# Overcompressive shocks in two-dimensional and three-dimensional magnetohydrodynamic bow shock flows

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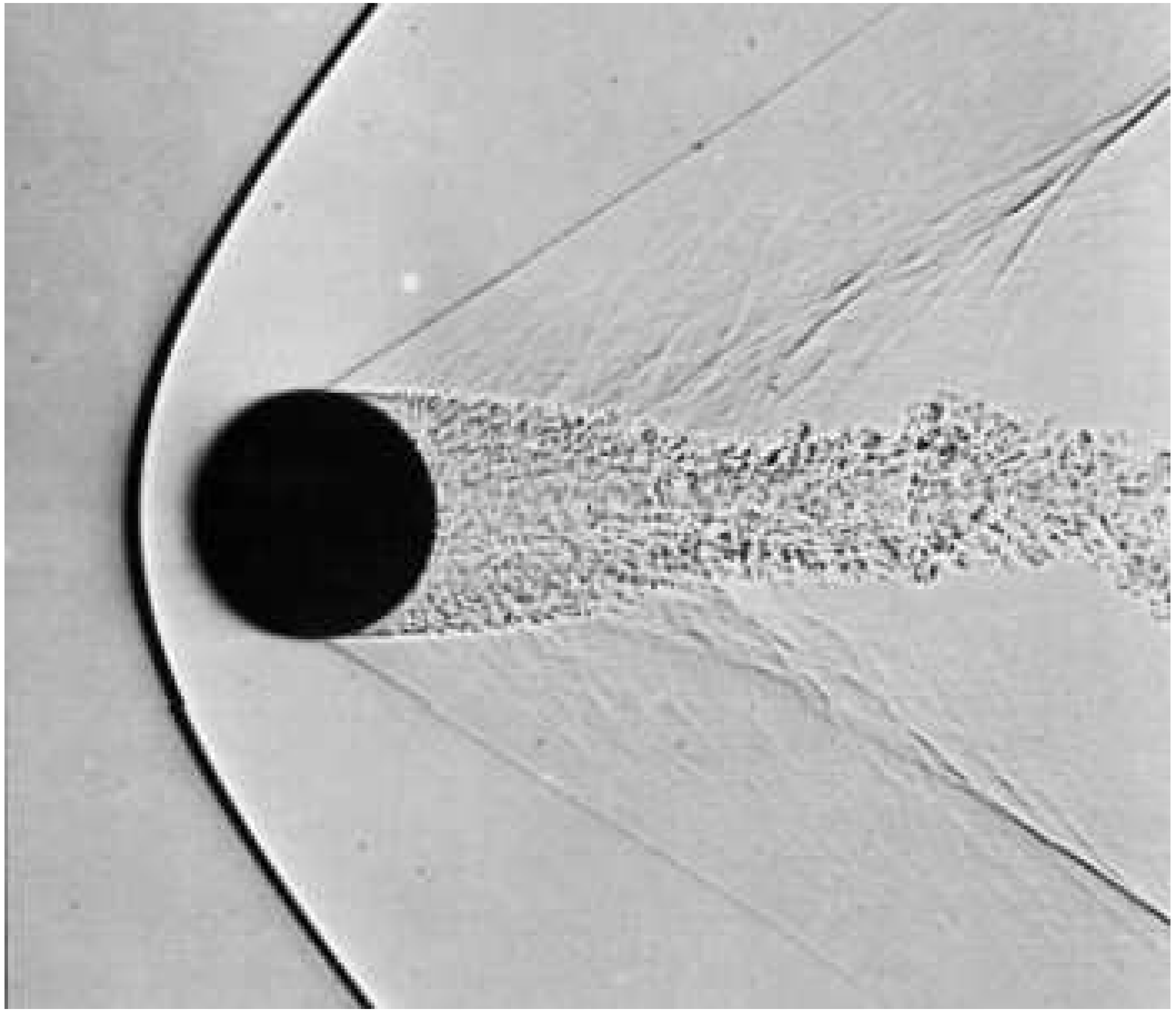
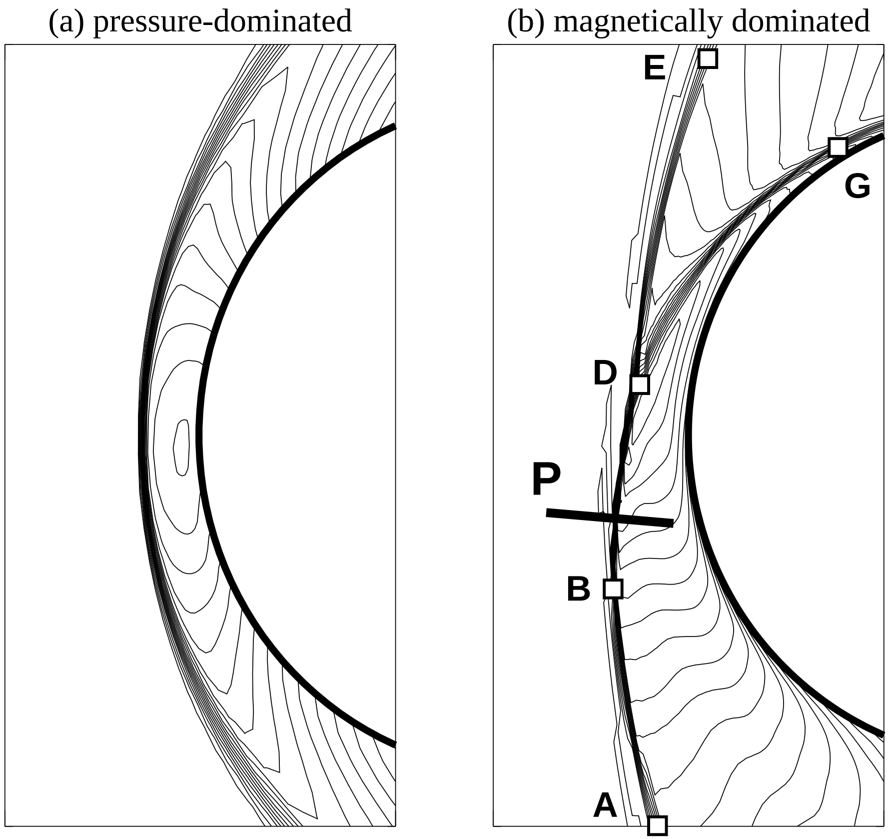


- supersonic flow of air over sphere (M=1.53)
- regular bow shock
- (An album of fluid motion, Van Dyke)

# 3D MHD bow shock fbws over conducting sphere

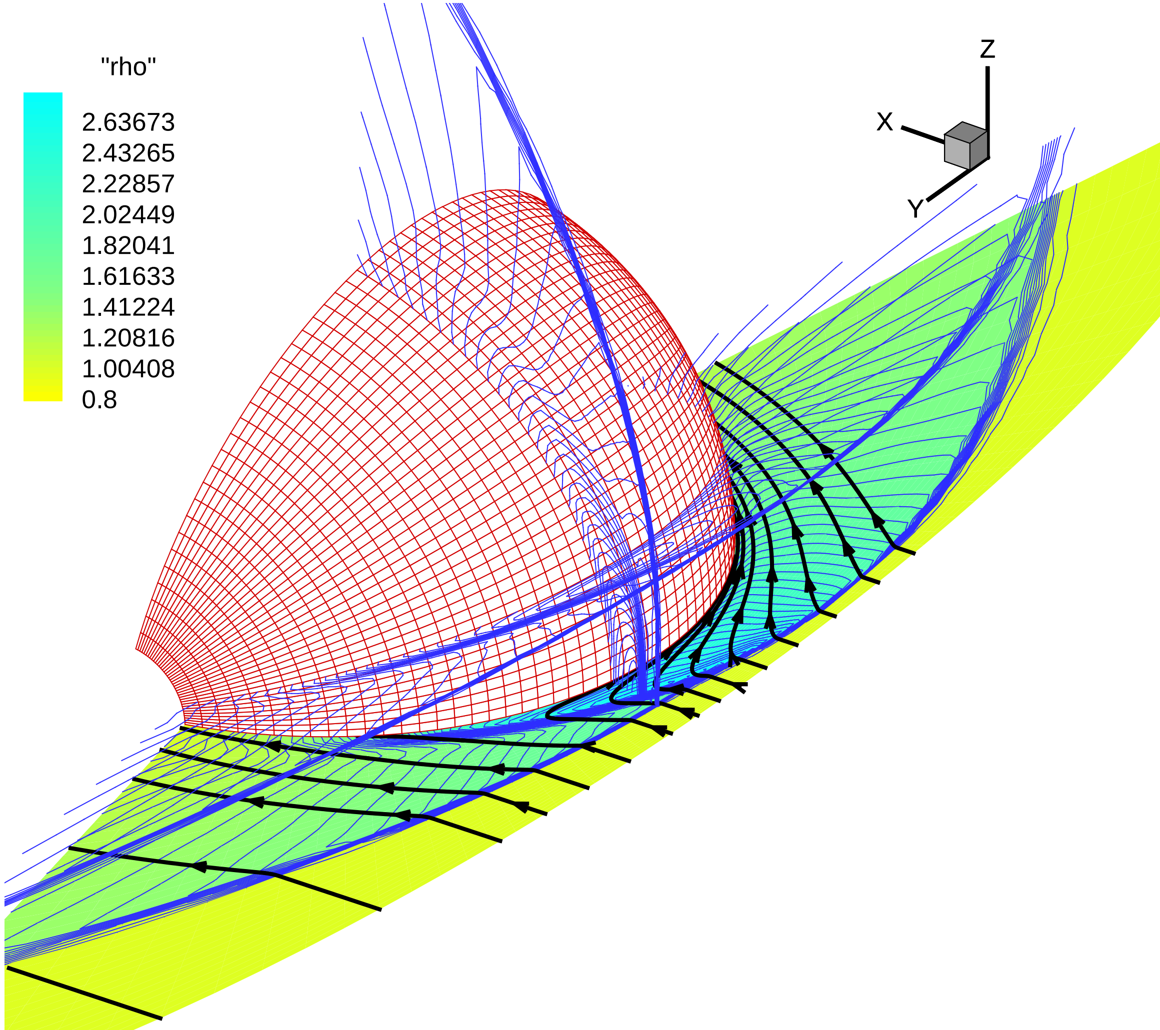



B weak

pressure-dominated

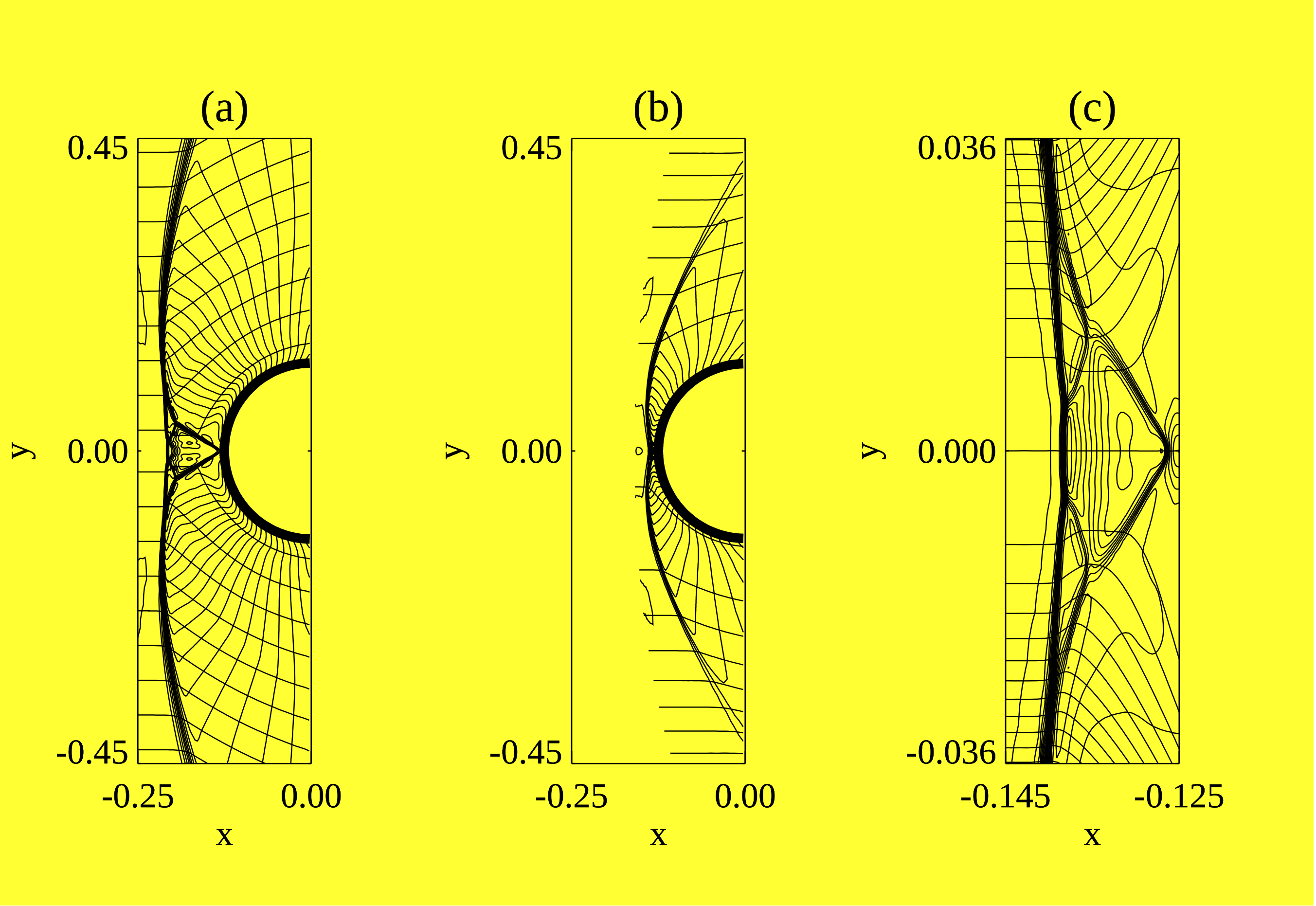
B strong

magnetically dominated




- strong upstream B
- dual-front bow shock
- overcompressive shock segments

## Symmetric 2D MHD bow shock flows over cylinder




2D flow over cylinder, and axisymmetric flow over sphere

- overcompressive shocks, compound shocks

## Conservative form ideal MHD equations

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho\vec{v} \\ \frac{\rho v^2}{2} + \rho e + \frac{B^2}{2} \\ \vec{B} \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\vec{v} \\ \rho\vec{v}\vec{v} + \left(p + \frac{B^2}{2}\right)\vec{I} - \vec{B}\vec{B} \\ \left(\frac{\rho v^2}{2} + \rho e + p\right)\vec{v} - (\vec{v}\times\vec{B})\times\vec{B} \\ \vec{v}\vec{B} - \vec{B}\vec{v} \end{bmatrix} = 0$$

