capturing numerical schemes.

The SMHD equations are presently being used in the study of the dynamics of layers in the solar interior,[1,5] and they may also be applicable to problems involving the free surface flow of conducting fluids in laboratory and industrial environments.

ACKNOWLEDGMENTS

The author thanks the High Altitude Observatory (HAO) for its hospitality during a research visit. Discussions with P. Gilman, D. Schecter, and M. Dikpati were very interesting. The properties of linear small-amplitude SMHD waves and the SMHD wave normal diagram were first investigated by J. Boyd during his research stay at HAO as a summer student.

APPENDIX: A ROE-TYPE LINEARIZATION

1. Linearized approximate Riemann solvers

Finite volume numerical techniques based on solving 1D Riemann problems at computational cell interfaces have become a method of choice for the numerical simulation of conservation laws in 2D and 3D, because using these techniques robust second-order accurate schemes can be constructed that remain oscillation-free at discontinuities without introducing excessive numerical dissipation.[3,4,12,34] Indeed, the decomposition in characteristic waves allows us to apply to each wave mode separately the minimum amount of numerical dissipation that keeps it from becoming numerically unstable.

Solving 1D Riemann problems at cell interfaces exactly is computationally expensive, and therefore the Riemann problems are often only solved approximately. One approach to solving a Riemann problem with left and right states $\mathbf{U}_l$ and $\mathbf{U}_r$ in an approximate way is to linearize the conservation law as

$$\frac{\partial \mathbf{U}}{\partial t}+\tilde{\mathsf{A}}_c(\mathbf{U_l},\mathbf{U_r})\cdot\frac{\partial \mathbf{U}}{\partial x}=0, \tag{A1}$$

with $\tilde{\mathsf{A}}_c(\mathbf{U_l},\mathbf{U_r})$ the linearized Jacobian matrix of the conservation law—written in terms of the conserved variables—chosen to be a function of the left and right states $\mathbf{U}_l$ and $\mathbf{U}_r$.

This linearized system of equations can easily be solved exactly for the Riemann problem,[3] and a procedure that solves these linearized equations to approximate the true nonlinear Riemann problem solution is called a linearized approximate Riemann solver.

There is a choice in how one defines the linearized Jacobian matrix as a function of the left and right states $\mathbf{U}_l$ and $\mathbf{U}_r$. Consistency with the original conservation law requires that

$$\tilde{\mathsf{A}}_c(\mathbf{U},\mathbf{U})=\mathsf{A}_c(\mathbf{U}), \tag{A2}$$

with $\mathsf{A}_c(\mathbf{U})$ the exact Jacobian for conserved variables. The linearized Jacobian matrix $\tilde{\mathsf{A}}_c(\mathbf{U_l},\mathbf{U_r})$ needs to have linearly independent eigenvectors in order for the linearized system to be hyperbolic.

A simple choice for $\tilde{\mathsf{A}}_c(\mathbf{U_l},\mathbf{U_r})$ that satisfies these requirements is

$$\tilde{\mathsf{A}}_c(\mathbf{U_l},\mathbf{U_r})=\mathsf{A}_c\left(\frac{\mathbf{U_l}+\mathbf{U_r}}{2}\right). \tag{A3}$$

Roe[34] advocates a more sophisticated choice, and proposes to look for a linearization that satisfies

$$\Delta\mathbf{F}=\tilde{\mathsf{A}}_c(\mathbf{U_l},\mathbf{U_r})\cdot\Delta\mathbf{U}, \tag{A4}$$

with $\Delta\mathbf{F}=\mathbf{F}(\mathbf{U_r})-\mathbf{F}(\mathbf{U_l})$ and $\Delta\mathbf{U}=\mathbf{U_r}-\mathbf{U_l}$.

