CO 355 Mathematical Optimization (Fall 2010)
Assignment 5

Due: Thursday, Dec 2nd.

Policy. No collaboration is allowed. You may use the course notes / textbook, lecture slides, and any solutions to previous assignments but **please be very specific** when using citing results found there. (Don't just say "from some claim in class we know....".) Every other resource that you might stumble upon must be properly referenced. You are welcome to seek help from the current instructor and TAs for CO 355.

Question 1: Prove that if the matrix A is totally unimodular, then so are the following matrices.

(a): A^T

(b): $\begin{pmatrix} A & I \end{pmatrix}$

(c): any matrix obtained by multiplying a row or column of A by -1

(d): $\begin{pmatrix} A & -A \end{pmatrix}$

(e): $\begin{pmatrix} A \\ -A \\ I \\ -I \end{pmatrix}$

Question 2: Let M be a matrix of size $m \times n$ such that, in every row i, the first r_i entries are 1 and the remainder are 0. In other words, there exist integers $r_1, ..., r_m$ such that

$$M_{i,j} = \begin{cases} 1 & \text{(if } j \leq r_i) \\ 0 & \text{(otherwise)} \end{cases}.$$

Prove that M is totally unimodular.

Question 3: Define

$$\begin{aligned} F &= \{ \, x \, : \, Ax \leq b, x \in \mathbb{Z}^n \, \} \\ P &= \operatorname{conv}(F). \end{aligned}$$

Here we assume that F is non-empty. Consider the *integer* program

$$\text{(IP)} \qquad \max \left\{ \, c^\mathsf{T} x \, : \, x \in F \, \right\}.$$

Here is a common approach to solving such integer programs. (We implicitly used this approach in class for, e.g., the bipartite matching problem.) Let $P = \operatorname{conv}(F)$ be the convex hull of F. Suppose that P is a polyhedron. (For example, this holds if F is bounded — see Theorem 2.32 in the Course Notes.) Consider the following *linear* program:

$$\text{(LP)} \qquad \max \left\{ \, c^\mathsf{T} x \, : \, x \in P = \operatorname{conv}(F) \, \right\}$$

(a): Prove that every basic feasible solution for (LP) is a feasible solution for (IP).

(b): Prove that every optimal BFS for (LP) is an optimal solution for (IP).

Question 4: Consider the Max Cut SDP.

(a): Is there a graph such that the optimal value of the SDP is strictly *less* than the size of the maximum cut in the graph? If so, give an example. If not, explain why not.

(b): Is there a graph such that the optimal value of the SDP is strictly *greater* than the size of the maximum cut in the graph? If so, give an example. If not, explain why not.

Question 5: Let $G = (V, E)$ be a graph and, for every $v \in V$, let w_v be a weight on vertex v, where $w_v \in \mathbb{R}$ and $w_v \geq 0$.

(a): Suppose G is bipartite. Describe an efficient (polynomial-time) algorithm to find a vertex cover C that minimizes $\sum_{v \in C} w_v$.

(b): Suppose G is not bipartite. Let C^* be a vertex cover that minimizes $\sum_{v \in C^*} w_v$. Describe an efficient (polynomial-time) algorithm to find a vertex cover C such that

$$\sum_{v \in C} w_v \; \leq \; 2 \cdot \sum_{v \in C^*} w_v.$$