# Mathematical nature MHD equations

- system of nonlinear hyperbolic conservation laws:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \qquad \frac{\partial U}{\partial t} + \mathsf{A}(U) \cdot \frac{\partial U}{\partial x} = 0$$

the system is a hyperbolic system of equations

$$\Updownarrow$$

$\mathsf{A}(U)$ has $n$ real eigenvalues and a complete set of eigenvectors $\forall U$

$$\Updownarrow$$

the system has $n$ real characteristic curves

$$\Rightarrow \mathsf{A}(U) = \mathsf{R}(U) \cdot \Lambda(U) \cdot \mathsf{L}(U)$$

- nonlinear hyperbolic system: $\Rightarrow$ waves, shocks

- HD (Euler): ($n = 5$)
  - $\lambda = u, u, u, u + c, u - c$
  - one nonlinear wave mode
  - isotropic
  - one type of shock

- MHD: ($n = 8$)
  - $\lambda = u, u, u + c_f, u - c_f,$
    $u + c_A, u - c_A, u + c_s, u - c_s$
  - three wave modes: fast, Alfven, slow
  - strongly anisotropic
  - three types of shocks

- hyperbolic theory of MHD:
  - non-strictly hyperbolic
  - non-convex $\Rightarrow$ compound shocks
  - rotationally invariant $\Rightarrow$ instability of (overcompressive) intermediate shocks

## Overview

1) MHD waves and shocks

2) Topology of bow shock flows

3) Non-convexity: compound shocks

4) Overcompressive shocks

5) ‘Every constant state is bordered by a simple wave’

# 1) MHD waves and shocks

## MHD waves

- three anisotropic waves

$$c_{fx} \geq c_{Ax} \geq c_{sx}$$

- non-strictly hyperbolic

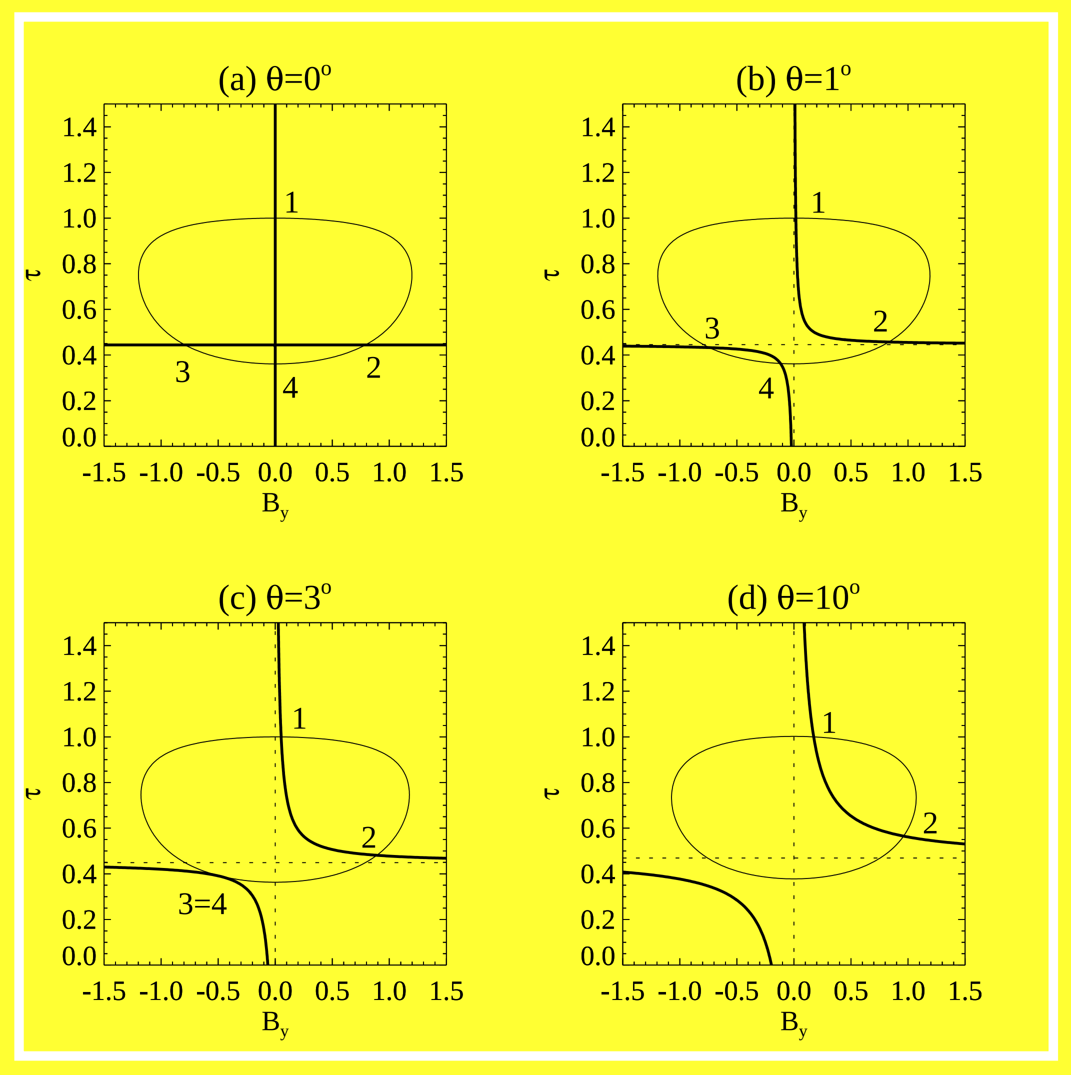
  - $x$ along $\vec{B}$: $c_{fx} = c_{Ax}$ or $c_{Ax} = c_{sx}$

    (exceptionally $c_{fx} = c_{Ax} = c_{sx}$)

  - $x$ perpendicular to $\vec{B}$: $c_{Ax} = c_{sx} = 0$

# MHD shock types

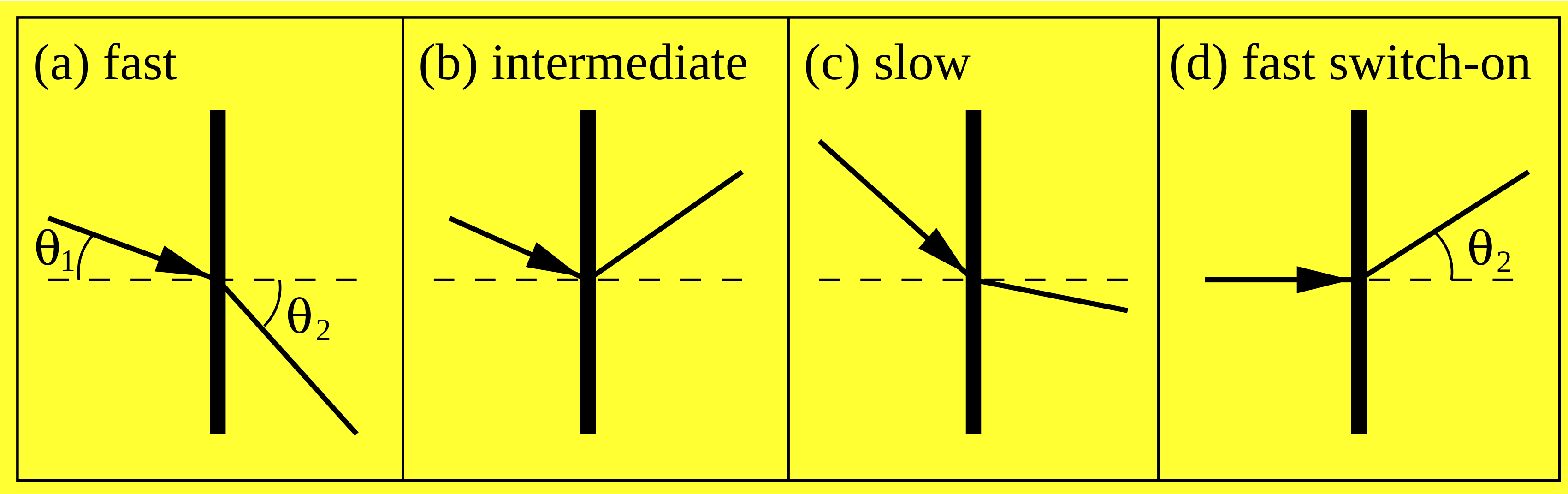
- MHD Rankine-Hugoniot relations
- up to 4 fixed points in phase space diagram



- 4 possible positions of flow speed in direction $x$:

$$\boxed{\boxed{1} \geq c_{fx} \geq \boxed{2} \geq c_{Ax} \geq \boxed{3} \geq c_{sx} \geq \boxed{4}}$$

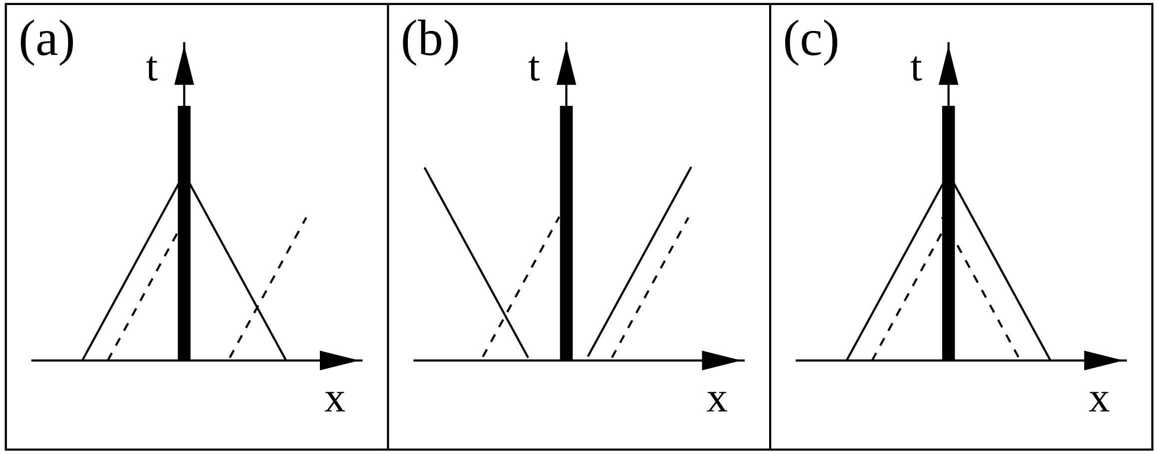
$\Rightarrow$ three types of MHD shocks:




1–2     1–3, 1–4, 2–3, 2–4     3–4     1–2=3

- 1–3, 2–4 and 1–4 intermediate shocks are overcompressive

$\Rightarrow$ instability can arise, see later




(slope of $i$th characteristic in $xt$ plane = $\lambda_i$ = $v_n - c_{fn}$ or $v_n - c_{An}$ or . . . )

- (a) Lax shock: $2n - 1$ characteristics impinging
- (b) undercompressive shock
- (c) overcompressive shock

- three wave speeds depend on direction

$\Rightarrow$ three Mach numbers too:

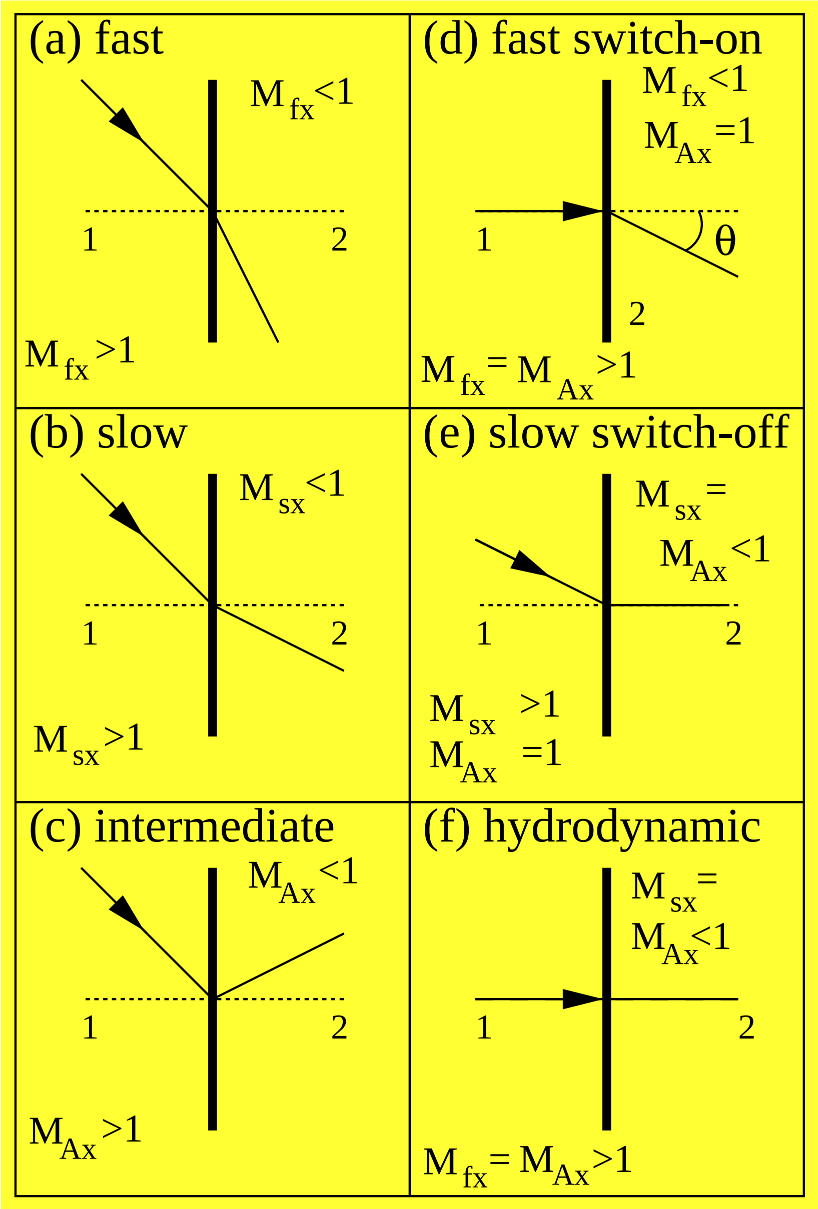
- $M_{fx} = \dfrac{|v_x|}{c_{fx}}$
- $M_{Ax} = \dfrac{|v_x|}{c_{Ax}}$
- $M_{sx} = \dfrac{|v_x|}{c_{sx}}$