This property is useful because it guarantees that the linearized approximate Rieman solver solution of a Riemann problem for which the exact solution is a single propagating discontinuity, is exact. Indeed, the Rankine–Hugoniot relations $\Delta\mathbf{F}=s\Delta\mathbf{U}$ are satisfied for an isolated discontinuity propagating with speed s, and if the linearized Jacobian matrix satisfies $\Delta\mathbf{F}=\tilde{\mathsf{A}}_c\cdot\Delta\mathbf{U}$, then

$$\tilde{\mathsf{A}}_c \cdot \Delta \mathbf{U} = s \Delta \mathbf{U} \tag{A5}$$

holds, which means that the jump over the isolated discontinuity is an eigenvector of $\tilde{\mathsf{A}}_c$, with eigenvalue $\lambda_i = s$. This means that the propagating isolated discontinuity is the solution found by the linearized Riemann solver. The linearized approximate Riemann solver that satisfies Roe's condition (A4) is thus exact for isolated discontinuities.[34] Numerical schemes that are built on linearized approximate Riemann solvers that satisfy this condition can capture discontinuities more accurately than schemes that use the simple linearization (A3).

In this Appendix we derive a linearized Jacobian matrix $\tilde{\mathsf{A}}_c(\mathbf{U_l}, \mathbf{U_r})$ for the SMHD equations which satisfies Roe's condition (A4). We use the general procedure for deriving such a linearization that was described by Balsara.[38–40] Balsara applied this systematic approach to derive Roe linearizations for adiabatic and isothermal MHD,[38] and for radiation hydrodynamics and radiation MHD.[39,40] In our derivation we follow the notation that was used by Balsara in Ref. 38.

2. The SMHD Jacobian for conservative variables

For purpose of comparison with the expressions to be derived below, we first give the expressions for the SMHD Jacobian A_c for conservative variables,

$$\mathsf{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ gh + \phi_x^2/h^2 - m_x^2/h^2 & 2m_x/h & 0 & 0 \\ -m_x m_y/h^2 + \phi_x \phi_y/h^2 & m_y/h & m_x/h & -\phi_x/h \\ -m_x \phi_y/h^2 + \phi_x m_y/h^2 & \phi_y/h & -\phi_x/h & m_x/h \end{bmatrix}, \tag{A6}$$

and the expressions for its right and left eigenvectors,

$$\mathsf{R}_c = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & m_x/h - c_{gx} & m_x/h + c_{gx} \\ 1 & -1 & m_y/h & m_y/h \\ 1 & 1 & \phi_y/h & \phi_y/h \end{bmatrix}, \tag{A7}$$

$$\mathsf{L}_c = \begin{bmatrix} -(\phi_y + m_y)/(2h) & 0 & 1/2 & 1/2 \\ -(\phi_y - m_y)/(2h) & 0 & -1/2 & 1/2 \\ (m_x/h + c_{gx})/(2c_{gx}) & -1/(2c_{gx}) & 0 & 0 \\ -(m_x/h - c_{gx})/(2c_{gx}) & 1/(2c_{gx}) & 0 & 0 \end{bmatrix}. \tag{A8}$$

Note that we have left out the evolution equation for $\phi_x = h\, B_x$, because this quantity is a constant in space and time for the 1D problem we consider here. In what follows we write ϕ in stead of ϕ_x to emphasize that $\phi_x = \phi$ is a constant.

3. Roe linearization

Following Balsara,[38] we start out by choosing a so-called parameter vector $\mathbf{W}$,

$$\mathbf{W} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} \sqrt{h} \\ \sqrt{h}\, v_x \\ \sqrt{h}\, v_y \\ \sqrt{h}\, B_y \end{bmatrix}. \tag{A9}$$

Note that this parameter vector $\mathbf{W}$ is to be distinguished from the vector of characteristic variables $\mathbf{W}$ that was introduced in Sec. III B.

We express the vector of conserved variables $\mathbf{U}$ and the flux vector $\mathbf{F(U)}$ as functions of the parameter vector,

$$\mathbf{U} = \begin{bmatrix} w_1^2 \\ w_1 w_2 \\ w_1 w_3 \\ w_1 w_4 \end{bmatrix}, \quad \mathbf{F(U)} = \begin{bmatrix} w_1 w_2 \\ w_2^2 + g/2 w_1^4 - \phi^2/w_1^2 \\ w_2 w_3 - \phi w_4/w_1 \\ w_2 w_4 - \phi w_3/w_1 \end{bmatrix}. \tag{A10}$$

We find the linearized Jacobian $\tilde{\mathsf{A}}_c$ of Roe-type by calculating the matrices $\tilde{\mathsf{B}}$ and $\tilde{\mathsf{C}}$ as defined by

$$\Delta \mathbf{U} = \tilde{\mathsf{B}} \cdot \Delta \mathbf{W}, \qquad \Delta \mathbf{F} = \tilde{\mathsf{C}} \cdot \Delta \mathbf{W}, \qquad \Delta \mathbf{F} = \tilde{\mathsf{C}} \cdot \tilde{\mathsf{B}}^{-1} \cdot \Delta \mathbf{U} = \tilde{\mathsf{A}}_c \cdot \Delta \mathbf{U}. \tag{A11}$$

Matrices $\tilde{\mathsf{B}}$ and $\tilde{\mathsf{C}}$ are given by

$$\tilde{\mathsf{B}}=\begin{bmatrix} 2\overline{w}_1 & 0 & 0 & 0 \\ \overline{w}_2 & \overline{w}_1 & 0 & 0 \\ \overline{w}_3 & 0 & \overline{w}_1 & 0 \\ \overline{w}_4 & 0 & 0 & \overline{w}_1 \end{bmatrix}, \tag{A12}$$

and

$$\tilde{\mathsf{C}}=\begin{bmatrix} \overline{w}_2 & \overline{w}_1 & 0 & 0 \\ 2g\langle w_1^2\rangle\overline{w}_1+2\phi^2\overline{w}_1/w_1^{*4} & 2\overline{w}_2 & 0 & 0 \\ \phi\overline{w}_4/w_1^{*2} & \overline{w}_3 & \overline{w}_2 & -\phi\overline{w}_1/w_1^{*2} \\ \phi\overline{w}_3/w_1^{*2} & \overline{w}_4 & -\phi\overline{w}_1/w_1^{*2} & \overline{w}_2 \end{bmatrix}, \tag{A13}$$

and the resulting expression for the linearized Jacobian $\tilde{\mathsf{A}}_c$ is

$$\tilde{\mathsf{A}}_c=\begin{bmatrix} 0 & 1 & 0 & 0 \\ g\langle w_1^2\rangle+\phi^2/w_1^{*4}-\overline{w}_2^2/\overline{w}_1^2 & 2\overline{w}_2/\overline{w}_1 & 0 & 0 \\ -\overline{w}_2\overline{w}_3/\overline{w}_1^2+\phi\overline{w}_4/\overline{w}_1 1/w_1^{*2} & \overline{w}_3/\overline{w}_1 & \overline{w}_2/\overline{w}_1 & -\phi/w_1^{*2} \\ -\overline{w}_2\overline{w}_4/\overline{w}_1^2+\phi\overline{w}_3/\overline{w}_1 1/w_1^{*2} & \overline{w}_4/\overline{w}_1 & -\phi/w_1^{*2} & \overline{w}_2/\overline{w}_1 \end{bmatrix}. \tag{A14}$$

Here we have used the following notation for various kinds of averages of the components of the parameter vector **W**:

$$\begin{aligned} \overline{w}_i &= 1/2(w_{il}+w_{ir}), \\ \langle w_1^2\rangle &= 1/2(w_{1l}^2+w_{1r}^2)=((\overline{w}_1)^2+1/4(\Delta w_1)^2), \\ w_1^* &= \sqrt{w_{1l}w_{1r}}=\sqrt{(\overline{w}_1)^2-1/4(\Delta w_1)^2}. \end{aligned} \tag{A15}$$