- fast switch-on shocks = intrinsic magnetic effect
- only occur when:

1) $B_1^2 > \gamma p_1$

2) $B_1^2 > \rho_1 v_{x,1}^2 \dfrac{\gamma - 1}{\gamma(1-\beta_1)+1}$

$\Rightarrow$ upstream flow is *magnetically dominated*



## Numerical simulation technique

- MHD is hyperbolic, like Euler $\Rightarrow$ use CFD techniques
- finite volume, structured grid
- second order in space (limited slope reconstruction)
- second order in time (explicit two-stage Runge-Kutta)
- parallel on 32 processors using message passing (MPI) (up to 1 000 000 cells)
- $\nabla \cdot \vec{B}$ constraint:

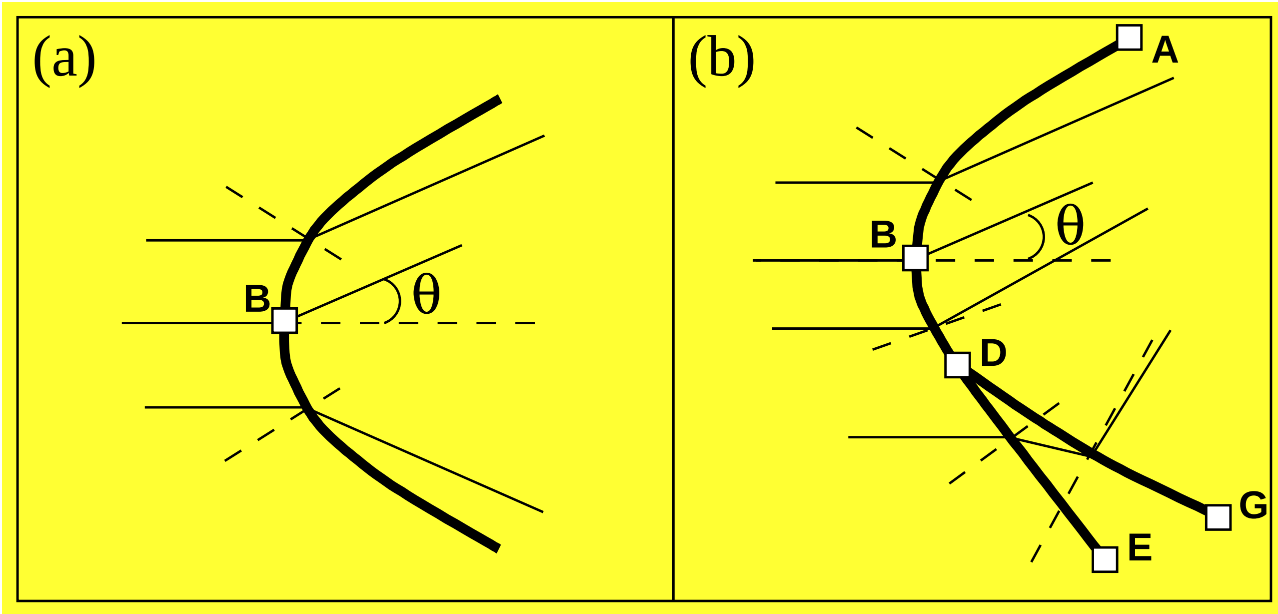
  MHD has singular Jacobians, which leads to numerical instabilities

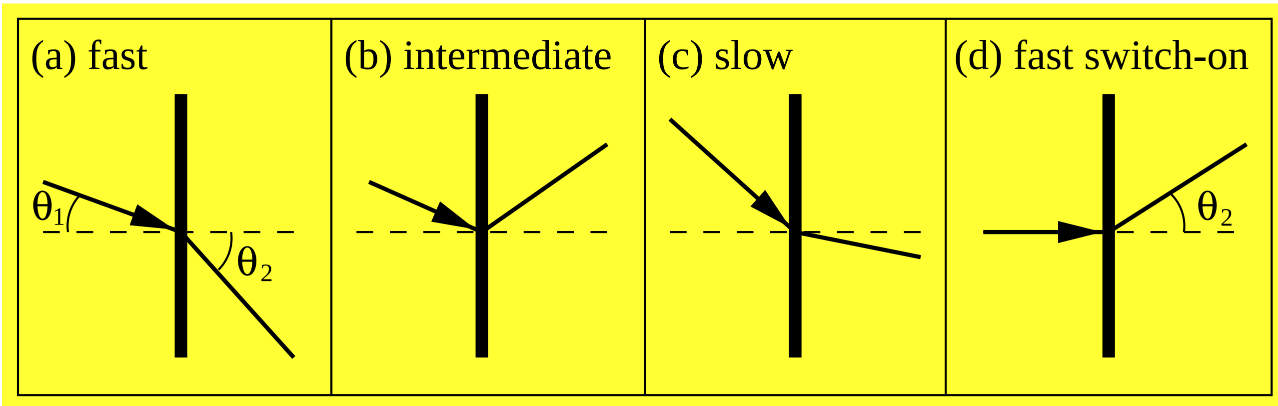
  $\Rightarrow$ add source term

  - makes equations symmetrizable
  - makes equations Galilean invariant
  - makes numerical scheme stable

# 2) Topology of bow shock flows

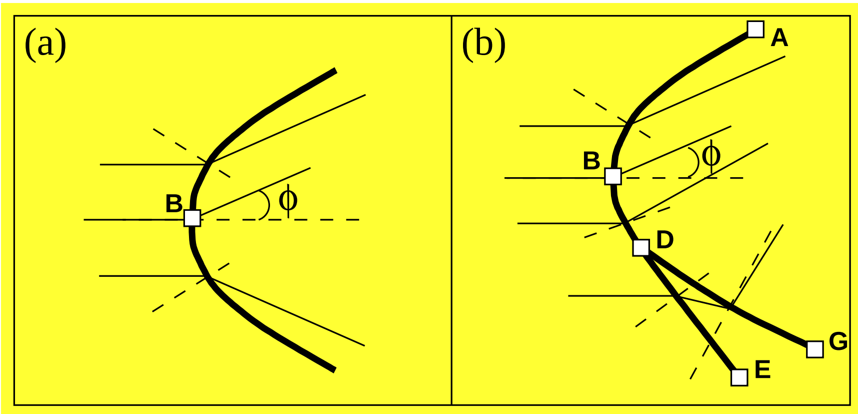
B weak    B strong

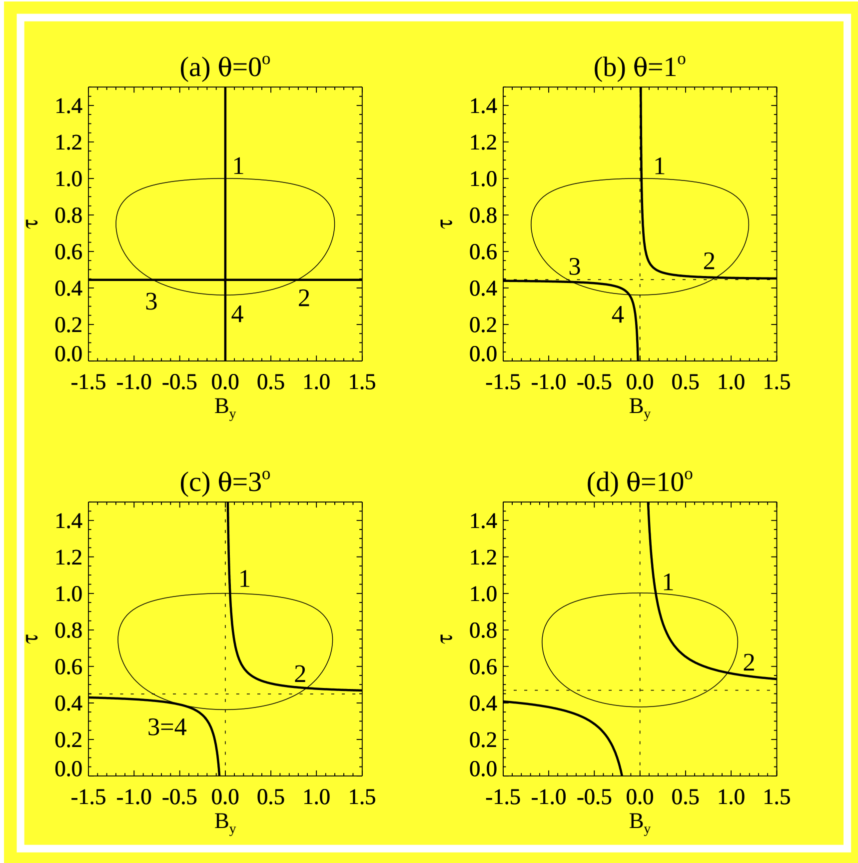


# 2) Topology of bow shock flows

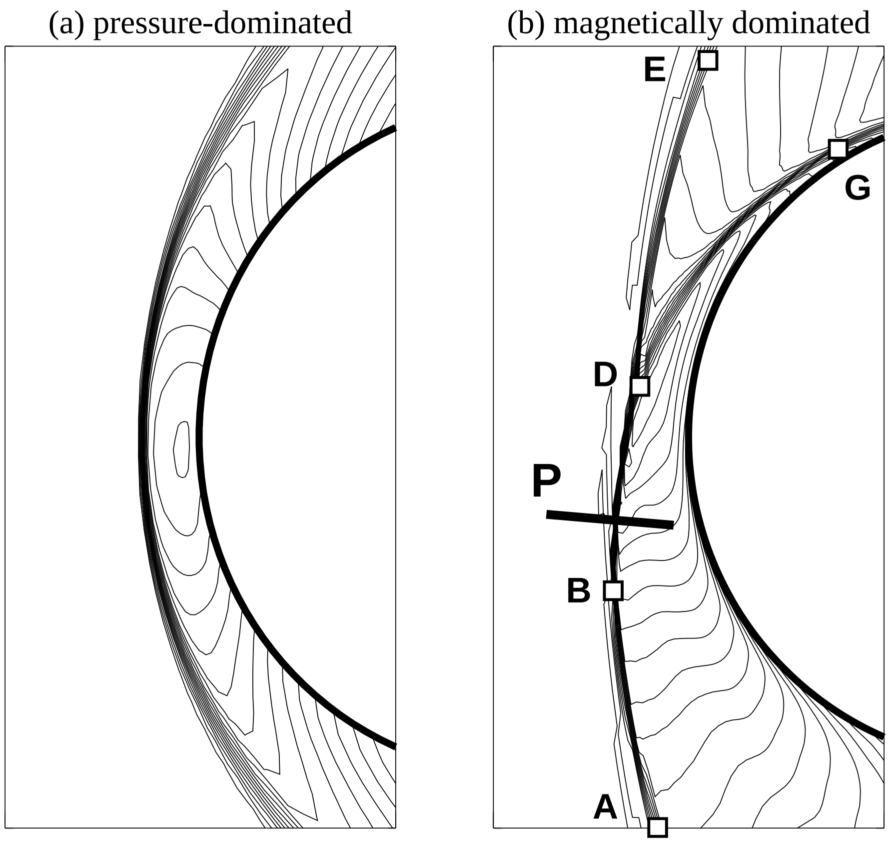
- why does the shock
  split up at point D?

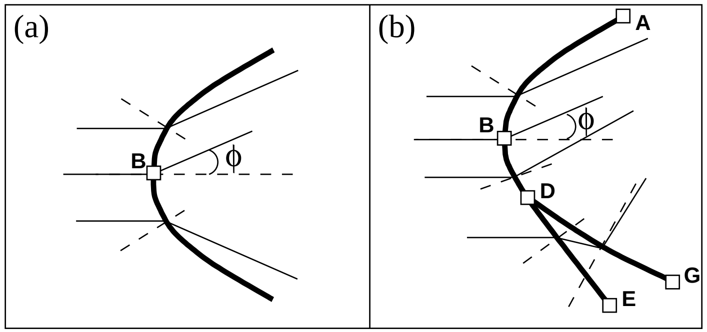


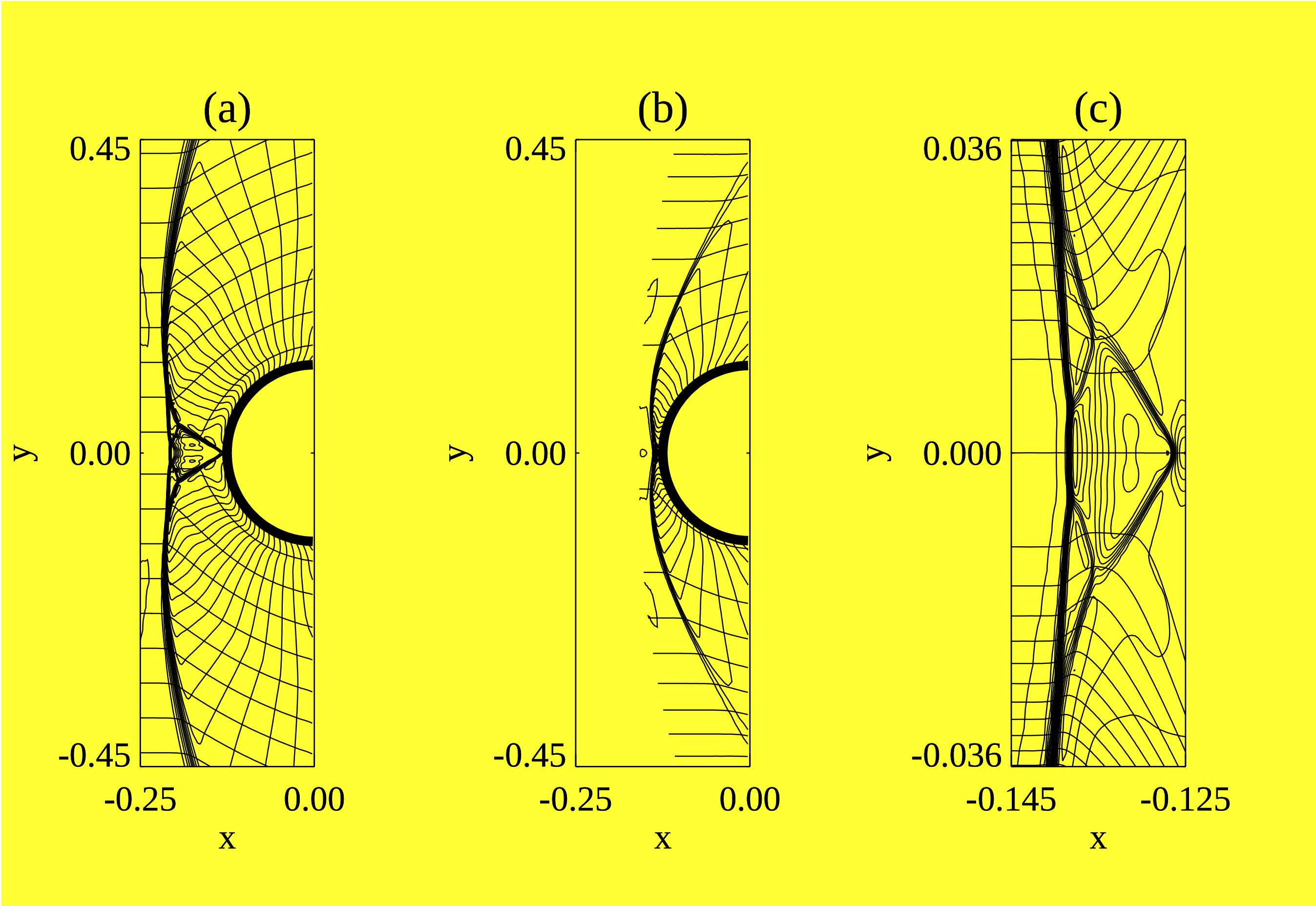

- A–B, D–E: fast 1–2
- B–D: intermediate 1–3
- D–G: intermediate 2–4 evolving into slow 3–4

# this topology in 3D MHD bow shock flows



this topology also in 2D MHD bow shock flows




2D flow over cylinder, and axisymmetric flow over sphere

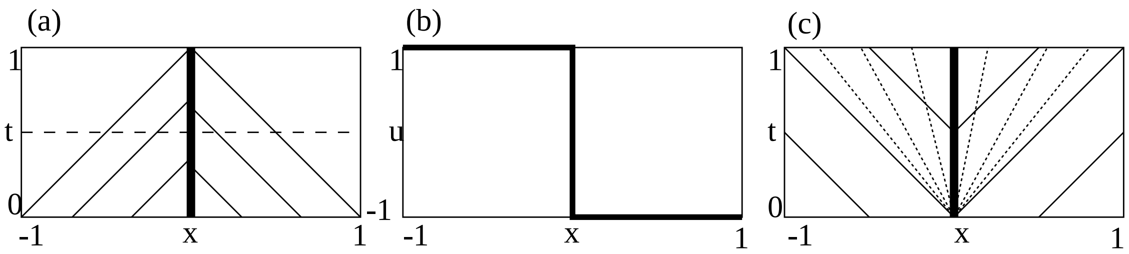
# 3) Non-convexity: compound shocks

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} = 0$$

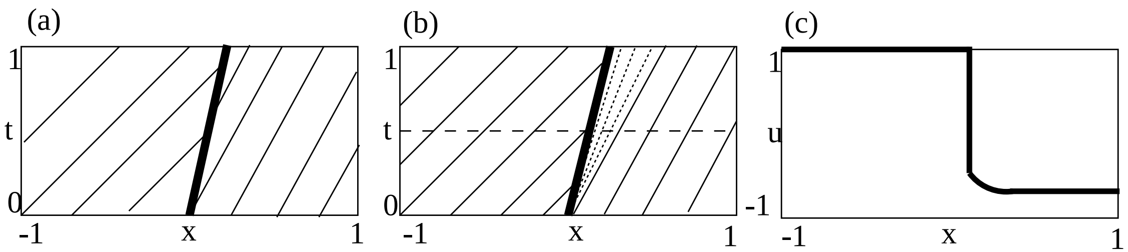
$f$ is convex $\Leftrightarrow$ $f'$ is monotone $\Leftrightarrow$ $f''$ does not change sign

- f is convex ($f = u^2$, Burgers, Euler) $\Rightarrow$

$$f'(u_1) \geq s = (f(u_2) - f(u_1))/(u_2 - u_1) \geq f'(u_2)$$




- f is non-convex ($f = u^3$) $\Rightarrow f'(u_1) \geq f'(u_2) \geq s$




compound shock: on rarefaction side: sonic ($M = 1$), tangent characteristic

# 3) Non-convexity: compound shocks

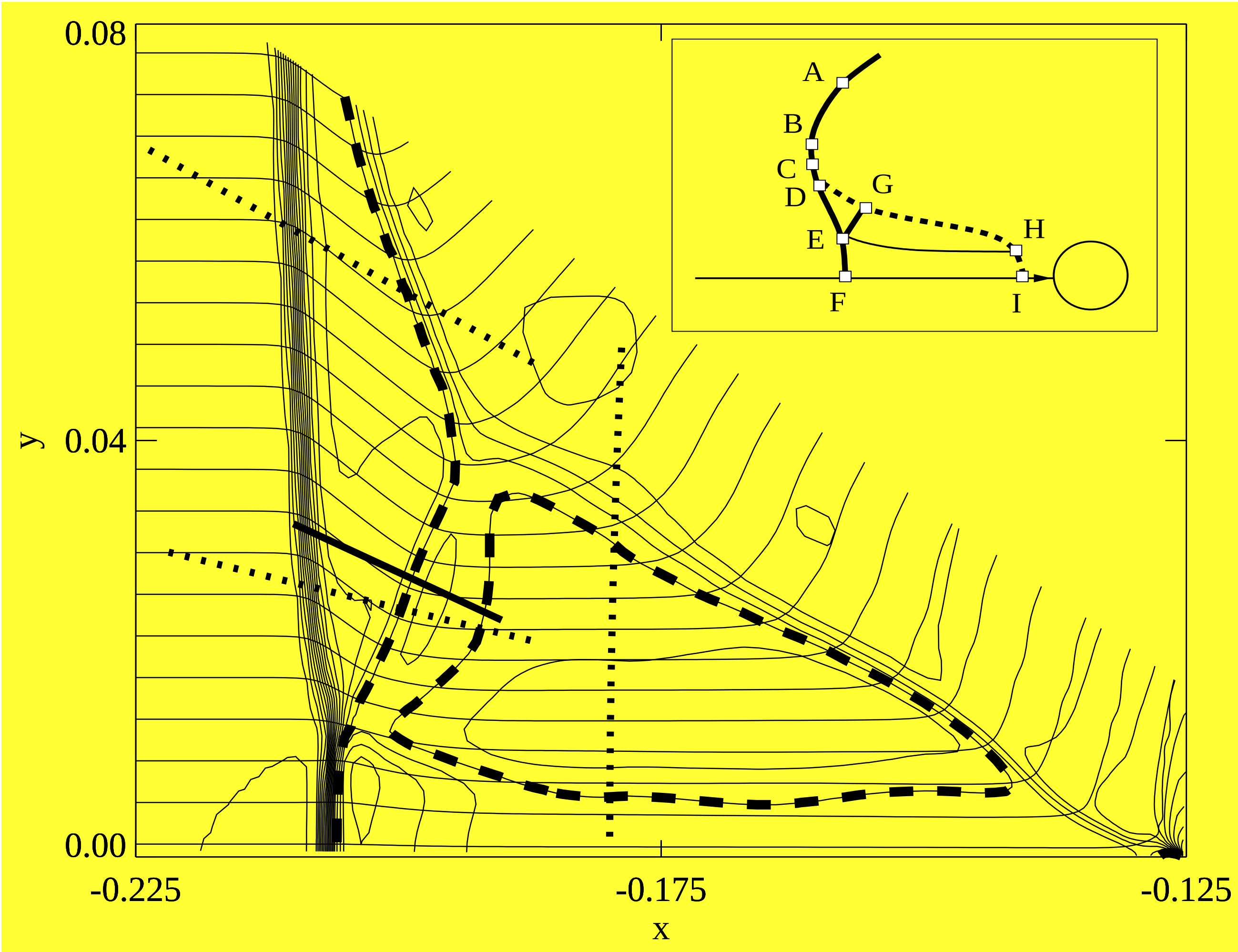
$$\boxed{\boxed{1} \geq c_{fx} \geq \boxed{2} \geq c_{Ax} \geq \boxed{3} \geq c_{sx} \geq \boxed{4}}$$

- $c_f$, $c_s$ non-convex :