The eigenvalues of the linearized Jacobian are given by

$$\begin{aligned} \tilde{\lambda}_{1,2} &= \overline{w}_2/\overline{w}_1 \mp \phi/w_1^{*2} = \tilde{\mathbf{v}}_x \mp \tilde{c}_{ax}, \\ \tilde{\lambda}_{3,4} &= \overline{w}_2/\overline{w}_1 \mp \sqrt{g\langle w_1^2\rangle+\phi^2/w_1^{*4}} = \tilde{\mathbf{v}}_x \mp \tilde{c}_{gx}, \end{aligned} \tag{A16}$$

with $\tilde{\mathbf{v}}_x=\overline{w}_2/\overline{w}_1$, $\tilde{c}_{ax}=\phi/w_1^{*2}$, and $\tilde{c}_{gx}=\sqrt{g\langle w_1^2\rangle+\phi^2/w_1^{*4}}$. The right and left eigenvectors are given by

$$\tilde{\mathsf{R}}_c=\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & \overline{w}_2/\overline{w}_1-\tilde{c}_{gx} & \overline{w}_2/\overline{w}_1+\tilde{c}_{gx} \\ 1 & -1 & \overline{w}_3/\overline{w}_1 & \overline{w}_3/\overline{w}_1 \\ 1 & 1 & \overline{w}_4/\overline{w}_1 & \overline{w}_4/\overline{w}_1 \end{bmatrix} \tag{A17}$$

and

$$\tilde{\mathsf{L}}_c=\begin{bmatrix} -(\overline{w}_3+\overline{w}_4)/(2\overline{w}_1) & 0 & 1/2 & 1/2 \\ -(-\overline{w}_3+\overline{w}_4)/(2\overline{w}_1) & 0 & -1/2 & 1/2 \\ (\overline{w}_2/\overline{w}_1+\tilde{c}_{gx})/(2\tilde{c}_{gx}) & -1/(2\tilde{c}_{gx}) & 0 & 0 \\ -(\overline{w}_2/\overline{w}_1-\tilde{c}_{gx})/(2\tilde{c}_{gx}) & 1/(2\tilde{c}_{gx}) & 0 & 0 \end{bmatrix}. \tag{A18}$$

Our choice of parameter vector (A9) is justified *a posteriori* by the fact that we obtain expressions for eigenvalues (A16) and eigenvectors (A17) and (A18) for the linearized Jacobian that are formally very similar to the expressions for the eigenvalues (6) and eigenvectors (A7) and (A8) of the exact Jacobian, with eigenvectors that are linearly independent. It can be seen that the eigenvalues and eigenvectors are a function of the arithmetic averages $\overline{w_i}$ of the components of the parameter vectors in the left and right states, and of other types of averages of w_{1l} and w_{1r}, as defined in (A15).

The expressions for the eigenvalues and eigenvectors of the Roe-type linearization are only slightly more complicated than the expressions for the exact Jacobian. The cost of using the Roe-type linearization in order to obtain more accurate discontinuities is thus not substantially larger than the cost of using the simple arithmetic linearization (A3).

4. Consistent linearization

For some hyperbolic conservation laws, including the Euler equations, a unique parameter vector can be found such that the Roe-type linearization is only a function of the components of the arithmetic average of the parameter vectors in the left and right states. In this case the linearized Jacobian is equal to the exact Jacobian evaluated in a state defined by this arithmetic average.[34] Roe calls such a linearized Jacobian consistent.[34] A consistent linearization is obtained when a parameter vector can be found which is such that both the conserved variables and the flux function are quadratic functions of the parameter vector.

For equations with different types of nonlinearity, which include the shallow water equations, the MHD equations,[23,38,41] and also the SMHD equations, it is not possible to find a parameter vector that leads to a linearization that is consistent in the sense of Roe.[34] A linearized Jacobian that satisfies Roe's condition (A4) can still be found, but this linearized Jacobian is necessarily also a function of other types of averages of the parameter vector components than arithmetic averages. Balsara has given a general procedure to find such a Roe-type linearization,[38–40] and we have used this procedure in this appendix to derive a Roe-type average for the SMHD equations.

It is intuitively clear that when a linearization cannot be found that is consistent in the sense of Roe, the parameter vector should be chosen such that the resulting expressions for eigenvalues and eigenvectors are formally close to the expressions for the exact Jacobian.[38] This is of pertinent practical concern when eigenvectors can become degenerate, as is the case for the MHD equations.[38] In the case of SMHD degenerate eigenvectors are of no concern because eigenvalues cannot coincide. Our choice of parameter vector (A9) leads to resulting expressions for eigenvalues and eigenvectors of the linearized Jacobian that are formally close to the expressions for the exact Jacobian.

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