  - 1=2–3 and 2–3=4 sonic shocks in compound shocks

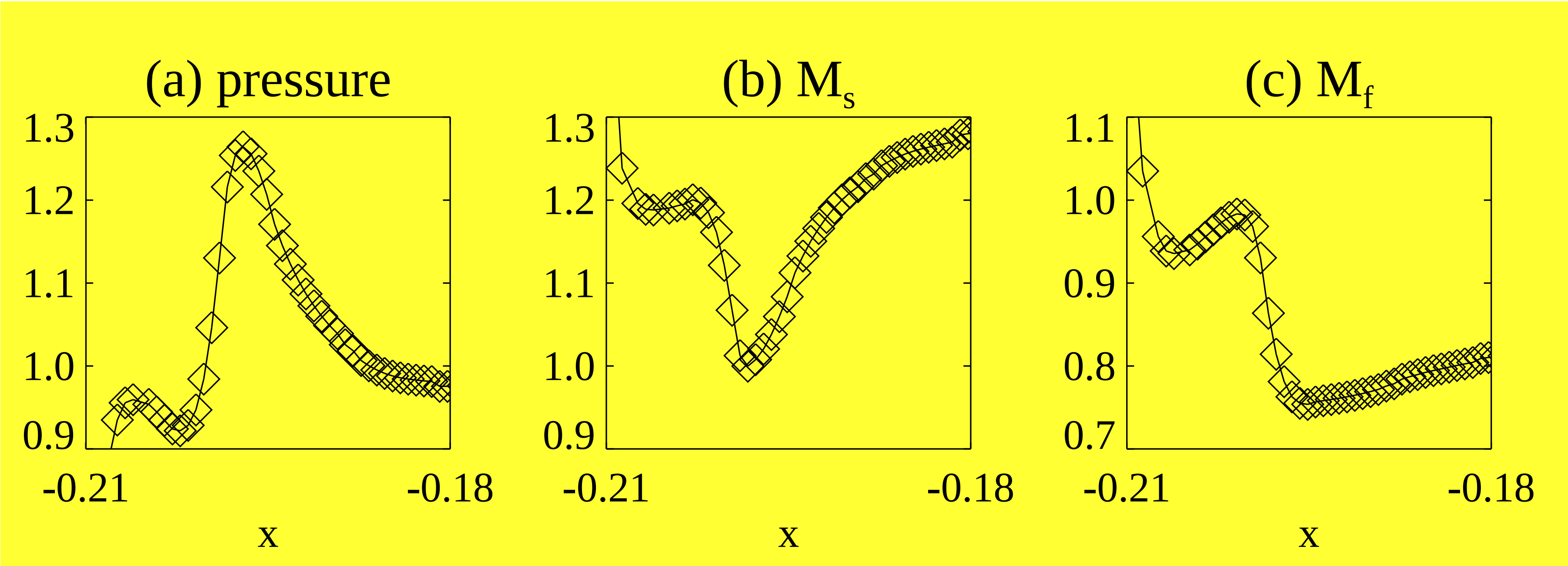
  - 1=2–3=4 double compound shock

## compound shocks in 2D flow




*Magnetic field lines and $M_A$ contour lines*

## 3) Non-convexity: compound shocks



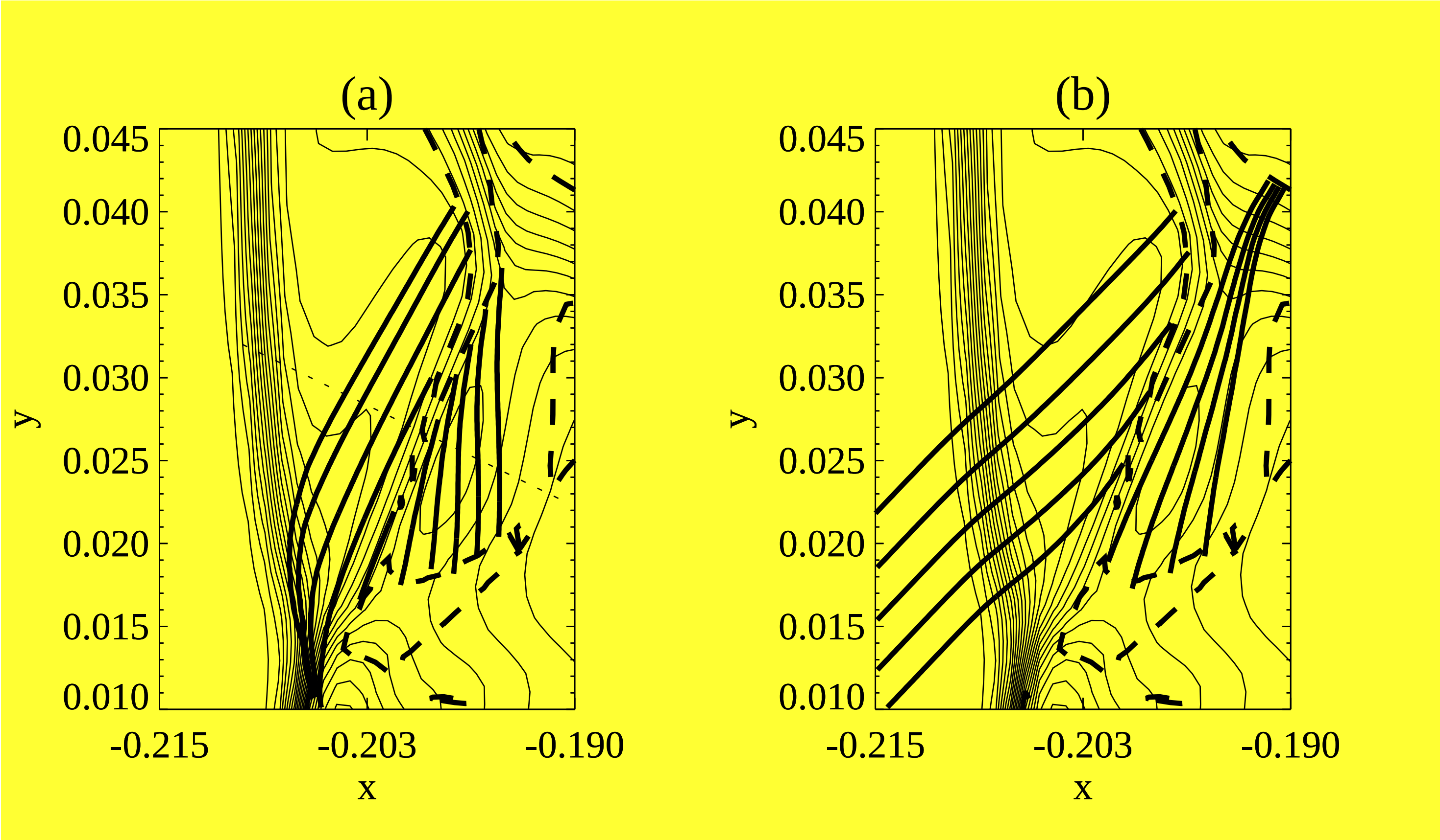

*Cut along solid line*

- E-G shock is preceded and followed by rarefaction regions
- $M_f = 1$ where upstream (left) rarefaction is attached to shock
- $M_s = 1$ where downstream (right) rarefaction is attached to shock

$\Rightarrow$ E-G: 1=2–3=4 shock

$\Rightarrow$ stationary double compound shock!

# 3) Non-convexity: compound shocks

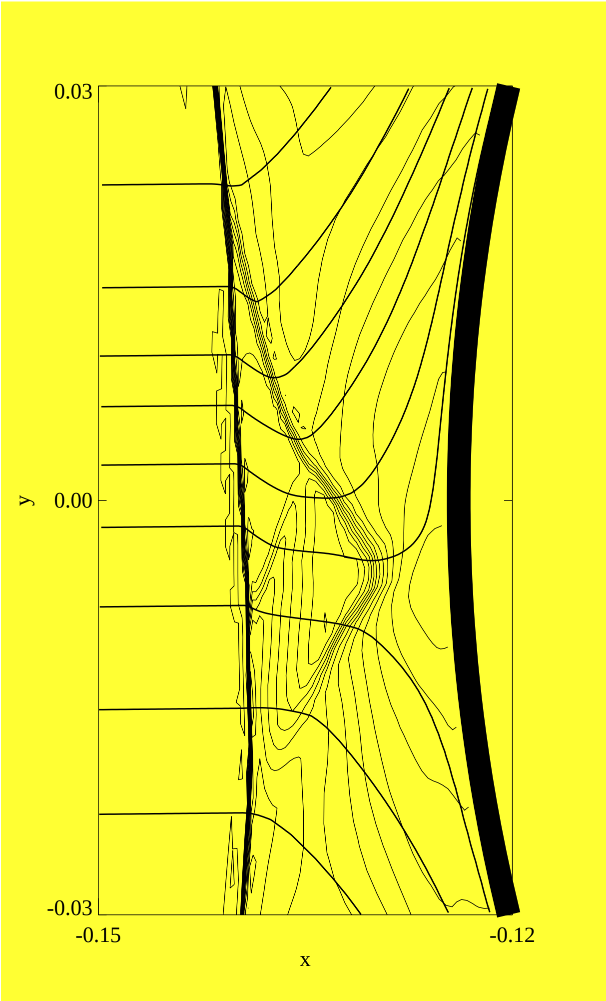



*Two families of characteristics near E-G shock*

$\Rightarrow$ characteristic analysis $\Rightarrow$ stationary double compound shock!

$\Rightarrow$ $xy$ mathematical equivalent of known $xt$ double compound shocks

## compound shocks in 3D flow

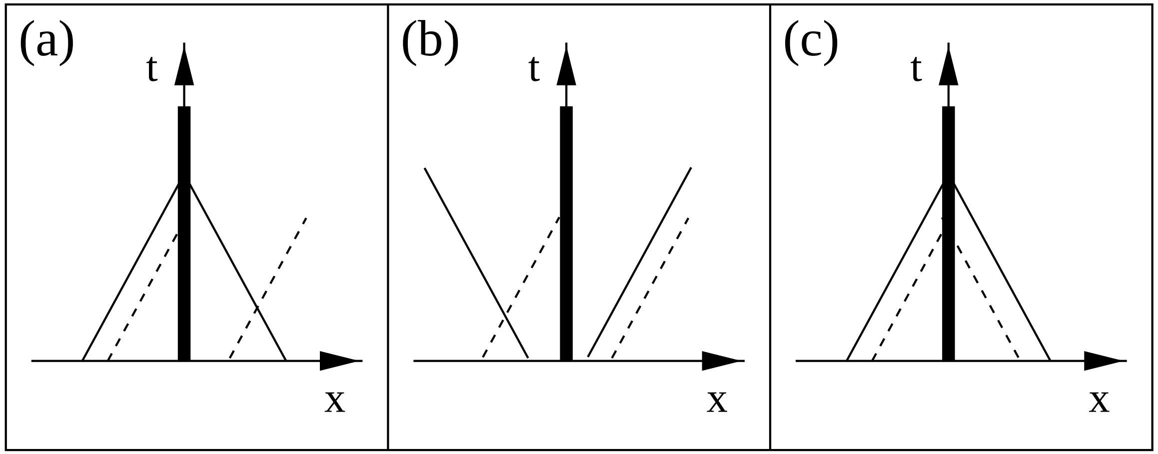



- conclusion: effects of non-convexity of MHD in 2D and 3D flows

# 4) Overcompressive shocks

$2 \times 2$ system ($n = 2$)

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix} = 0$$

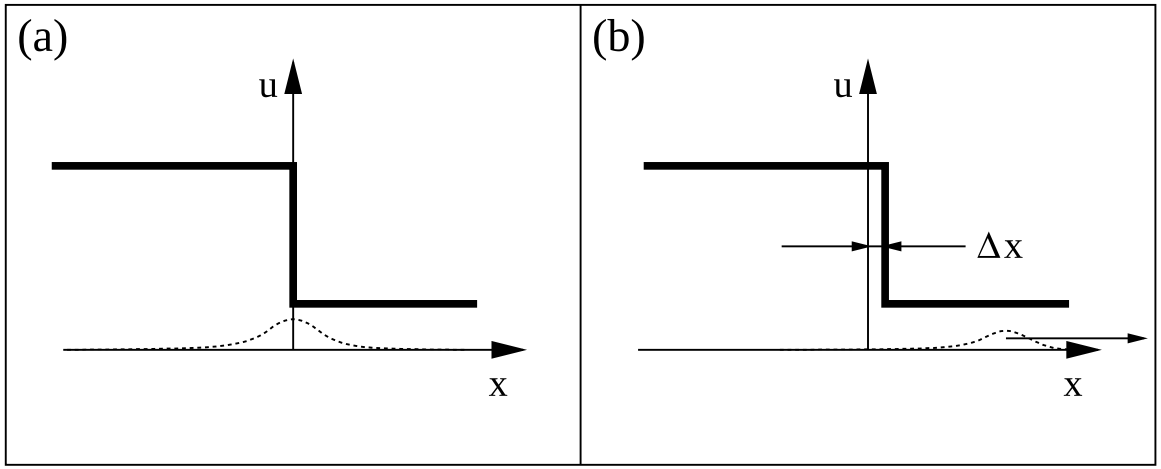



- (a) Lax shock: $2n - 1$ characteristics impinging
- (b) undercompressive shock
- (c) overcompressive shock

## Strictly hyperbolic system

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} v \\ v^2/u + c^2\, u \end{bmatrix} = 0$$

- isothermal Euler
- $\lambda = \frac{v}{u} \pm c$
- perturbation: $\int u\, dx$ and $\int v\, dx$ remain constant
- shock is displaced, wave is sent out



## *4) Overcompressive shocks*

- ideal (hyperbolic) system:

  - Lax shock is stable against perturbations

- dissipative (parabolic) system:

  - unique viscous profile connects unstable node with saddle

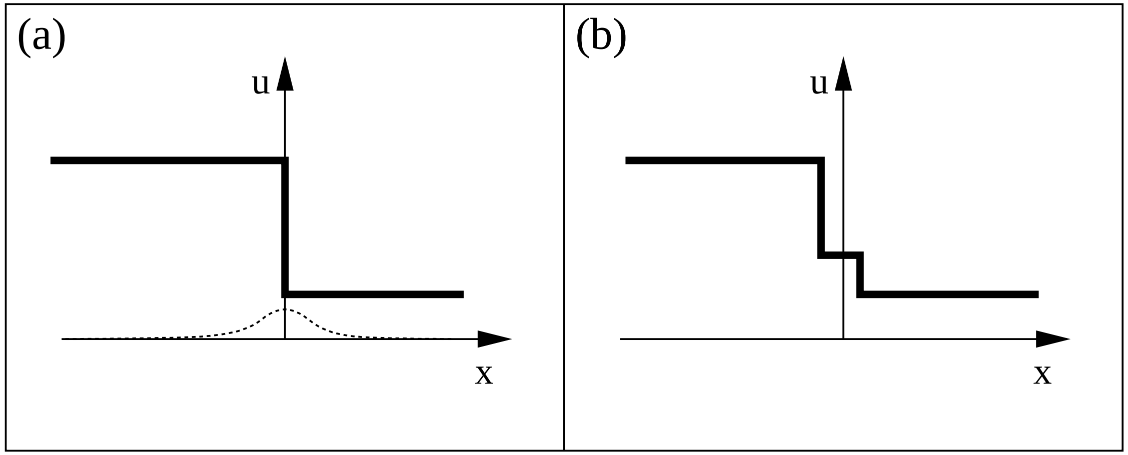
  - Lax shock has stable dissipative (viscous) profile

$\Rightarrow$ vanishing viscosity limit:

  - shock admissibility is the same in ideal and dissipative system

## System with overcompressive shocks: Type A, non-rotational

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} a\,u^2/2 + b\,v \\ v^2/2 \end{bmatrix} = 0$$




- perturbation: $\int u\,dx$ and $\int v\,dx$ remain constant
- shock can be displaced, wave CANNOT be sent out: problem!

## *4) Overcompressive shocks*

- ideal (hyperbolic) system:

  - overcompressive shock splits under generic perturbation = unstable

  - BUT: resulting shocks have same speed as original shock = L1 stable

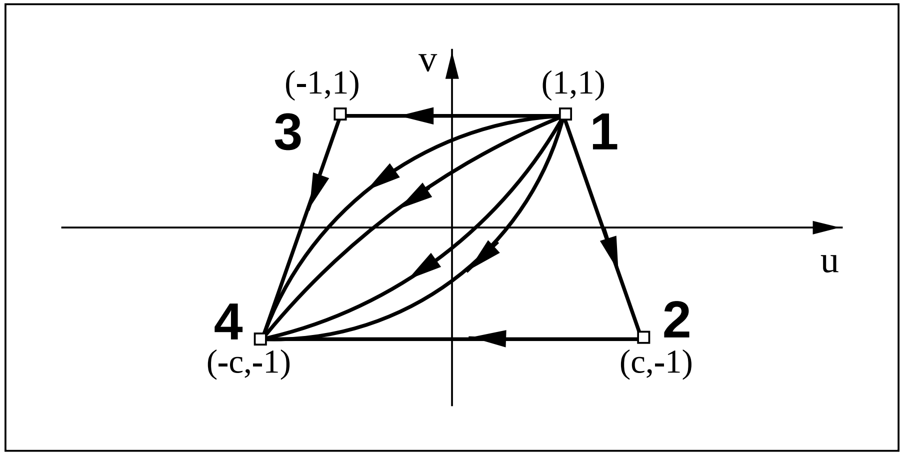
- dissipative (parabolic) system:

  - family of viscous profiles connecting unstable node with stable node

  - upon perturbation, viscous shock is translated, and different profile is assumed

  - viscous shock is stable under arbitrary perturbation

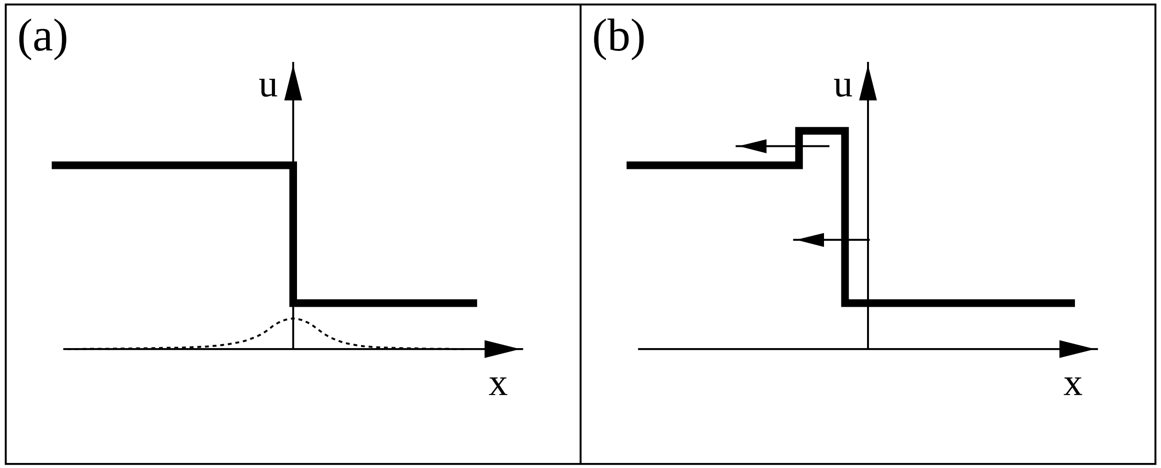
$\Rightarrow$ ideal $\neq$ dissipative !! (vanishing viscosity limit?)



## System with overcompressive shocks: Type B, rotational

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} u\,(u^2+v^2) \\ v\,(u^2+v^2) \end{bmatrix} = 0$$

- rotational invariance: $F(U) = \Phi(||U||^2)\,U$
- perturbation: $\int u\,dx$ and $\int v\,dx$ remain constant
- shock can be displaced, wave CANNOT be sent out: problem!



## 4) Overcompressive shocks

- ideal (hyperbolic) system:

  - overcompressive shock splits under generic perturbation = unstable

  - AND: resulting shocks have speed different from original shock = not L1 stable

  - generically unstable

- dissipative (parabolic) system:

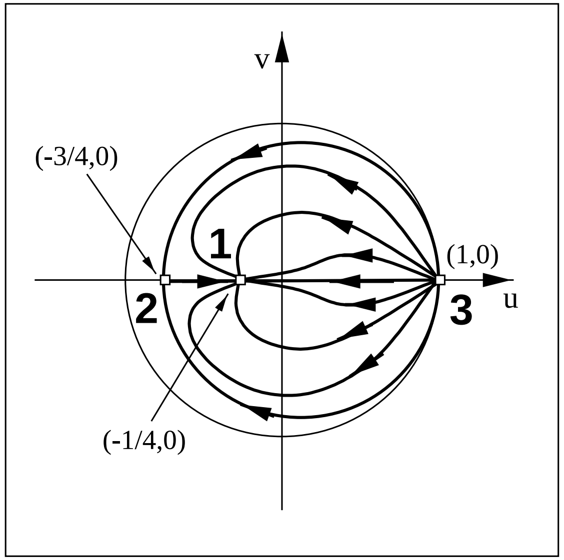
  - family of viscous profiles connecting unstable node with stable node

  - upon perturbation, viscous shock is translated, and different profile is assumed

  - BUT: if $\int v\,dx > M_{crit}$, shock disintegrates

  - BUT: $M_{crit}$ vanishes for vanishing dissipation

  $\Rightarrow$ viscous shock is conditionally stable




$\Rightarrow$ ideal $\neq$ dissipative !! (vanishing viscosity limit???)

# Overcompressive shocks in MHD

$$\boxed{1} \geq c_{fx} \geq \boxed{2} \geq c_{Ax} \geq \boxed{3} \geq c_{sx} \geq \boxed{4}$$

- planar MHD (2D)

  $B_z \equiv v_z \equiv 0$, $c_A$ drops out

  - 1–3 and 2–4 NOT overcompressive, but Lax

  - 1–4 overcompressive (Type A)

- full MHD

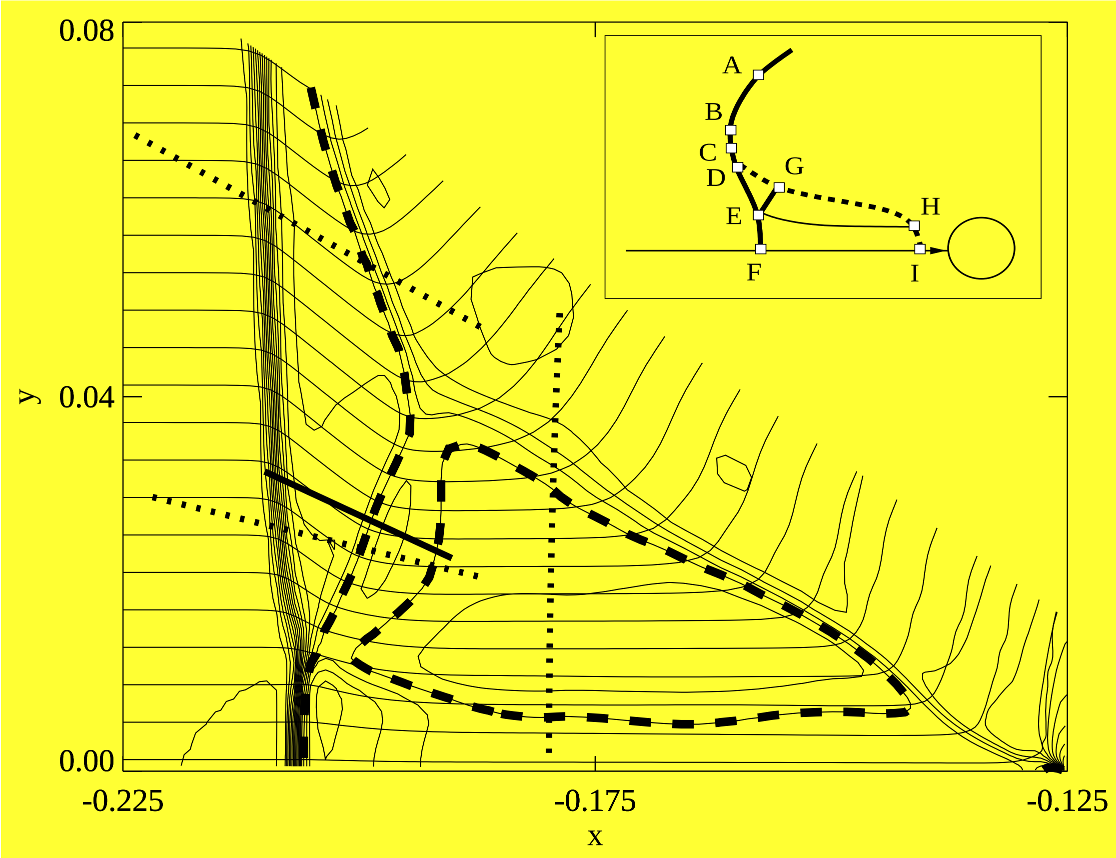
  - 1–3 and 2–4 overcompressive (Type B)

  - 1–4 overcompressive (Type B)

  - complication: bifurcation in existence of viscous profiles as function of the dissipative parameters

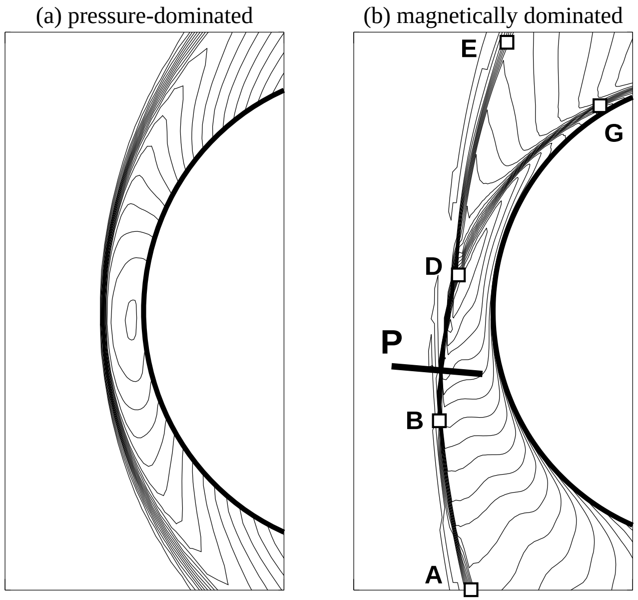
⇒ intermediate shocks (1–3, 2–4 and 1–4) can arise in MHD flows for small dissipation

## 2D bow shock: intermediate shocks

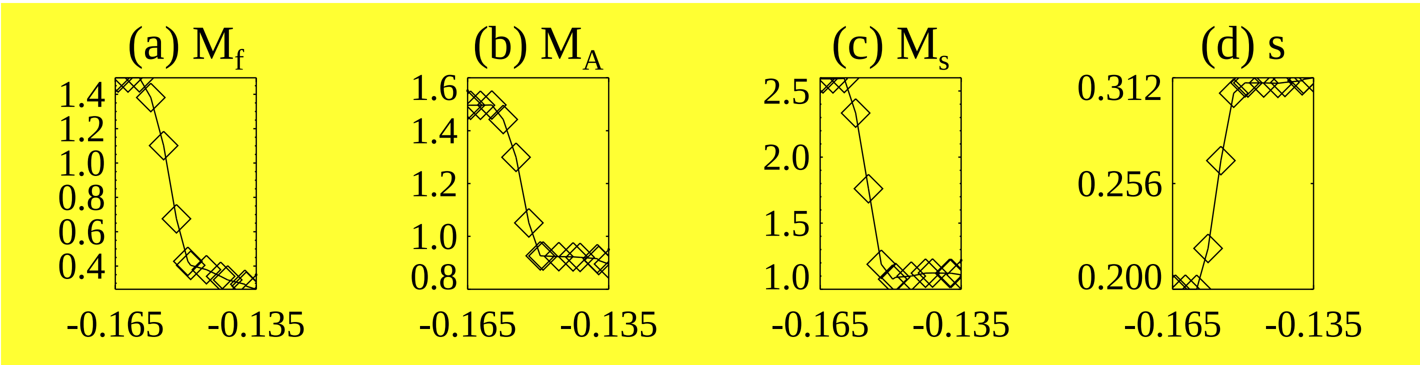



| | | | |
|---|---|---|---|
| D-E: | fast shock (almost switch-on) | E-H: | tangential discontinuity |
| E-F: | 1–4 intermediate shock | E-G: | intermediate shock |
| at F: | 1–4 hydrodynamic shock | D-G-H-I: | 2–4 and 2=3–4 intermediate shock |

## 3D bow shock: intermediate 1–3 shock in the leading front

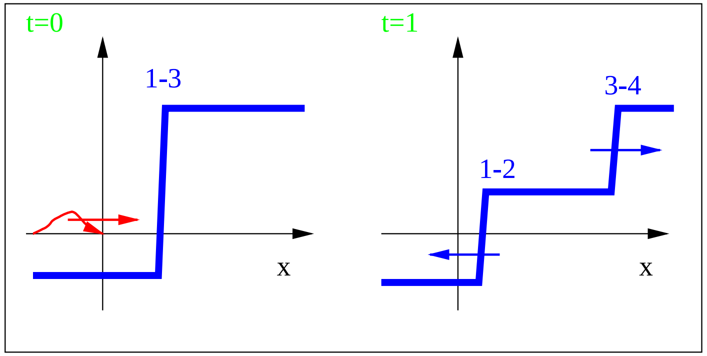



- along cut P: fast and Alfvénic Mach number pass 1



# Large-scale stability of 3D bow shock flows

# with intermediate shock segments



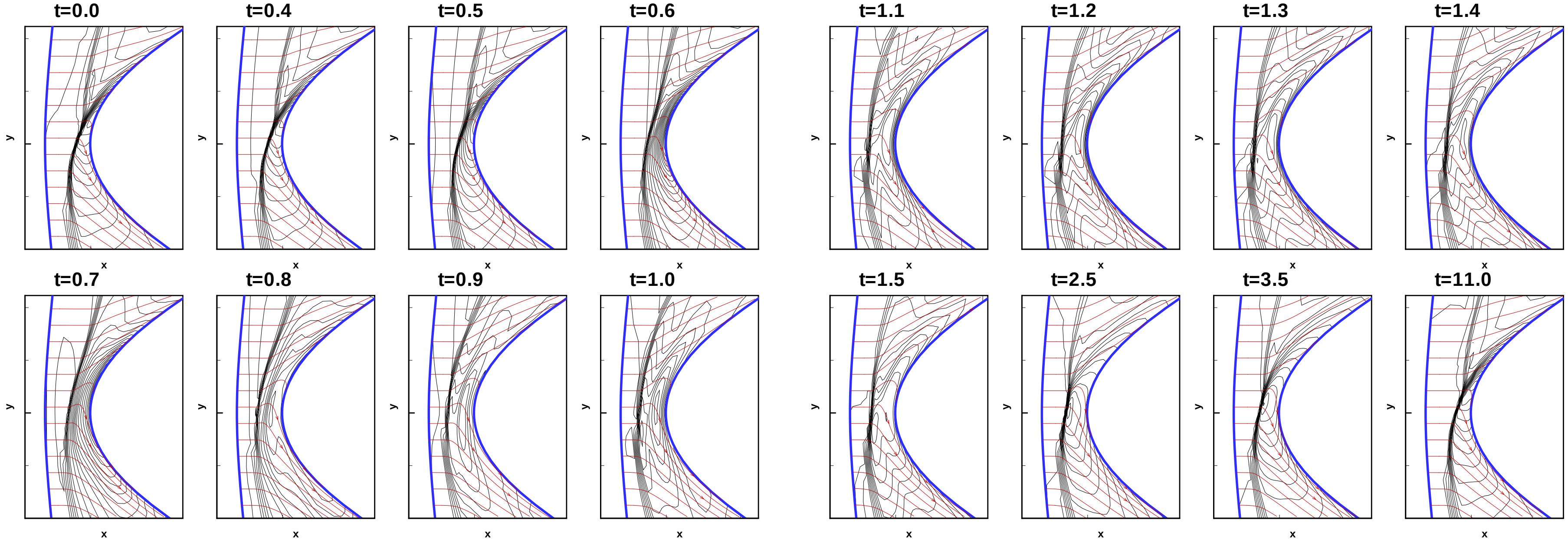

• **question**: can an intermediate shock survive in a plasma where perturbations are present?

**first thought**: no, the perturbations will destabilize the shock

**second thought**: YES: in DRIVEN problems an intermediate shock can be *reformed* (through nonlinear steepening) after the perturbation has passed

• this is proved in the following numerical experiments which are perturbations of the above described 3D stationary flows

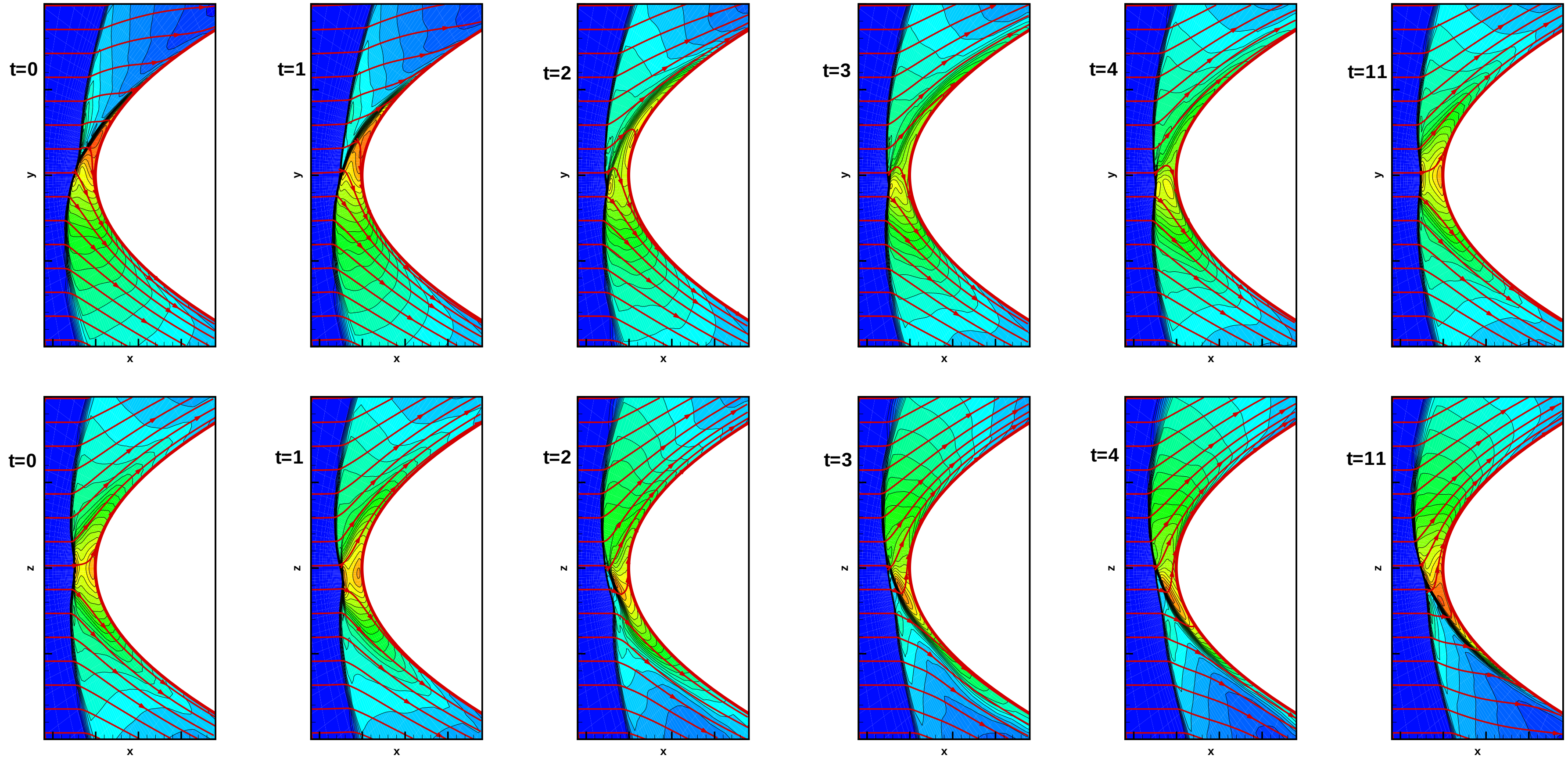
- **EXPERIMENT A**: perturb $B_z = \exp(-(t-0.5)^2/0.04)$




$\Rightarrow$ the 1–3 intermediate shock segment splits up in other shocks, but is *REFORMED* when the perturbation has passed

# *4) Overcompressive shocks*

● **EXPERIMENT B**: rotate $v$ around the $x$–axis from $v_z = 0$ to $v_y = 0$ between $t = 0$ and $t = 1$




$\Rightarrow$ the 1–3 intermediate shock segment in the $xy$ plane splits up into two other shocks, but is *REFORMED* at a different location – in the $xz$ plane – corresponding to the new (steady) inflow conditions

## remaining questions

- what is the ideal MHD solution for the 3D bow shock problem?

  - (without intermediate (overcompressive) shocks?)

- what is the solution for ranges of dissipative parameters for which some intermediate shocks do not exist?

- in physical plasmas: can intermediate shocks be observed?

  - will the supercritical perturbations be infrequent enough such that the intermediate shocks get time to form ? (turbulence)

  - what are the dissipation mechanisms and coefficient magnitudes?

- simulations with explicit discretization of dissipative terms

  - (huge parameter space)

# 5) ‘Every constant state is bordered by a simple wave’

## Simple waves

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \qquad n \times n \text{ system}$$

Solutions $U(x,t)$ that are constant on curves $x(t)$ in a region of the $(x,t)$ plane

$$\text{simple wave} \Leftrightarrow \frac{\mathrm{d}U(x(t),t)}{\mathrm{d}t} \equiv 0$$

($U$ constant on curves $x(t)$)

## Simple waves in systems

$$\frac{\partial U}{\partial t} + F'(U) \cdot \frac{\partial U}{\partial x} = 0 \qquad \text{and curve } x(t) : \frac{\partial x(t)}{\partial t} = \xi$$

simple wave: $$\frac{\mathrm{d}U(x(t),t)}{\mathrm{d}t} = \frac{\partial U(x(t),t)}{\partial t} + \frac{\partial U(x(t),t)}{\partial x} \frac{\partial x(t)}{\partial t} \equiv 0$$

$$\frac{\partial U(x(t),t)}{\partial t} + \frac{\partial U(x(t),t)}{\partial x} \xi = 0$$

$$F'(U) \cdot \frac{\partial U(x(t),t)}{\partial t} + F'(U) \cdot \frac{\partial U(x(t),t)}{\partial x} \xi = 0 \qquad \Rightarrow$$

$$\boxed{F'(U) \cdot \frac{\partial U(x(t),t)}{\partial t} = \xi \, \frac{\partial U(x(t),t)}{\partial t}}$$

## 5) 'Every constant state is bordered by a simple wave'

$$F'(U) \cdot \frac{\partial U(x(t),t)}{\partial t} = \xi \, \frac{\partial U(x(t),t)}{\partial t}$$

$$\Rightarrow \xi = \lambda_k(u)$$

simple wave of the $k$th family

curve $x(t)$ is characteristic curve, and because $U$ is constant on $x(t)$, the $k$–characteristic is a straight line

$$\Rightarrow \frac{\partial U(x(t),t)}{\partial t} = \alpha \, R_k$$

time derivative proportional to $k$th right eigenvector

$$\Rightarrow \frac{\partial U(x(t),t)}{\partial x} = -\frac{1}{\xi} \, \frac{\partial U(x(t),t)}{\partial t} = \beta \, R_k$$

space derivative proportional to $k$th right eigenvector

$\Rightarrow$ leads to equivalent definition of simple wave:

| simple wave of $k$th family $\Leftrightarrow$ $U$ is constant on $k$–characteristic (straight line) |
|---|

$\Rightarrow$ only $k$th wave mode is 'active' in simple wave of $k$th family

## $k$–Riemann Functions ($k$–RF)

for a given $k$, k–Riemann Functions are scalar functions

$$v_s^{(k)}(U) \quad s = 1 \dots n-1$$

with property

$$\frac{\partial v_s^{(k)}(U)}{\partial U} \cdot R_k(U) = 0$$

$$(\nabla v_s^{(k)}(U) \cdot R_k(U) = 0)$$

theorem: there exist $n-1$ independent $v_s^{(k)}(U) \quad \forall k \quad (s = 1 \dots n-1)$

the $\nabla v_s^{(k)}(U)$ span the orthogonal complement of $R_k$

## $k$–Riemann Functions in k–simple wave

$k$–Riemann Functions:

$$\frac{\partial v_s^{(k)}(U)}{\partial U} \cdot R_k(U) \equiv 0$$

- $k$–simple wave:

$$\frac{\partial U}{\partial t} = \alpha\, R_k \quad \text{and} \quad \frac{\partial U}{\partial x} = \beta\, R_k$$

$$\Rightarrow \frac{\partial v_s^{(k)}(U)}{\partial t} = \frac{\partial v_s^{(k)(U)}}{\partial U} \cdot \frac{\partial U}{\partial t} = \alpha\, \frac{\partial v_s^{(k)(U)}}{\partial U} \cdot R_k = 0$$

$$\Rightarrow \frac{\partial v_s^{(k)}(U)}{\partial t} = \frac{\partial v_s^{(k)(U)}}{\partial U} \cdot \frac{\partial U}{\partial x} = \beta\, \frac{\partial v_s^{(k)(U)}}{\partial U} \cdot R_k = 0$$

$\Rightarrow$ in $k$–simple wave the $v_s^{(k)}(U)$ are constant in space and time ($s = 1 \ldots n-1$)

## 5) *'Every constant state is bordered by a simple wave'*

Property: every constant region is bordered by a characteristic in continuous flow

assume region in $(x, t)$ plane with $U$ constant in space and time, bordered by curve C

- if C is not characteristic $\Rightarrow$ the solution away from C can uniquely be continuated

  constant continuation is certainly possible $\Rightarrow$ the unique continuation is the constant solution!

- if the solution is NOT constant away from C, then C has to be a characteristic

  (also, C will be a straight characteristic because $U$ is constant on C)

## 5) 'Every constant state is bordered by a simple wave'

Property: every constant region is bordered by a simple wave region

e.g., Lax '57

border curve C is $k$–characteristic $\Rightarrow$ $k$–simple wave

$j$–characteristic $x_j(t) : \dfrac{\partial x_j(t)}{\partial t} = \lambda_j$

$$\mathsf{L}(U) \cdot \frac{\partial U}{\partial t} + \Lambda(U) \cdot \mathsf{L}(U) \cdot \frac{\partial U}{\partial x} = 0$$

$L_j \cdot (\dfrac{\partial U}{\partial t} + \lambda_j \dfrac{\partial U}{\partial x}) = 0$ or $L_j \cdot \dfrac{dU}{dx_j} = 0$

also $L_j \cdot R_k = 0 \quad (j \neq k)$

and thus $L_j = \sum_{s=1}^{n-1} b_{js} \nabla v_s^{(k)}$

because $\nabla v_s^{(k)}$ span orthogonal complement of $R_k$

## 5) 'Every constant state is bordered by a simple wave'

$$\Rightarrow \sum_{s=1}^{n-1} b_{js} \, \nabla v_s^{(k)} \, \frac{dU}{dx_j} = 0$$

$$\Rightarrow \sum_{s=1}^{n-1} b_{js} \frac{dv_s^{(k)}}{dx_j} = 0 \quad \forall j \neq k \quad (n-1 \text{ system})$$

$$\Rightarrow \mathsf{B} \cdot \frac{\partial v_s^{(k)}}{\partial t} + \Lambda \cdot \mathsf{B} \cdot \frac{\partial v_s^{(k)}}{\partial x} = 0$$

$$\Rightarrow \frac{\partial v_s^{(k)}}{\partial t} + \mathsf{B}^{-1} \cdot \Lambda \cdot \mathsf{B} \cdot \frac{\partial v_s^{(k)}}{\partial x} = 0$$

linear hyperbolic system with $\lambda = \lambda_j \quad (j \neq k)$

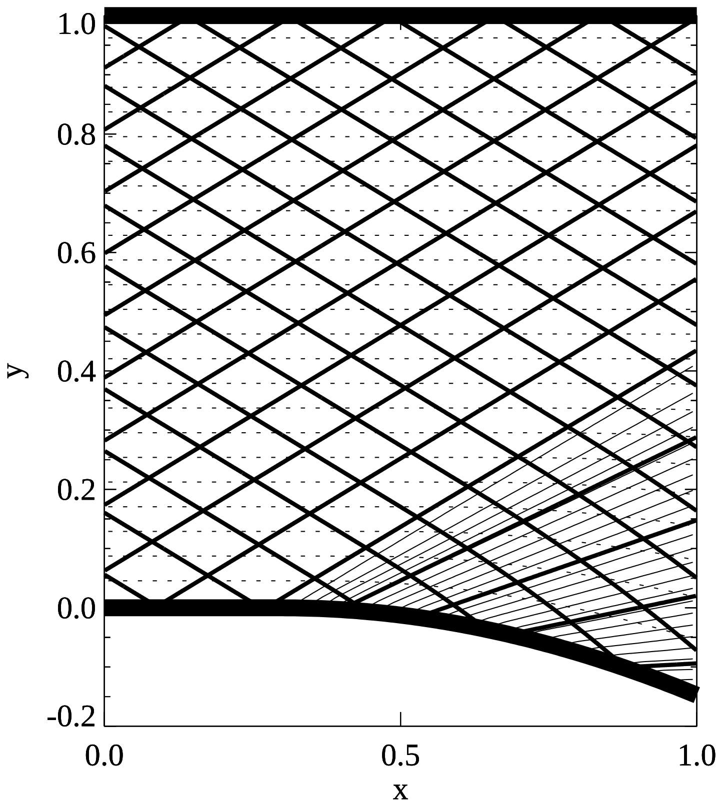
border curve $C$ is not a $k$–characteristic of this system!

we can uniquely continue the $v_s^{(k)}$ as a constant from the constant region

$v_s^{(k)}$ are constant on the other side of curve C

$\Rightarrow$ the constant region is bordered by a $k$–simple wave region!

# 5) 'Every constant state is bordered by a simple wave'



## 5) *'Every constant state is bordered by a simple wave'*

this is stated as a general result in textbooks on hyperbolic systems

BUT: is this always true?

NO! what if curve C is a DOUBLE characteristic

$$x_C(t) : \frac{\partial x_C(t)}{\partial t} = \lambda_k = \lambda_{k+1}$$

$\Rightarrow$ proof is not valid

- in Euler: strictly hyperbolic: $v + c \neq v - c$!

  this case can never happen; property is generally valid

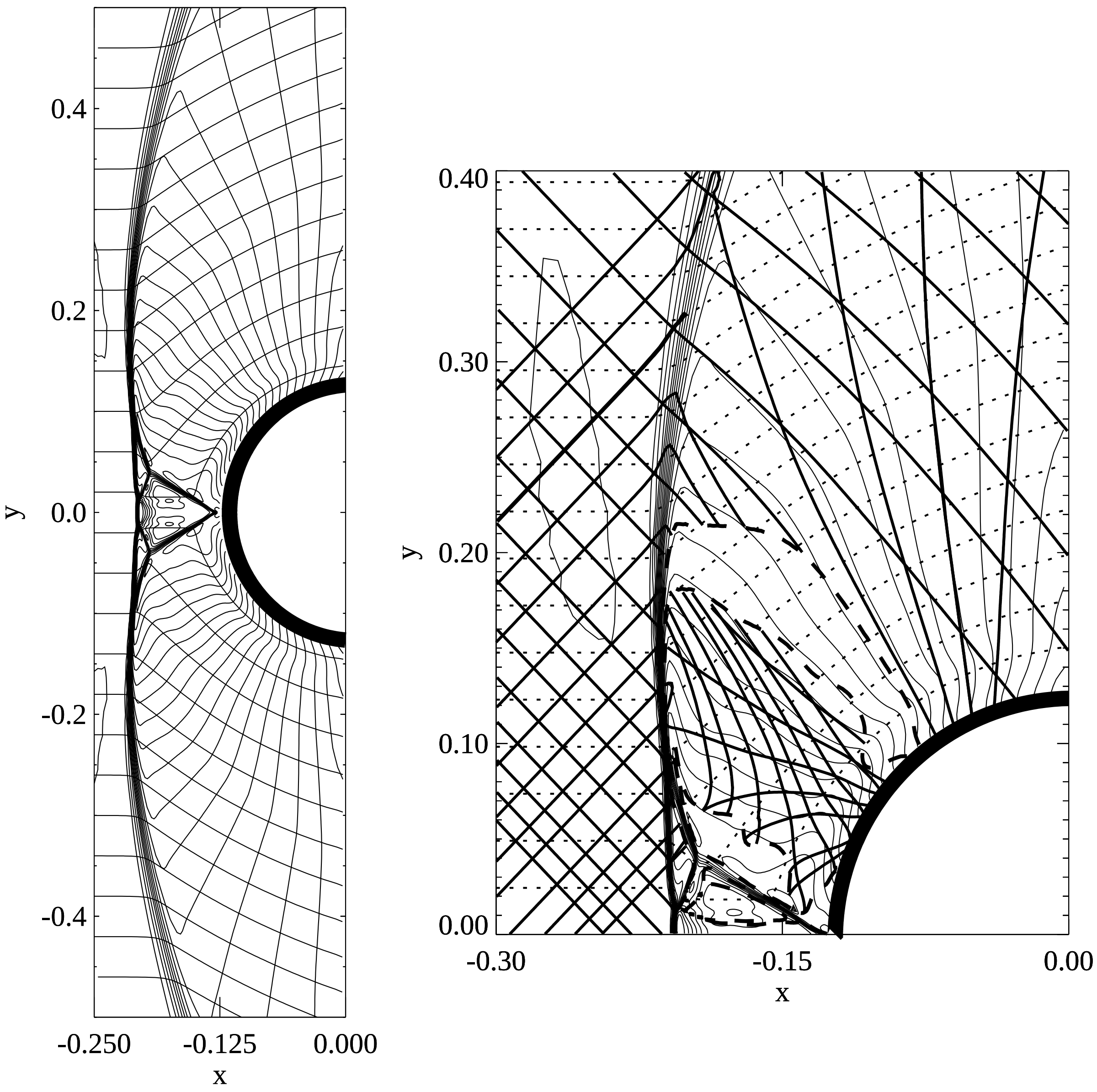
- in MHD: non-strictly hyperbolic: $v + c_s = v + c_f$ is possible

  theorem not generally valid, but remains special case

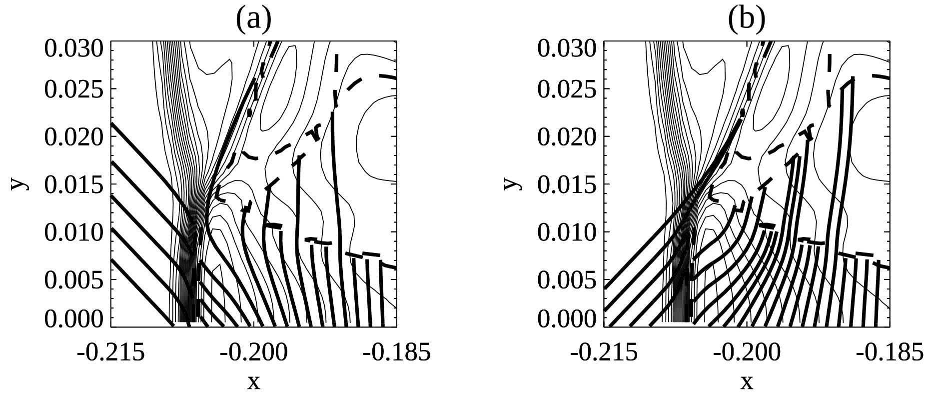
  example in 2D simulation results of MHD bow shock

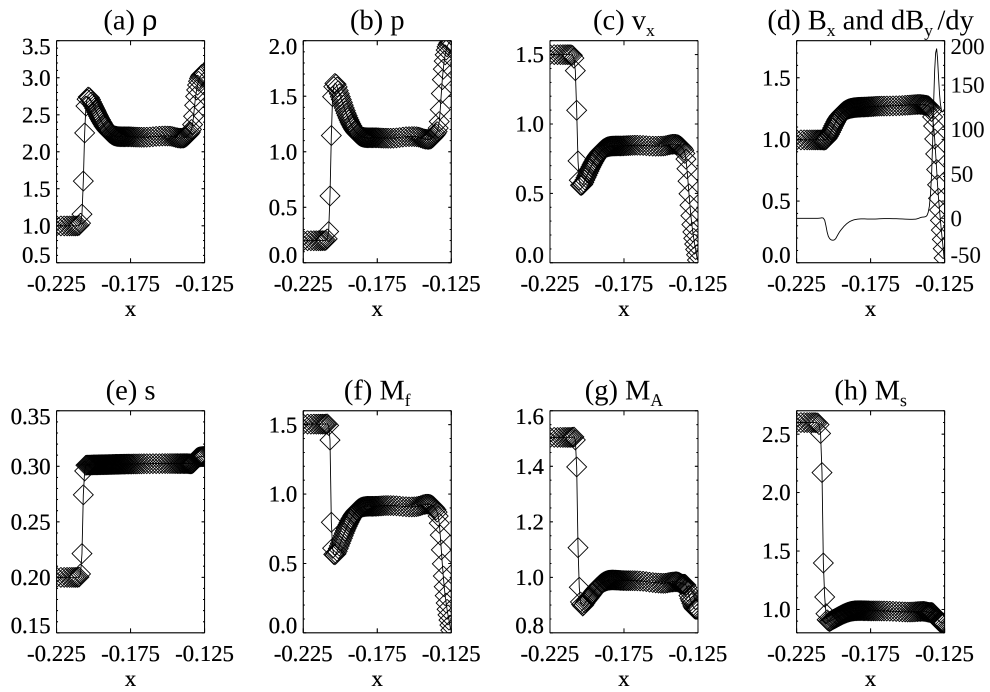
  (stationary characteristics $\sim$ space-time characteristics)

# 5) ‘Every constant state is bordered by a simple wave’



# 5) 'Every constant state is bordered by a simple wave'



# Conclusions

- hyperbolic theory of MHD:

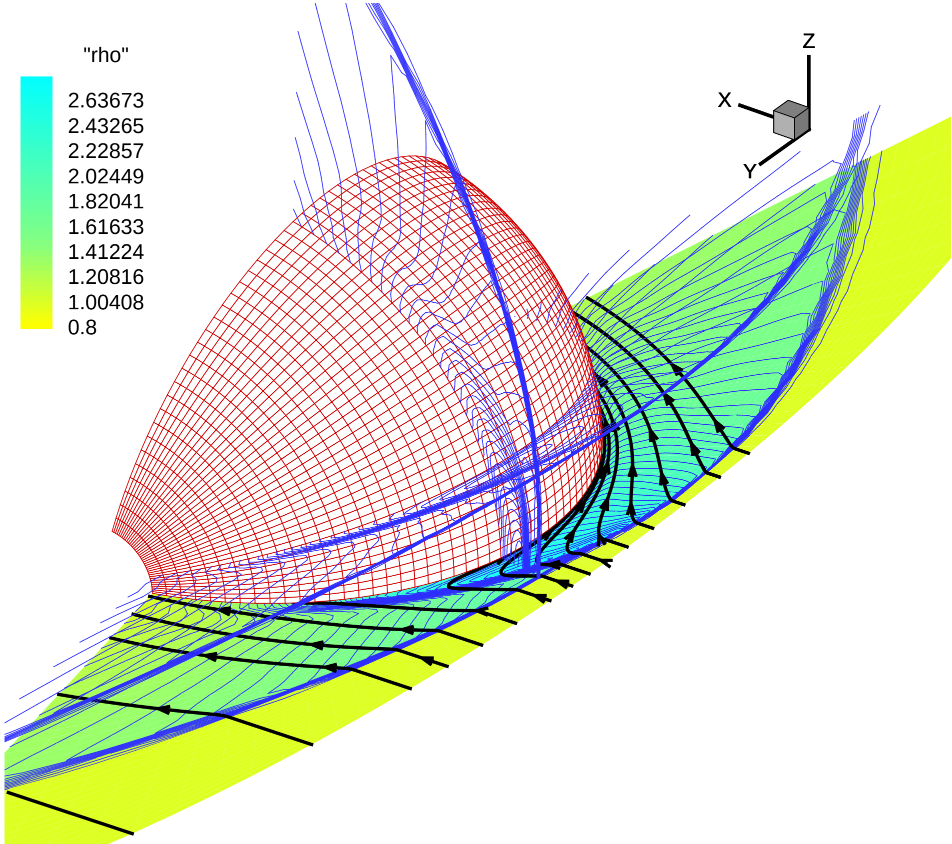
  - non-strictly hyperbolic
  - non-convex $\Rightarrow$ compound shocks
  - rotationally invariant $\Rightarrow$ instability of (overcompressive) intermediate shocks

has been illustrated in 2D and 3D stationary bow shock flows

- compound shocks in 2D and 3D
- overcompressive (intermediate) shocks in 2D and 3D
- non-simple wave region bordering constant region

## *Conclusions*

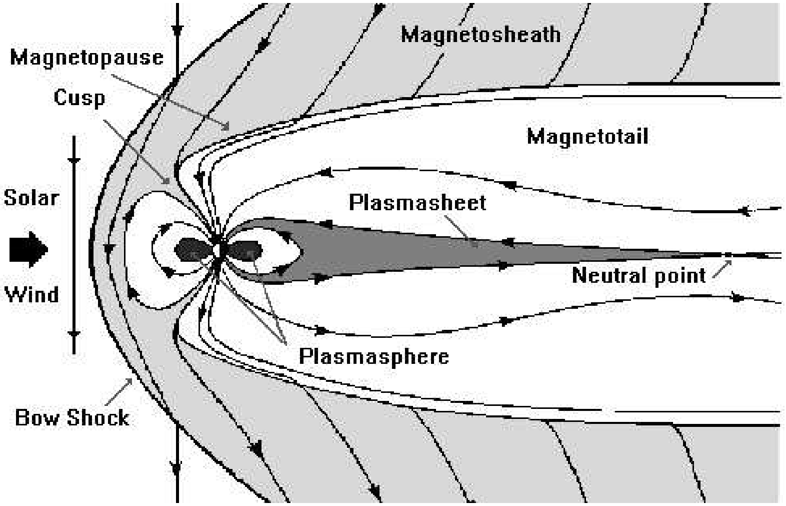
- (overcompressive) intermediate shocks arise naturally in 3D MHD flows
- perturbations destroy shocks, but shocks are reformed in a driven problem



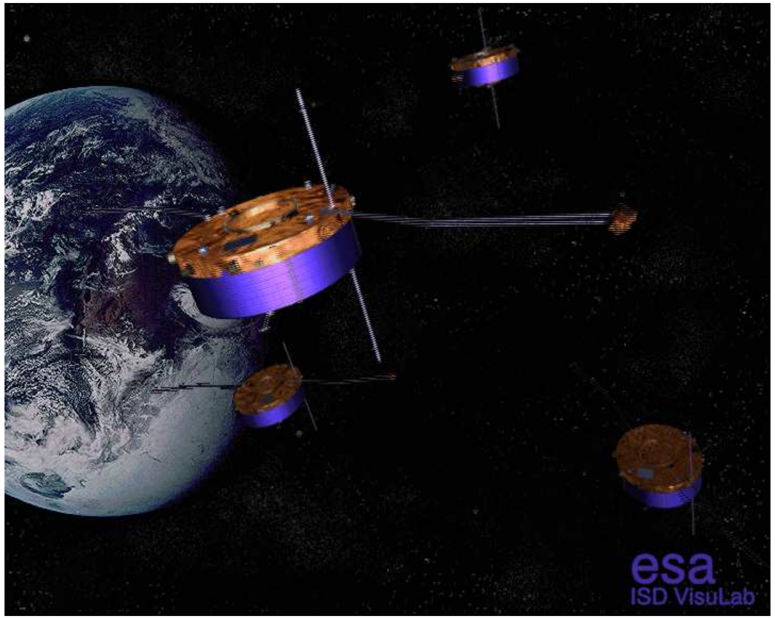

- remaining questions
  - vary dissipation: other solutions?
  - validity of ideal MHD, solution for ideal MHD 3D flow?

## *Conclusions*

- observation of intermediate shocks in Space Physics flows?




CLUSTER II (2000)




STEREO (2003)